Packing chromatic number for square and hexagonal lattices

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Packing Chromatic Number

Definition

Graph $G = (V, E)$, $X_d \subseteq V$ is *d-packing* if
$\forall u, v \in X_d : \mathrm{distance}(u, v) > d$.

1-packing is an independent set

Definition

Packing chromatic number is the minimum k such that
$V = X_1 \cup X_2 \cup ... \cup X_k$; denoted by $\chi_\rho(G)$.

Also known as the *broadcast chromatic number*.

Example with path P_∞

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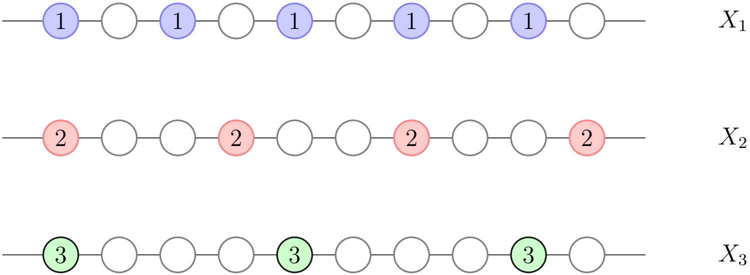

informally, density of X_d is $|X_d|/|V|$

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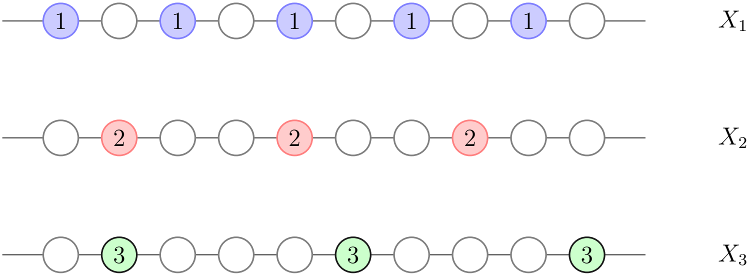

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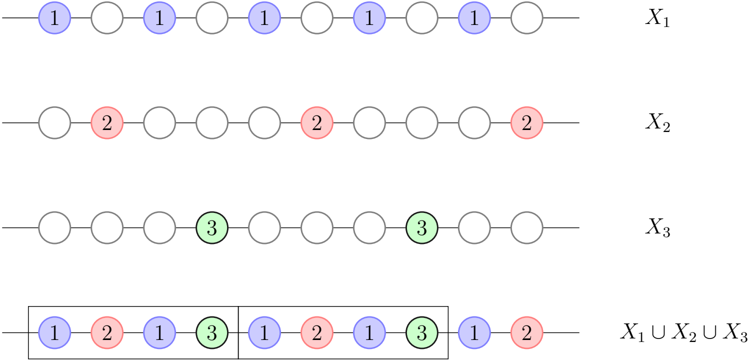
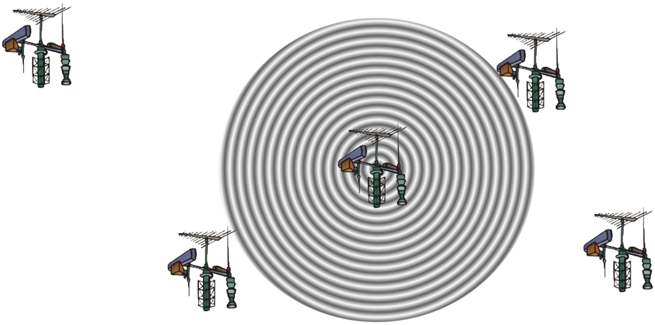
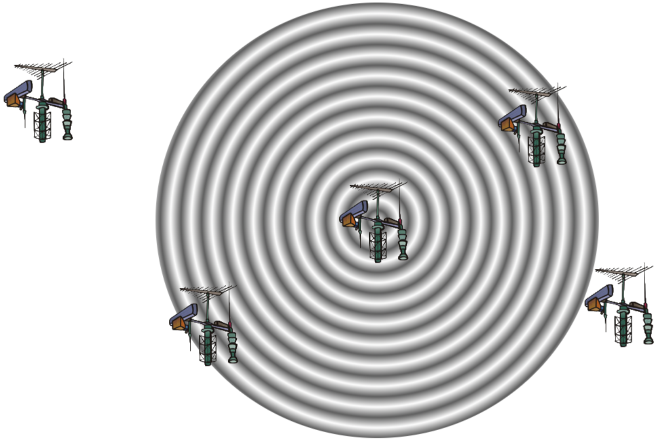
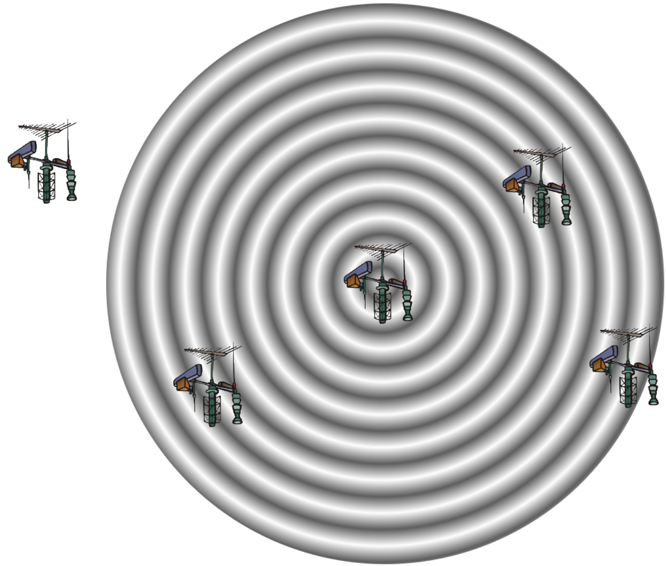
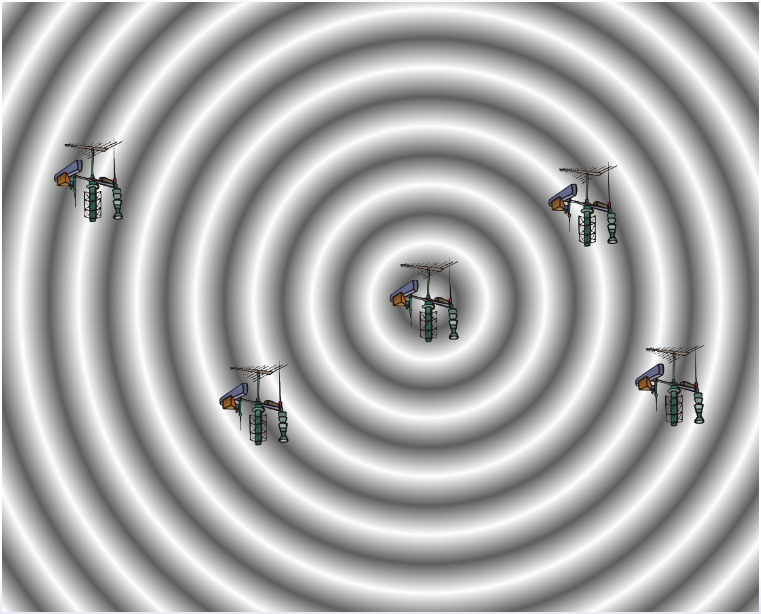


