Packing Coloring and Grids

Martin Böhm, Jan Ekstein, Jiří Fiala, Přemek Holub,
Sandi Klavžar, Lukáš Lánský and Bernard Lidický

Charles University
University of Ljubljana and
University of West Bohemia

7th Slovenian International Conference on Graph Theory - Bled'11

Packing Chromatic Number

Definition

Graph $G = (V, E)$, $X_d \subseteq V$ is *d-packing* if
$\forall u, v \in X_d : \mathrm{distance}(u, v) > d$.

1-packing is an independent set

Definition

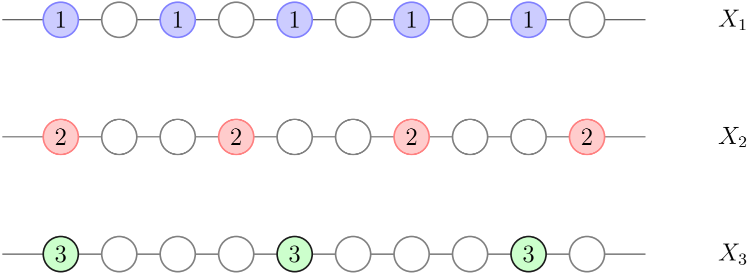
Packing chromatic number is the minimum k such that
$V = X_1 \cup X_2 \cup ... \cup X_k$; denoted by $\chi_\rho(G)$.

Also known as the *broadcast chromatic number*.

Example with path P_∞

Definition

Packing chromatic number is the minimum k such that $V = X_1 \cup X_2 \cup \ldots \cup X_k$; denoted by $\chi_\rho(G)$.

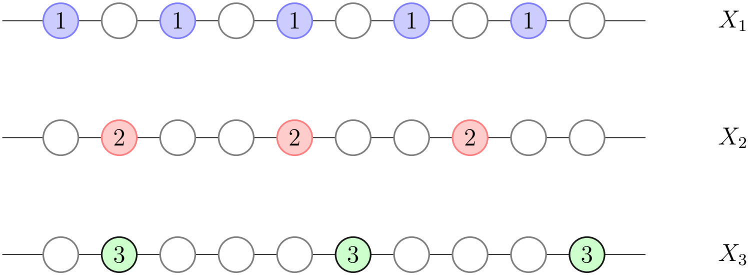


informally, density of X_d is $|X_d|/|V|$

Example with path P_∞

Definition

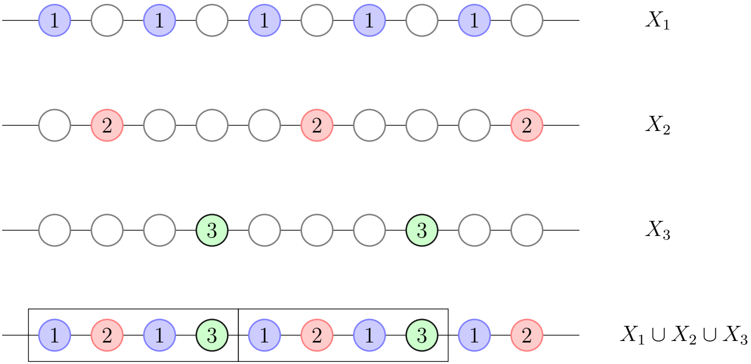
Packing chromatic number is the minimum k such that $V = X_1 \cup X_2 \cup \ldots \cup X_k$; denoted by $\chi_\rho(G)$.



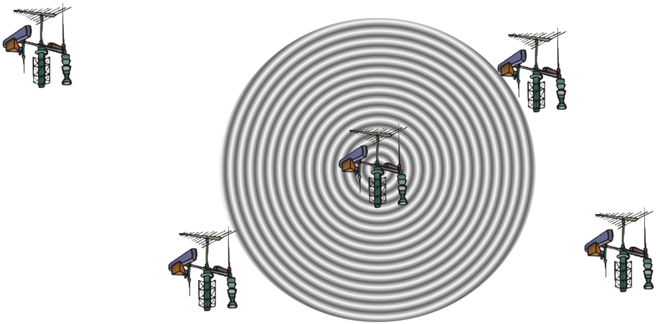
informally, density of X_d is $|X_d|/|V|$

