Computational Complexity of Locally Injective Homomorphisms to Weight Graphs: A Full Classification of Simple Weights

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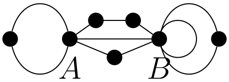
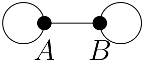
Definitions

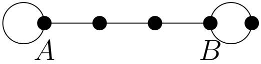
A *weight graph* is a connected multi-graph G with two vertices A, B of degree at least three and all other of degree two. Moreover, $G - A$ and $G - B$ contain a cycle.



A weight graph is *simple* if $\deg(A) = \deg(B) = 3$.
Also known as dumbbell or barbell.



$W(1,1,1)$



$W(1,2,3)$

Simple weight graphs

Name: H-LIHOM
Parameter: graph H
Input: graph G
Question: Does exist a locally injective homomorphism $f : V(G) \to V(H)$?

Theorem

If H is a bipartite simple weight graph (A, B in different parts), then H-LIHOM is solvable in polynomial time.

Theorem

If H is a non-bipartite simple weight graph, then H-LIHOM is NP-complete.

Simple overview of the polynomial algorithm

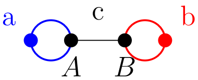
simple bipartite weight graph H
a graph G as input

- fix a bipartition of G
- compute possible mappings of paths of 2-vertices in G
- replace paths by gadgets and obtain G'
- compute a "matching" in G'
- construct a mapping $G \to H$ according to the matching

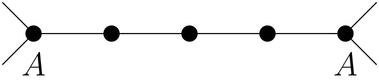
Simple bipartite case (polynomial time)

Let $H = W(a, b, c)$ be a bipartite weight graph, G an input graph

$W(2,2,1)$



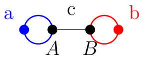
fix a bipartition of 3-vertices in G (2 possibilities)



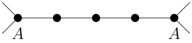
for every path decide possible mappings on ends
(a-c, b-c, a-a, a-b, c-a, c-b, c-c)

Simple bipartite case (polynomial time)

$H = W(a, b, c)$ be a bipartite weight graph



fix a bipartition of 3-vertices of G



for every path decide possible mappings on ends
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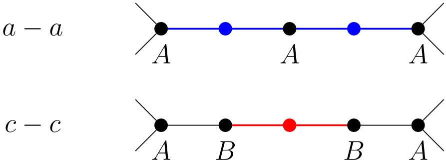


Simple bipartite case (polynomial time)

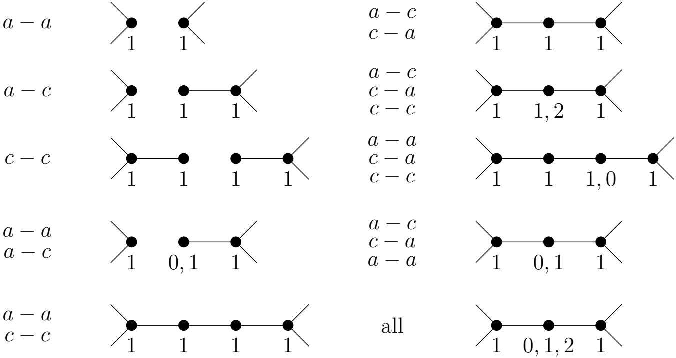
We want to determine the mapping of c around every 3-vertex one at every vertex - like matching

Lemma

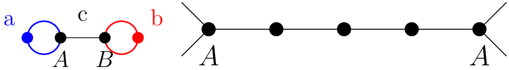
Let G be a graph and a mapping $f : V(G) \to I$, where I is a set of intervals. A subgraph G' of G such that $\deg_{G'}(v) \in f(v)$ for all $v \in V(G)$ can be found in polynomial time.

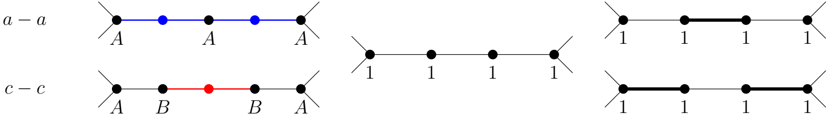
replace paths by gadgets (includes f) - G'
assign $f(v) = 1$ for all 3-vertices of G'

replace paths  by gadgets (includes *f*)



Simple bipartite case (polynomial time) - Example

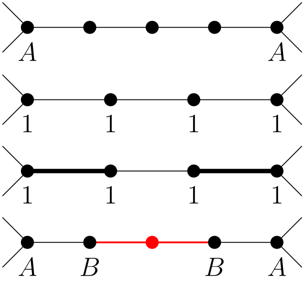




Replacing path by a gadget and getting a factor.