$\chi_\rho(P_\infty) = 3$









Complexity of χ_ρ

Theorem (Goddard et al. '08)

Let G be a graph.

- *Decide if* $\chi_\rho(G) \leq k$ *is* $\mathcal{NP}$*-complete (k on input).*
- *Decide if* $\chi_\rho(G) \leq 3$ *is in* $\mathcal{P}$*.*
- *Decide if* $\chi_\rho(G) \leq 4$ *is* $\mathcal{NP}$*-complete.*

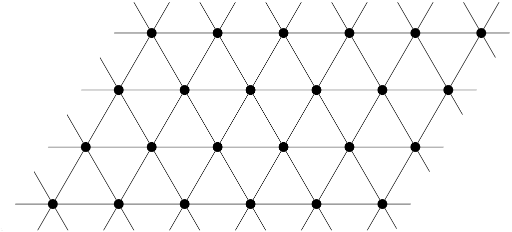
Theorem (Fiala and Golovach '09)

Decide if $\chi_\rho(G) \leq k$ *for trees is* $\mathcal{NP}$*-complete (k on input).*

Triangular lattice $\mathcal{T}$

Theorem (Finbow and Rall '07)

Infinite triangular lattice $\mathcal{T}$ cannot be colored by a finite number of colors.



We use notation $\chi_\rho(\mathcal{T}) = \infty$.

