Locally injective homomorphism to the simple Weight graphs *

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Abstract. A Weight graph is a connected (multi)graph with two vertices u and v of degree at least three and other vertices of degree two. Moreover, if any of these two vertices is removed, the remaining graph contains a cycle. A Weight graph is called simple if the degree of u and v is three. We show full computational complexity characterization of the problem of deciding the existence of a locally injective homomorphism from an input graph G to any fixed simple Weight graph by identifying some polynomial cases and some NP-complete cases.

Keywords: computational complexity; locally injective homomorphism; Weight graph

1 Introduction

Graphs in this paper are generally simple. The only graphs with allowed loops and parallel edges are the Weight graphs. We denote the set of vertices of a graph G by $V(G)$ and the set of edges by $E(G)$. The degree of a vertex v in a graph G is denoted by $\deg_G(v)$ (recall that in multigraphs, degree of a vertex v is defined as the number of edges going to other vertices plus twice the number of loops at v) and the set of all neighbors of v by $N_G(v)$. We omit G in the subscript if G is clear from the context. By $[n]$ we denote the set of integers $\{1, \ldots, n\}$.

A connected (multi)graph H is a *Weight graph* (or sometimes known also as a dumbbell graph) if it contains two vertices u and v of degree at least three and all other vertices are of degree two. Moreover, both $H - u$ and $H - v$ contain a cycle or a loop. The Weight graph H is *simple* if both u and v have degree 3. Note that a simple Weight graph consists of a path connecting u and v and two cycles. A simple Weight graph H is denoted by $\mathcal{W}(a, b, c)$ if H is a union of cycles of lengths a and b and a path of length c. We call $\mathcal{W}(a, b, c)$ *reduced* if the greatest common divisor of a, b and c is one.

Let G and H be graphs. A *homomorphism* $f : G \to H$ is an edge preserving mapping from $V(G)$ to $V(H)$. If H is not a multigraph then homomorphism is *locally injective (resp. surjective, bijective)* if $N_G(v)$ is mapped to $N_H(f(v))$

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injectively (resp. surjectively, bijectively) for every $v \in V(G)$. Locally bijective homomorphism is also known as a *covering projection* or simply a *cover*. Similarly, locally injective homomorphism is known as a *partial covering projection* or a *partial cover*. In this paper we denote locally injective homomorphism as *LI-homomorphism*. In case that H is multigraph (and G is simple) LI-homomorphism generalizes as a mapping $f : V(G) \cup E(G) \to V(H) \cup E(H)$ such that:

i) for every $u \in V(G)$: $f(u) \in V(H)$
ii) for every $e = \{u, v\} \in E(G)$: $f(e) \in E(H)$ and $f(e) = \{f(u), f(v)\}$
iii) for every $u \in V(G)$ and every non loop edge $e \in E(H)$ such that $f(u) \in e$, there is at most one edge $e' \in E(G)$ such that $u \in e'$ and $f(e') = e$
iv) for every $u \in V(G)$ and every loop edge $e \in E(H)$ on vertex $f(u)$, there are at most two edges $e', e'' \in E(G)$ such that $u \in e'$, $u \in e''$ and $f(e') = f(e'') = e$

We can generalize the definition of locally surjective, resp. locally bijective homomorphism from simple graph to multigraph by simply changing the phrase "at most" by "at least", resp. "exactly" in *iii*) and *iv*). In the following text we denote homomorphism f from G to H simply as $f : G \to H$.

We consider the following decision problem. Let H be a fixed graph and G be an input graph. Determine the existence of a locally injective (surjective, bijective) homomorphism $f : G \to H$. We denote the problem by H-LIHom (resp. H-LSHom, H-LBHom). If there is no local restriction on the homomorphism, the problem is called H-Hom.

In this paper we focus on the H-LIHom problem.

Problem: H-LIHom
Input: Graph G
Task: Determine the existence of a LI-homomorphism $f : G \to H$.