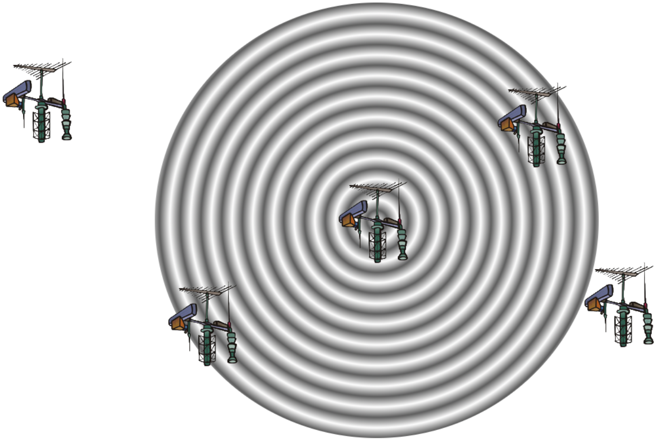
Definition

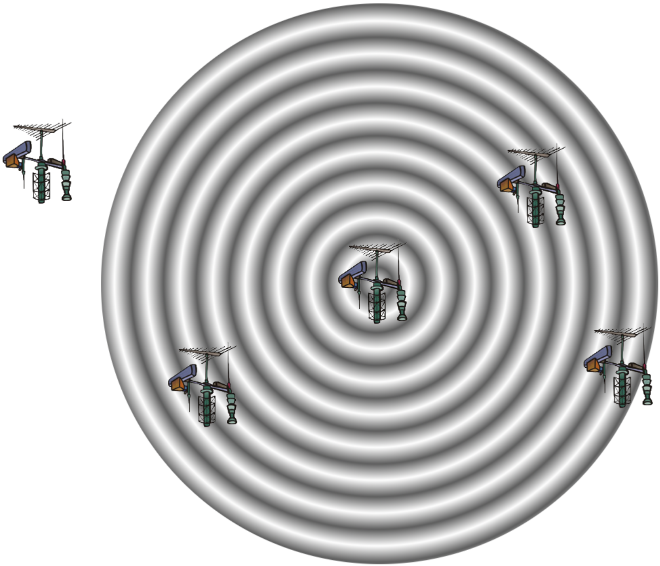
Packing chromatic number is the minimum k such that $V = X_1 \cup X_2 \cup \ldots \cup X_k$; denoted by $\chi_\rho(G)$.

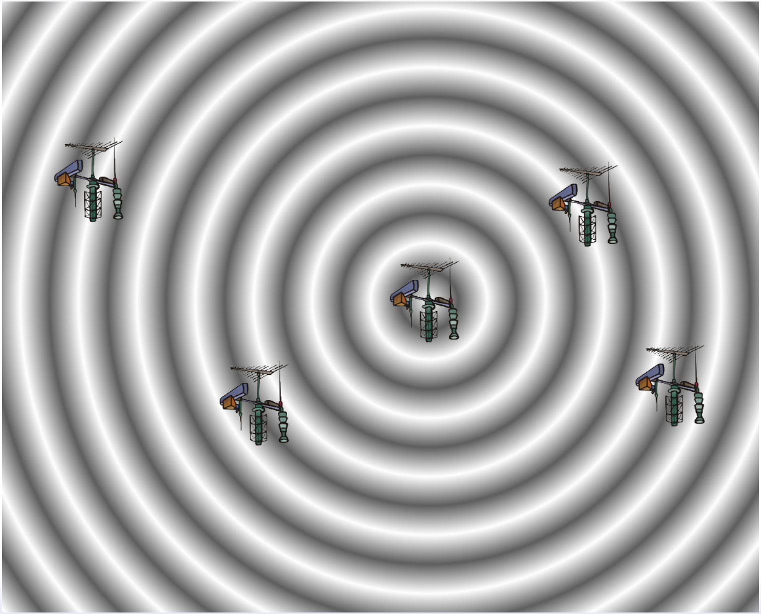


$\chi_\rho(P_\infty) = 3$









Complexity of χ_ρ

Theorem (Goddard, Hedetniemi, Hedetniemi, Harris, Rall '08)

Let G be a graph.