Simple bipartite case (polynomial time) - Overview

- fix a bipartition of G
- compute possible mappings of paths in G
- replace paths by gadgets (G')
- compute f-factor in (G')
- map c in G according to edges in f-factor of G'
- map a, b around A and B
- map remaining vertices



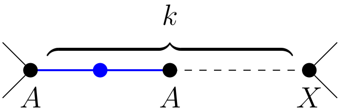
Simple non-bipartite case (NP-complete)

$G = W(a, b, c)$, assume $GCD(a, b, c) = 1$ and $GCD(a, b, 2c) = 1$

Lemma

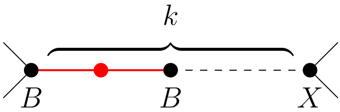
Let $a, b, c \in \mathbb{N}$ such that $GCD(a, b, 2c) = 1$. Then exist $x, y, z \in \mathbb{N}$ such that $ax = by + 2cz + c$ and $x, y \geq z$. $\square$

Let $k \in \mathbb{N}$ be the smallest such that exists a mapping of path of length k:



$A \sim k \sim X$

$A \sim k - X$



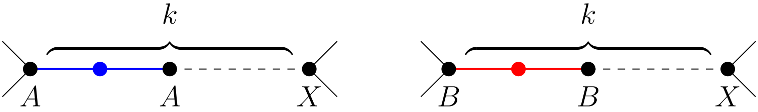
$B \sim k - X$

$B \sim k \sim X$

$X \in \{A, B\}$

Simple non-bipartite case (NP-complete)

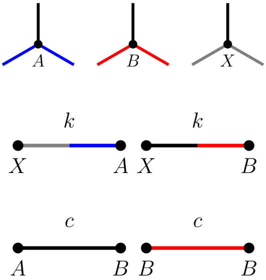
decide reduction according to what the path of length *k* allows

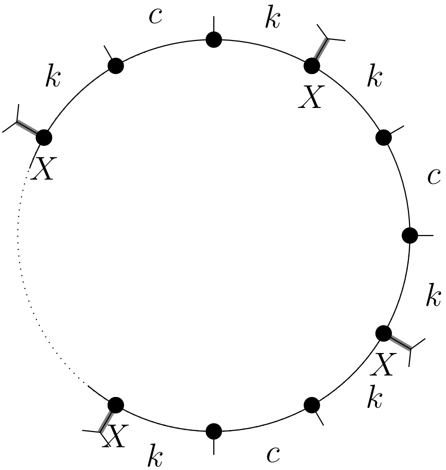


$A \sim k \sim X$		2-IN-3-SAT
$B \sim k - X$		
$A \sim k \sim X$	$A \sim k \sim Y$	NAEQ-SAT
$B \sim k - X$	$B \sim k - Y$	
$A \sim k \sim X$	$A \sim k - Y$	NAEQ-SAT
$B \sim k - X$	$B \sim k \sim Y$	

$$Y \neq X,\; X, Y \in \{A, B\}$$

Variable gadget - 2-IN-3-SAT





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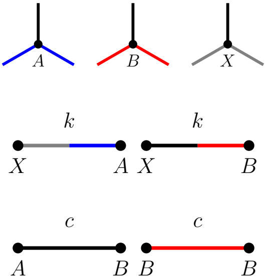

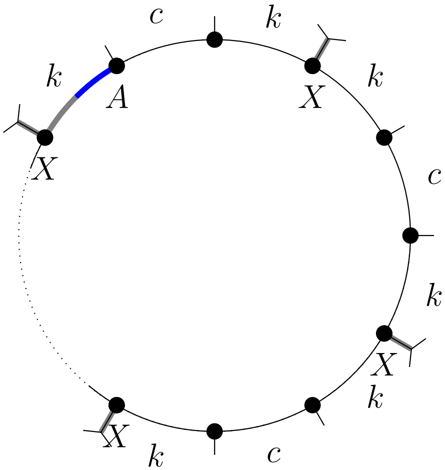

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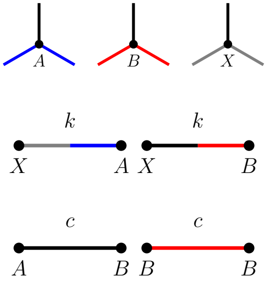