Locally injective homomorphisms are closely related to $H(2, 1)$-labelings, which have an applications in frequency assignment. Let H be a graph. An $H(2, 1)$-*labeling* of a graph G is a mapping $f : V(G) \to V(H)$ such that image of every pair of adjacent vertices is two distinct and nonadjacent vertices. Moreover, image of every pair of vertices in distance two is two distinct vertices. For simple graphs the mapping f corresponds to a LI-homomorphism from G to the complement of H.

The computational complexity of H-Hom was fully determined by Hell and Nešetřil [10]. They showed that the problem is solvable in polynomial time if H is bipartite and it is $\mathcal{NP}$-complete otherwise.

The study of H-LSHom was initiated by Kristiansen and Telle [14] and completed by Fiala and Paulusma [9] who gave a full characterization by showing that H-LSHom is $\mathcal{NP}$-complete for every connected graph on at least three vertices.

The computational complexity of locally bijective homomorphisms was first studied by Bodlaender [2] and Abello et al. [1]. Despite of the effort [11–13] the complete characterization is not known.

Similarly, for the locally injective homomorphism the dichotomy for the computational complexity is not known. Some partial results can be found in [5, 6, 8]. Fiala and Kratochvíl [7] also considered a list version of the problem and showed dichotomy. Note that no direct consequences of complexity of H-Hom or H-LSHom to H-LIHom are known.

The authors of [6] show that H-LBHom is reducible in polynomial time to H-LIHom. Hence it makes sense to study the complexity of H-LIHom where H-LBHom is solvable in polynomial time. In particular, many graphs with at most two vertices of degree more than two belong to this class. It is also the case for Theta graphs.

Theta graph is a graph with exactly two vertices u and v of degree at least three and several paths connecting them. Note that u and v may be connected by parallel edges. Study of Theta graphs started by a work of Fiala and Kratochvíl [5], continued in [8] and was finished by showing a complete dichotomy by Lidický and Tesař [15].

Also Weight graphs, considered in this paper, have at most two vertices of degree more than two. Study of Weight graphs was initiated by Fiala [4] by showing the following theorems.

Theorem 1. $\mathcal{W}(a, a, a)$-LIHom *is* $\mathcal{NP}$*-complete.*

Theorem 2. $\mathcal{W}(a, a, b)$-LIHom *is polynomial time solvable, whenever a is divisible by strictly higher power of two than b, and is* $\mathcal{NP}$*-complete otherwise.*

Fiala and Kratochvíl [5] observed that it is sufficient to consider only reduced Weight graphs.

Proposition 1. *Let W be a Weight graph and d be the greatest common divisor of lengths of simple paths in W. Let W' be obtained from W by shortening each of its simple path by a factor of d. Then* W-LIHom *is reducible to* W'-LIHom *in polynomial time.*

In this paper we continue the study of Weight graphs by showing complete dichotomy for simple Weight graphs.

Theorem 3. *Let H be a bipartite simple reduced Weight graph. Then* H-LIHom *problem is solvable in polynomial time.*

Theorem 4. *Let H be a non-bipartite simple reduced Weight graph. Then* H-LIHom *problem is* $\mathcal{NP}$*-complete.*

In the following section, we introduce several definitions and observations. Then we give the proof of Theorem 3 and finish with Section 4 containing the proof of Theorem 4. Most of the proofs and all figures are moved to Appendix due to page limit.

2 Preliminaries

Let G be a graph and H be a spanning subgraph of G. We say that H is a *2-factor* of G if for all $v \in V(G) : \deg_H(v) = 2$. Let C be a set of colors. A mapping $\varphi : E(G) \to C$ is called an *edge coloring* if for every $e_1, e_2 \in E(G)$ that share a common vertex holds $\varphi(e_1) \neq \varphi(e_2)$.