- *Decide if* $\chi_\rho(G) \leq k$ *is* $\mathcal{NP}$*-complete (k on input).*
- *Decide if* $\chi_\rho(G) \leq 3$ *is in* $\mathcal{P}$*.*
- *Decide if* $\chi_\rho(G) \leq 4$ *is* $\mathcal{NP}$*-complete.*

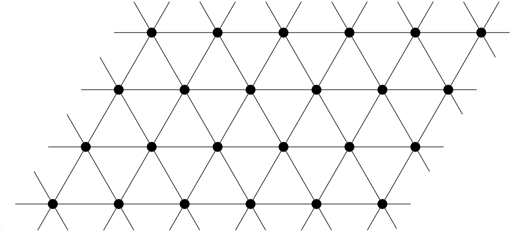
Theorem (Fiala, Golovach '09)

Decide if $\chi_\rho(G) \leq k$ *for trees is* $\mathcal{NP}$*-complete (k on input).*

Triangular lattice $\mathcal{T}$

Theorem (Finbow, Rall '07)

Infinite triangular lattice $\mathcal{T}$ cannot be colored by a finite number of colors.



We use notation $\chi_\rho(\mathcal{T}) = \infty$.

Hexagonal Lattice $\mathcal{H}$

Theorem (Brešar, Klavžar, Rall '07)

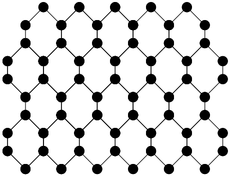
For hexagonal lattice $\mathcal{H}$: $6 \leq \chi_\rho(\mathcal{H}) \leq 8$

Theorem (Vesel '07)

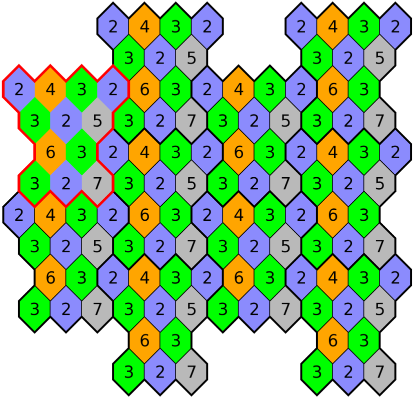
$7 \leq \chi_\rho(\mathcal{H})$

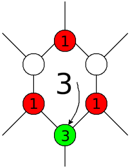
Theorem (Fiala, Klavžar, L. '09)

$\chi_\rho(\mathcal{H}) \leq 7$



$\chi_\rho(\mathcal{H}) \leq 7$





d-packing	density
1	1/2
2	1/6
3	1/6
4	1/24
5	1/24
6	1/24
7	1/24

Square lattice $\mathbb{Z}^2 (= \mathbb{Z} \,\square\, \mathbb{Z})$

Theorem (Goddard et al. '08)

For infinite planar square lattice $\mathbb{Z}^2$:
$9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$

Theorem (Schwenk '02)

$\chi_\rho(\mathbb{Z}^2) \leq 22$

Theorem (Fiala, Klavžar, L. '09)

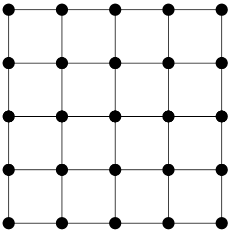
$10 \leq \chi_\rho(\mathbb{Z}^2)$

Theorem (Holub, Soukal '09)

$\chi_\rho(\mathbb{Z}^2) \leq 17$

Theorem (Ekstein, Holub, Fiala, L. '10)