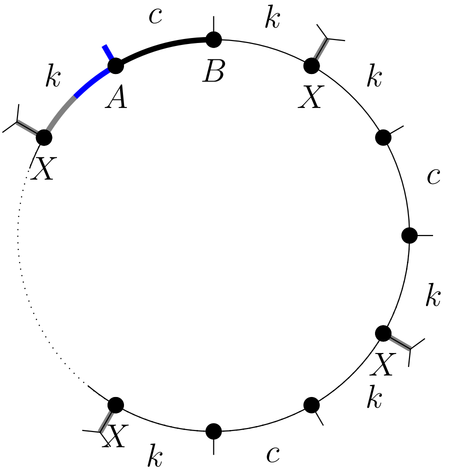

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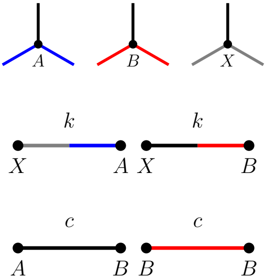
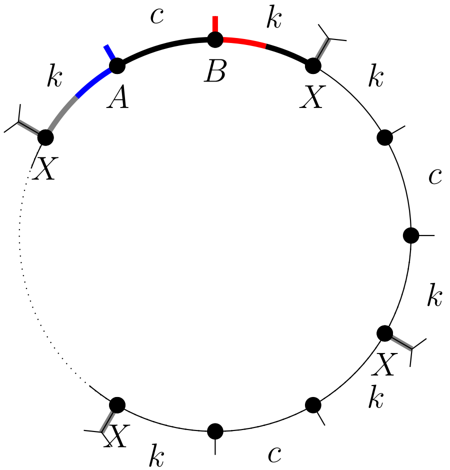
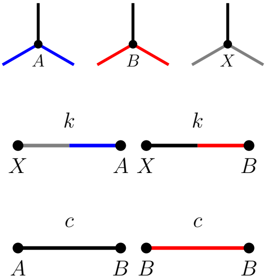
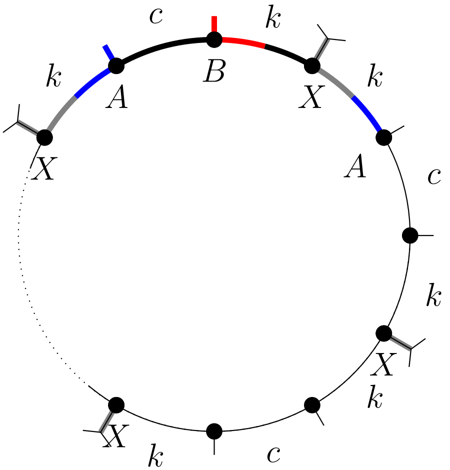


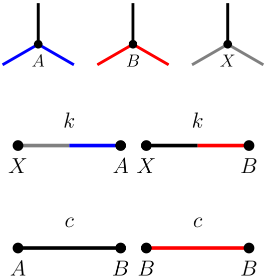


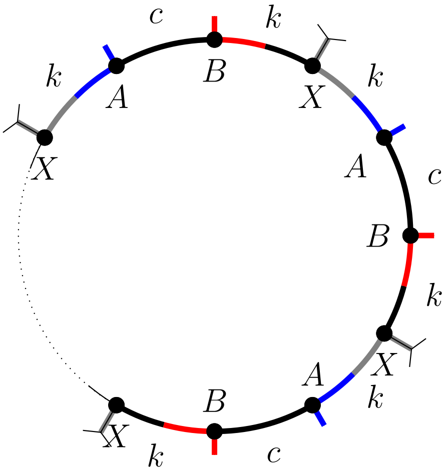
Variable gadget - 2-IN-3-SAT



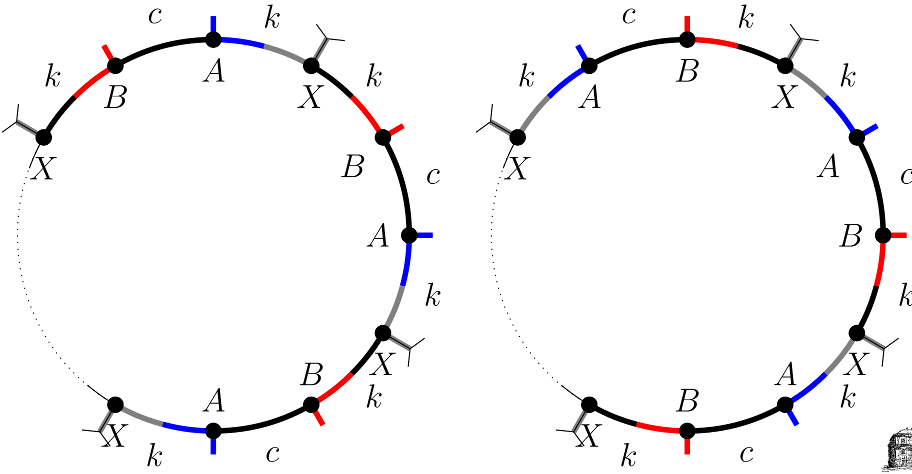


Variable gadget - 2-in-3-SAT

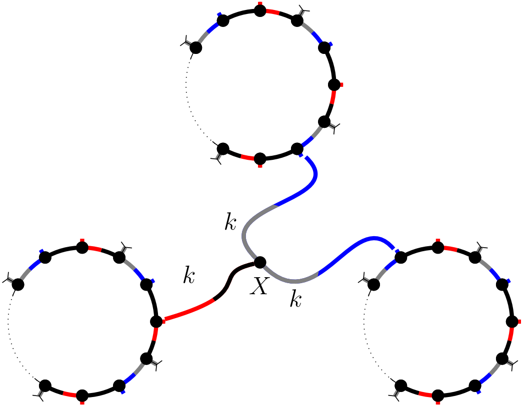




Variable gadget - 2-in-3-SAT



Clause gadget - 2-IN-3-SAT



Use three paths of length k to form a clause.

Other partial results

- symmetric weight graph, which contains $W(a, a, a)$ (NP-complete)
- generalization of the polynomial time algorithm
- some small cases for weight graphs(NP-complete)
- some small cases for graphs