Let G be a graph and $v_0v_1v_2 \dots v_n$ be a path P of length n in G. The path is *simple* if v_0 and v_n are vertices of degree at least three and all inner vertices of P have degree two. We denote a simple path of length n by SP_n.

Let H be a Weight graph with vertices w_A and w_B of degree at least three. Let $f : G \to H$ be a LI-homomorphism. Note that f must map all vertices of degree at least three to w_A or w_B. Hence every end vertex of every simple path of G must be mapped to w_A or w_B. We call a vertex *big* if it has degree at least three. We denote the set of big vertices of G by $B(G)$. Note that w_A and w_B are also big vertices.

We need to control what are the possible LI-homomorphism of simple paths. Hence we define a function $g^P_f(v_0, v_n) = x$ if the edge v_0v_1 is mapped by a LI-homomorphism f to an edge of SP_x in H. We also use notation $st^P_f(v_0, v_n)$ to denote $f(v_0v_1)$. We omit the superscript P if there is only one simple path containing v_0 and v_n and the subscript f if it is clear from the context.

We say that SP_n allows *decomposition* $x - y$ if there exists a graph G containing a simple path P of length n with end vertices v_0 and v_n and a LI-homomorphism $f : G \to H$ such that $g^P_f(v_0, v_n) = x$ and $g^P_f(v_n, v_0) = y$. We denote the decomposition by $x -_k y$ (resp. $x -_c y$) if it forces that $f(v_0) = f(v_n)$ (resp. $f(v_0) \neq f(v_n)$).

In the case of decomposition $x -_k y$ (resp. $x -_c y$) we say, that the decomposition keeps (resp. changes) the parity.

Let n be a positive integer and $\mathcal{E} \subseteq \{a_1, a_2, \dots a_n\}$. The following notation

$$\boldsymbol{n^H_{\mathcal{E}}} \textbf{ :} \;\; x_1 - y_1, x_2 - y_2, \dots, x_s - y_s, (z_1 - w_1), (z_2 - w_2), \dots, (z_t - w_t)$$

describes the list of all decompositions $x - y$ of SP_n where $x, y \in \mathcal{E}$. Decompositions $x_i - y_i$ must be possible and decompositions $z_j - w_j$ are optional for all $i \in [s]$ and $j \in [t]$. Moreover, $-_k$ and $-_c$ can be used instead of just $-$.

We call an edge e of H *bridge edge* if it is on a simple path with distinct end vertices and *loop edge* otherwise. If an edge e' is mapped by a LI-homomorphism f to a bridge (loop) edge, we call e' also a bridge (resp. loop) edge.

We denote the greatest common divisor of $x_1, \dots x_n$ by $GCD(x_1, \dots, x_n)$.

Proposition 2. *Let $a, b, d \in \mathbb{N}$ such that $GCD(a, b) = d$. Then for every $z \in \mathbb{Z}$ there exist $s, t \in \mathbb{Z}$ such that $as + bt = zd$.*

3 Proof of Theorem 3 (polynomial case)

In this section we give the proof of Theorem 3.

First, we define $\mathcal{W}(1^{n_a}, 1^{n_b}, 1^{n_c})$ to be a Weight graph with n_a, resp. n_b loops on vertices w_A, resp. w_B and n_c parallel edges between w_A and w_B. Next we define that Weight graph $\mathcal{W}(a^{n_a}, b^{n_b}, c^{n_c})$ is obtained from $\mathcal{W}(1^{n_a}, 1^{n_b}, 1^{n_c})$ by subdividing all loops on vertex w_A, resp. w_B $a-1$, resp. $b-1$ times and all parallel edges between w_A and w_B $c-1$ times. Theorem 3 is a special case of the following Theorem 5 for $n_a = n_b = n_c = 1$.