$12 \leq \chi_\rho(\mathbb{Z}^2)$



$\chi_\rho(\mathbb{Z}^2) \leq 17$

1	2	1	3	1	2	1	10	1	4	1	9	1	2	1	3	1	2	1	5	1	4	1	14
7	1	5	1	6	1	3	1	2	1	3	1	8	1	5	1	4	1	3	1	2	1	3	1
1	3	1	2	1	4	1	7	1	5	1	2	1	3	1	2	1	11	1	6	1	10	1	2
4	1	9	1	3	1	2	1	3	1	6	1	4	1	7	1	3	1	2	1	3	1	5	1
1	2	1	15	1	5	1	11	1	2	1	3	1	2	1	17	1	5	1	4	1	2	1	3
6	1	3	1	2	1	3	1	4	1	14	1	5	1	3	1	2	1	3	1	7	1	8	1
1	5	1	4	1	16	1	2	1	3	1	2	1	10	1	4	1	13	1	2	1	3	1	2
3	1	2	1	3	1	6	1	5	1	7	1	3	1	2	1	3	1	9	1	5	1	4	1
1	7	1	10	1	2	1	3	1	2	1	4	1	6	1	5	1	2	1	3	1	2	1	11
2	1	3	1	5	1	4	1	8	1	3	1	2	1	3	1	7	1	4	1	6	1	3	1
1	4	1	2	1	3	1	2	1	9	1	5	1	11	1	2	1	3	1	2	1	12	1	5
3	1	6	1	13	1	7	1	3	1	2	1	3	1	4	1	8	1	5	1	3	1	2	1
1	2	1	3	1	2	1	5	1	4	1	15	1	2	1	3	1	2	1	10	1	4	1	9
8	1	5	1	4	1	3	1	2	1	3	1	7	1	5	1	6	1	3	1	2	1	3	1
1	3	1	2	1	11	1	6	1	10	1	2	1	3	1	2	1	4	1	7	1	5	1	2
4	1	7	1	3	1	2	1	3	1	5	1	4	1	9	1	3	1	2	1	3	1	6	1
1	2	1	17	1	5	1	4	1	2	1	3	1	2	1	14	1	5	1	11	1	2	1	3
5	1	3	1	2	1	3	1	7	1	8	1	6	1	3	1	2	1	3	1	4	1	15	1
1	10	1	4	1	9	1	2	1	3	1	2	1	5	1	4	1	16	1	2	1	3	1	2
3	1	2	1	3	1	12	1	5	1	4	1	3	1	2	1	3	1	6	1	5	1	7	1
1	6	1	5	1	2	1	3	1	2	1	11	1	7	1	10	1	2	1	3	1	2	1	4
2	1	3	1	7	1	4	1	6	1	3	1	2	1	3	1	5	1	4	1	8	1	3	1
1	11	1	2	1	3	1	2	1	13	1	5	1	4	1	2	1	3	1	2	1	9	1	5
3	1	4	1	8	1	5	1	3	1	2	1	3	1	6	1	12	1	7	1	3	1	2	1

$\chi_\rho(\mathbb{Z}^2) \leq 12$

Wish (Conjecture)

If $\chi_\rho(\mathbb{Z}^2) = k$ then exist $X_1, \ldots, X_k$ such that $\forall i$ X_i has maximum possible density after fixing $\bigcup_{1 \leq j < i} X_j$.

Wish implies the lower bound 12.

No wish implies brute force computer search (backtracking).

Find a (small) part of $\mathbb{Z}^2$ that cannot be colored by 11 colors.

$\chi_\rho(\mathbb{Z}^2) \leq 12$

Wish (Conjecture)

If $\chi_\rho(\mathbb{Z}^2) = k$ then exist $X_1, \ldots, X_k$ such that $\forall i$ X_i has maximum possible density after fixing $\bigcup_{1 \leq j < i} X_j$.

Wish implies the lower bound 12.

No wish implies brute force computer search (backtracking).

Find a (small) part of $\mathbb{Z}^2$ that cannot be colored by 11 colors.