Hexagonal Lattice

Theorem (Brešar, Klavžar and Rall '07)

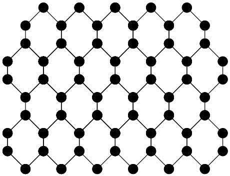
For hexagonal lattice $\mathcal{H}$: $6 \leq \chi_\rho(\mathcal{H}) \leq 8$

Theorem (Vesel '07)

$7 \leq \chi_\rho(\mathcal{H})$

Theorem (Fiala, Klavžar, L. '09)

$\chi_\rho(\mathcal{H}) \leq 7$



Square lattice

Theorem (Goddard et al. ’02)

For infinite planar square lattice $\mathbb{Z}^2$:
$9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$

Theorem (Schwenk ’02)

$\chi_\rho(\mathbb{Z}^2) \leq 22$

Theorem (Fiala, Klavžar, L. ’09)

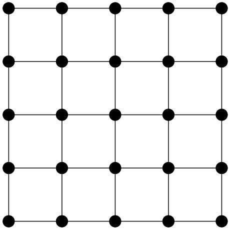
$10 \leq \chi_\rho(\mathbb{Z}^2)$

Theorem (Holub and Soukal ’09)

$\chi_\rho(\mathbb{Z}^2) \leq 17$

Theorem

$12 \leq \chi_\rho(\mathbb{Z}^2)$



How we did the lower bound 12

Wish (Conjecture)

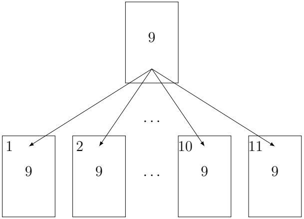
If $\chi_\rho(\mathbb{Z}^2) = k$ then exist $X_1, \ldots, X_k$ such that $\forall i$ X_i has maximum possible density after fixing $\bigcup_{1 \leq j < i} X_j$.

Wish implies the lower bound 12.

No wish implies brute force computer search (backtracking).

Find a (small) part of $\mathbb{Z}^2$ that cannot be colored by 11 colors.

Grid 15×9 cannot be colored using 11 colors when $[5, 5]$ is precolored by 9.



- *easy* to compute in parallel
- two implementations
- 115 days of CPU time (in 2009)
- 43112312093324 configurations tested

Layers of a lattice

Theorem (Finbow and Rall '07)

$\chi_\rho(\mathbb{Z}^3) = \infty$

Theorem (Fiala, Klavžar, L. '09)

$\chi_\rho(P_2 \,\square\, \mathbb{Z}^2) = \infty$

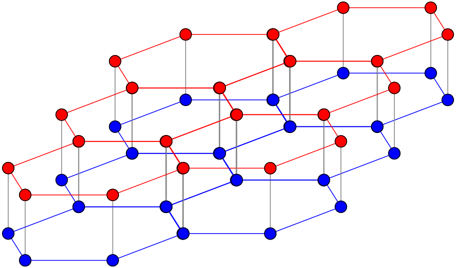
Theorem (Fiala, Klavžar, L. '09)

$\chi_\rho(P_6 \,\square\, \mathcal{H}) = \infty$

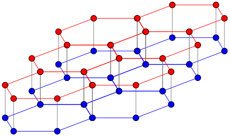
Theorem

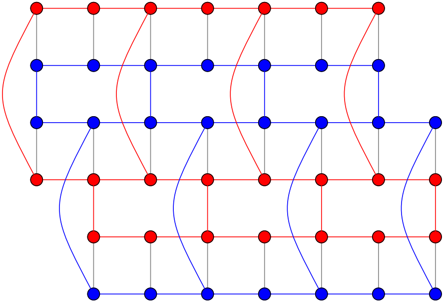
$\chi_\rho(P_2 \,\square\, \mathcal{H}) \leq 526$ *(large but finite)*

$P_2 \,\square\, \mathcal{H}$



$P_2 \square \mathcal{H}$





$P_2 \square \mathcal{H}$

Observations

- large colors have large period
- use small colors as much as possible (the Wish)
- there might be a lot of locally good patterns
- do not try to fill large area of zeros

Ideas

- glue small patterns with small colors to large pattern for large colors
- use some random while constructing pattern
- repeat search lot of times with crossed fingers

$P_2 \,\square\, \mathcal{H}$

Result

- pattern $(2\times)$ 768×768
- extra program for checking correctness of the result
- 526 colors (probably not optimal)
- lower bound for $\chi_\rho(P_2 \,\square\, \mathcal{H})$ around 20
  (more that $\chi_\rho(\mathbb{Z}^2)$, huge gap)

Open problems

- Is $\chi_\rho(\mathcal{H} \,\square\, P_3)$ finite?
- What is $\chi_\rho(\mathbb{Z}^2)$ for the infinite planar square lattice $\mathbb{Z}^2$?
- Is there c such that every cubic graph G has $\chi_\rho(G) \leq c$?
  - if G is planar?
  - if G has large girth?

Open problems

- Is there c such that every cubic graph G has $\chi_\rho(G) \leq c$?
  - if G is planar?
  - if G has large girth?

Theorem (Sloper '02)

3-regular infinite tree T_3: $\chi_\rho(T_3) = 7$

Theorem (Sloper '02)

4-regular infinite tree T_4: $\chi_\rho(T_4) = \infty$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