Theorem 5. *Let $H = \mathcal{W}(a^{n_a}, b^{n_b}, c^{n_c})$, where $a, b, c, n_a, n_b, n_c \in \mathbb{N}$ such that $GCD(a, b, c) = 1$, be bipartite. Then H-*LIHom *problem is solvable in polynomial time.*

Proof. Let H be a fixed Weight graph from the statement of the theorem. Let $w_A, w_B \in V(H)$ be of degrees $2n_a + n_c$ and $2n_b + n_c$ respectively. As H is bipartite and $GCD(a, b, c) = 1$, we conclude that c is odd and a and b are (not necessarily distinct) even numbers.

Let G be an input graph. We assume that G is bipartite. If not, there is no LI-homomorphism from G to H and the algorithm returns NO answer instantly.

First we partition big vertices of $V(G)$ to two sets of the bipartition A and B. Note that all vertices of A must be mapped to w_A and all vertices of B to w_B or vice versa. We try both possibilities and without loss of generality we assume that vertices of A are mapped to w_A and vertices of B are mapped to w_B.

We reduce H-LIHom problem to a *flag factor* problem of an auxiliary graph G' (of size polynomial in size of G).

Problem: Flag factor
Input: graph G' and functions $f_l : V(G') \to \mathbb{N}_0$ and $f_u : V(G') \to \mathbb{N}$
Output: spanning subgraph F of G' satisfying $f_l(v) \leq deg_F(v) \leq f_u(v)$ for all $v \in V(G')$.

We call the edges of F *matched edges*. The flag factor problem is solvable in polynomial time [5]. We use the flag factor to identify edges of G, which should be mapped to bridge edges of H. Let us now describe G'. The auxiliary graph G' contains two sets of vertices A' and B' corresponding to A and B. We define $f_u(v) = n_c$ for all $v \in B' \cup A'$, $f_l(v) = deg_G(v) - 2n_a$ for all $v \in A'$ and $f_l(v) = deg_G(v) - 2n_b$ for all $v \in B'$.

For every (simple) path between big vertices $u, v \in B(G)$ we construct a list L of all possible mappings under some LI-homomorphism (since H is bipartite, all these decompositions either change or keep parity) and we only distinguish if these decompositions begin, resp. end by loop edge or bridge edge. In list L we denote loop edge as "$\sim$" and bridge edge as "$-$". We join corresponding $u' \in V(G')$ and $v' \in V(G')$ by gadget according to L as depicted in Figure 1 (note that corresponding lists for vertices u, v and v, u are symmetric and so corresponding gadgets are symmetric as well).

We show that there exists a LI-homomorphism $f : G \to H$ if and only if there exists a flag factor F of G'.

If f exists, we can get F in a following way: for every edge $\{u, w\} \in E(G)$ such that $u \in B(G)$ and $\{u, w\}$ is mapped to the bridge edge, we take an edge incident to u' in a gadget corresponding to a simple path containing an edge $\{u, w\}$ to F. For such F for every $u \in B(G)$ holds that $f_l(u') \leq deg_F(u') \leq f_u(u')$ and it is not hard to prove (because of choosing of appropriate gadgets in construction of G') that if F is not a flag factor then we can add to F some edges which are not incident to any u' for $u \in B(G)$. So if a flag factor F does not exist, neither does a LI-homomorphism f.

If a flag factor F exists, we show that a LI-homomorphism $f : G \to H$ also exists. The choice of f_l and f_u assures that each vertex of A', resp. B' has at most n_c incident matched edges and at most $2n_a$, resp. $2n_b$ incident non-matched edges.

Let $v \in V(G)$ be big and v' be the corresponding vertex in G'. Then if $v \in A$ we prescribe $f(v) = w_A$ and otherwise $f(v) = w_B$. Let P be a simple path beginning with v and g be the gadget corresponding to P in G'. If there is a matched edge in g incident to v', then f will map the beginning of P to some bridge edge and to some loop edge incident to vertex $f(v)$ otherwise. We do not define f on P yet. We only prescribe one of decompositions $\{a-a, a-b, a-c, b-c, b-b, c-c\}$, which splits P into paths of lengths a, b and c and it prescribes which internal vertices of P are mapped to w_A and w_B. So, we have fixed which vertices of G are mapped to w_A and w_B.