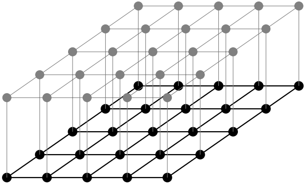
Layers of the square lattice - going 3D

Theorem (Finbow, Rall '07)

$\chi_\rho(\mathbb{Z}^3) = \infty$

Theorem (Fiala, Klavžar, L. '09)

$\chi_\rho(P_2 \,\square\, \mathbb{Z}^2) = \infty$



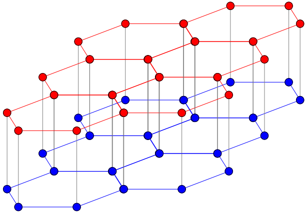
Layers of the hexagonal lattice - going 3D

Theorem (Fiala, Klavžar, L. '09)

$\chi_\rho(P_6 \square \mathcal{H}) = \infty$

Theorem (Böhm, Lánský, L. '10)

$\chi_\rho(P_2 \square \mathcal{H}) \leq 526$ *(large but finite)*

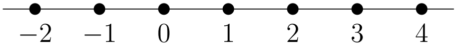


Layers summary

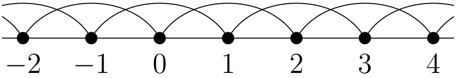
Lattice	Triangular	Square ($\mathbb{Z}^2$)	Hexagonal ($\mathcal{H}$)
Colorable layers l	0	1	$2 \leq l < 6$

Distance graphs

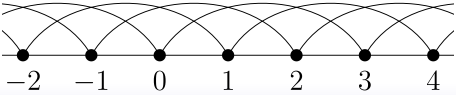
- $C \subset \mathbb{N}$
- A *distance graph* $D(C)$ is a graph on vertices $\mathbb{Z}$, uv adjacent if $|u - v| \in C$.
- $D(\{1\}) = P_\infty$



- $D(\{1,2\})$



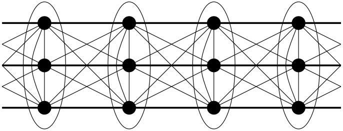
- $D(\{1,3\})$



Distance graphs - general bound

Theorem (Goddard et al. '08)

Let G be finite. Then $\chi_\rho(P_\infty \square G) < \infty$.



Corollary

$\chi_\rho(D(C)) < \infty$ *for any C.*

Distance graphs - $D(\{1,k\})$

Theorem (Togni '10)

$$\chi_\rho(D(\{1,t\})) \leq \begin{cases} 174 & t \text{ even}, \\ 86 & t \text{ odd} \end{cases}$$

if $t \geq 224$

special constructions

Theorem (Ekstein, Holub, L. '11)

$$\chi_\rho(D(\{1,t\})) \leq \begin{cases} 56 & t \text{ even}, \\ 35 & t \text{ odd} \end{cases}$$

if $t \geq 648$

using $\mathbb{Z}^2$

Distance graphs - $D(\{1, k\})$

Theorem (Togni '10)

$$\chi_\rho(D(\{1,t\})) \le \begin{cases} 174 & t \text{ even}, \\ 86 & t \text{ odd} \end{cases}$$

if $t \geq 224$

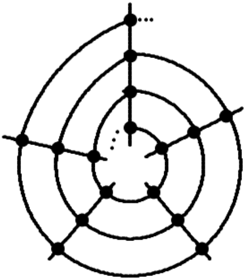
special constructions

Theorem (Ekstein, Holub, L. '11)

$$\chi_\rho(D(\{1,t\})) \le \begin{cases} 56 & t \text{ even}, \\ 35 & t \text{ odd} \end{cases}$$

if $t \geq 648$

using $\mathbb{Z}^2$



$D(\{1, 5\})$

Open problems

- Is $\chi_\rho(\mathcal{H} \,\square\, P_3)$ finite?
- What is $\chi_\rho(\mathbb{Z}^2)$? (12 – 17)
- Is there c such that every cubic graph G has $\chi_\rho(G) \leq c$?
  - if G is planar?
  - if G has large girth?

Open problems

- Is there c such that every cubic graph G has $\chi_\rho(G) \leq c$?
  - if G is planar?
  - if G has large girth?

Theorem (Sloper '02)

3-regular infinite tree T_3: $\chi_\rho(T_3) = 7$

Theorem (Sloper '02)

4-regular infinite tree T_4: $\chi_\rho(T_4) = \infty$

Thank you for your attention!