Let T be vertices of G which are mapped to big vertices of H. Note that also vertices of degree less than three may be in T. Also note that a path P where $T \cap P$ are the only endpoints of P has length in $\{a, b, c\}$. We just need to decide to which loop or bridge of H the path P will be mapped.

We create auxiliary graphs U_a, U_b, resp. U_c with vertex set $\{u \in T \mid f(u) = w_A\}$, $\{u \in T \mid f(u) = w_B\}$, resp. T. For all $x \in \{a, b, c\}$ two vertices of U_x are joined by an edge if they are connected by a path of length x which is internally disjoint with T and U_x contains a loop at vertex v if there is a cycle C of length x where $v = T \cap C$ for $x \in \{a, b\}$. The graph U_a can be made $2n_a$-regular by adding vertices and edges. It is well known that any $2n_a$-regular graph can be partitioned to n_a 2-factors in polynomial time. For each 2-factor Z of U_a we use one cycle C_a of length a of H and map vertices in cycles of Z to C_a.We treat U_b in analogous way. Note that there are no loops in U_c. Graph U_c is bipartite with maximum degree at most n_c and by König's theorem there exists an edge coloring $\varphi : E(U_c) \to [n_c]$. We assign one simple path of length c of H to each color class of φ. So f can be constructed from F.

All steps of the reduction can be computed in polynomial time and the flag factor problem is also solvable in polynomial time. Hence we conclude that f can be computed in polynomial time if exists or detect that it does not exist. □

4 Proof of Theorem 4 ($\mathcal{NP}$-complete case)

The goal of this section is to give a proof of Theorem 4 by showing a reduction from 1-IN-3 SAT or NAE-3-SAT. These problems ask for existence of an evaluation of formula in CNF with clauses of size exactly 3 such that in every clause there is exactly one positive literal, resp. in every clause there exist at least one positive as well as at least one negative literal. Both problems are $\mathcal{NP}$-complete by Schaefer [16]. Note that H-LIHOM is in $\mathcal{NP}$ as a description of the desired homomorphism is of linear size and can be verified in polynomial time. Hence we only need to proof $\mathcal{NP}$-hardness.

We use the same basic notation as in the previous section. Let H be a fixed non-bipartite Weight graph $\mathcal{W}(a,b,c)$ with big vertices w_A and w_B. Recall that Theorem 2 implies $a \neq b$.

We start by restricting a, b and c by the following corollary of Theorem 1.

Corollary 1. *If there exist $x, y \in \mathbb{N}$ such that $c = ax = by$, then H-LIHom is $\mathcal{NP}$-complete.*
(proof is in Appendix)

Lemma 1. *Let $a, b, c \in \mathbb{N}$ such that $GCD(a, b, 2c) = 1$. Then exist $s, t, z \in \mathbb{N}$ such that $as = bt + 2cz + c$ and $t > z$.*
(proof is in Appendix)

We use the notation $u \sim n - v$, where $u, v \in \{w_A, w_B\}$ and $n \in \mathbb{N}$ to denote existence of a LI-homomorphism f from a simple path $u = v_0 v_1 \dots v_n = v$ to H such that $g_f(v_0, v_1)$ is a loop edge and $g_f(v_n, v_{n-1})$ is a bridge edge. Variants where $\sim$ and $-$ are combined differently are defined similarly. Also $\simeq$ is used instead of $\sim$ and $-$ if the exact mapping is not known. We say that f is *beginning* with $g_f(v_0, v_1)$ and f is *ending* with $g_f(v_n, v_{n-1})$ on path $v_0 v_1 \dots v_n$.

The following corollary connects Lemma 1 and mappings of simple paths to H.

Corollary 2. *There exists $k \in \mathbb{N}$ such that there exist LI-homomorphisms f_1 and f_2 where $f_1 = w_A \sim k \sim x$, $f_2 = w_B \sim k - x$ and $x \in \{w_A, w_B\}$.*

Proof. As H is non-bipartite, a, b and c satisfy the assumptions of Lemma 1. Let $k = as + az = bt + 2cz + az + c$ and $x = w_A$. As k is divisible by a, f_1 can just use the the cycle of length a to achieve $w_A \sim k \sim w_A$. We construct the mapping f_2 starting from w_A by z repetitions of the pattern *cbca* and then by adding c and finally $(t - z)$ times b. As $t > z$, we have that $g_{f_2}(v_k, v_{k-1}) = b$. Hence we constructed $w_A - k \sim w_B$. $\square$

In the following, we assume that k is the smallest possible number, whose existence is guaranteed by the previous Corollary, such that there exists mappings $f_1 = w_A \sim k \simeq x$ and $f_2 = w_B \sim k \simeq x$. We use y for the vertex of H such that $\{x, y\} = \{w_A, w_B\}$.

Lemma 2. *There do not exist mappings*

1. *both $w_A \sim k - x$ and $w_B \sim k - x$*
2. *both $w_A \sim k \sim x$ and $w_B \sim k \sim x$*

(proof is in Appendix)

Without loss of generality, in what follows we assume that $f_1 = w_A \sim k \sim x$ and $f_2 = w_B \sim k - x$ (since Lemma 2 and the fact that $\mathcal{W}(a,b,c)$ is isomorphic to $\mathcal{W}(b,a,c)$).

Lemma 3. *There do not exist mappings*

1. $w_A \sim k \sim y$
2. $w_B \sim k - y$

(proof is in Appendix)

Let us now summarize the results of previous lemmas:

– mappings $w_A \sim k \sim x$ and $w_B \sim k - x$ exist
– mappings $w_A \sim k - y$ and $w_B \sim k \sim y$ may or may not exist
– mappings $w_A \sim k - x$, $w_A \sim k \sim y$, $w_B \sim k \sim x$, and $w_B \sim k - y$ do not exist

Next we introduce several gadgets. Let $z \in \{w_A, w_B\}$ then Z-gadget is a graph containing a vertex v_z of degree one such that any LI-homomorphism from Z-gadget to H maps v_z to z and the edge incident with v_z is mapped to a loop edge. We call v_z a *z-vertex*.

Lemma 4. *For every $z \in \{w_A, w_B\}$ there exists an Z-gadget.*
(proof is in Appendix)

If z is w_A, w_B, resp. x then we denote appropriate Z-gadget as A-gadget, B-gadget, resp. X-gadget and appropriate z-vertex as a-vertex, b-vertex, resp. x-vertex.

A *variable gadget* $VG(i)$ for $i \in \mathbb{N}$ is a graph containing two subsets of vertices A and B such that $|A| = |B| = i$ and vertices of $A \cup B$ have degree two. Moreover, if a graph G contains a copy of $VG(i)$ and all vertices of $A \cup B$ are big, then for every LI-homomorphism $f : G \to H$ holds that edges adjacent to vertices of $A \cup B$ not contained in $VG(i)$ are mapped to loop edges and either $\forall v \in A : f(v) = w_A$ and $\forall v \in B : f(v) = w_B$ or in the other case $\forall v \in A : f(v) = w_B$ and $\forall v \in B : f(v) = w_A$.

Lemma 5. *If c is not divisible neither by a nor b then there exists a* variable gadget *$VG(i)$ for all $i \in \mathbb{N}$.*
(proof is in Appendix)

Lemma 6. *If c is divisible by exactly one of the numbers a or b then there exists a* variable gadget *$VG(i)$ for all $i \in \mathbb{N}$.*
(proof is in Appendix)

Recall that from the proofs of Lemmas 5 and 6 we know that for every $i \in \mathbb{N}$ there exist variable gadget $VG(i)$ such that there exist LI-homomorphisms $f_1, f_2 : VG(i) \to H$ such that for every vertex $v \in A \cup B$ one edge incident to v is mapped to loop edge and one is mapped to bridge edge (for both f_1 and f_2) and $\forall u \in A, \forall v \in B : f_1(u) = f_2(v) = w_A$ and $f_1(v) = f_2(u) = w_B$. In the following text we use only such variable gadgets.

The last gadget is a gadget for representing clauses, so called *CL-gadget.* It contains three vertices of degree one z_1, z_2 and z_3 which are connected by three internally disjoint paths of length k to a vertex z_4 of degree three.

Lemma 7. *Let f be a locally injective homomorphism from CL-gadget to H such that $f(z_1), f(z_2), f(z_3) \in \{w_A, w_B\}$ and $st(z_1, z_4), st(z_2, z_4)$ and $st(z_3, z_4)$ are loop edges. Then $\{f(z_1), f(z_2), f(z_3)\} = \{w_A, w_B\}$.*

Proof. Suppose for contradiction that $|\{f(z_1), f(z_2), f(z_3)\}| = 1$. First, let $f(z_1) = f(z_2) = f(z_3) = w_A$. If $f(z_4) = x$, then we get a contradiction with Lemma 2 as f implies existence of a decomposition $w_A \sim k - x$. If $f(z_4) = y$, then we get existence of $w_A \sim k \sim y$ contradicting Lemma 3.

The case $f(z_1) = f(z_2) = f(z_3) = w_B$ is analogous. Both cases $f(z_4) = x$ and $f(z_4) = y$ contradict Lemma 2 or Lemma 3. □

Let ϱ be a formula in CNF with clauses $C_1, \ldots, C_m$ and variables $p_1, \ldots, p_n$ where all clauses contain exactly three literals. We denote the number of occurrences of a variable p in ϱ by $\mathrm{occ}(p)$.

Now describe a construction of a graph G_ϱ. First take a copy CL_i of CL-gadget for every $i \in [m]$ and then take a copy VG_j of $VG(\mathrm{occ}(p_j))$-gadget for every $j \in [n]$ (see Figure 3 in Appendix for an example.). We denote A and B of VG_j by A_j and B_j for every $j \in [n]$ and $\{z_1, z_2, z_3\}$ of CL_i by Z_i and z_j by z_j^i for every $j \in \{1, 2, 3, 4\}$ and $i \in [m]$.

Next we identify some vertices. If p_j occurs as a positive literal in C_i, we identify one vertex of Z_i with one vertex of A_j and if the occurrence is negative, we identify one of Z_i with one of B_j. The identification can be done such that every vertex is identified at most once as $\mathrm{occ}(p_j) \leq |A_j| = |B_j|$. Finally, for every $w \in A_j \cup B_j$ of degree 2, we add a new vertex of degree one adjacent to w (so w is big vertex).

We prove Theorem 4 as a consequence of the following Lemmas 8 and 9.

Lemma 8. *If there exist both decompositions $w_A \sim k - y$ and $w_B \sim k \sim y$, then H-LIHom is $\mathcal{NP}$-hard.*

Proof. Let ϱ be an instance of NAE-3-SAT.

If there exists a LI-homomorphism $f : G_\varrho \to H$, then we evaluate p_j true if $f(A_j) = w_B$ and false otherwise for all $j \in [n]$ (this evaluation is well defined because of definition of variable gadget). Lemma 7 implies that there is no clause with all literals equal.

On the other hand let φ be an NAE evaluation of ϱ. We predefine a LI-homomorphism $f : G_\varrho \to H$ by mapping VG_j while requiring $f(A_j) = w_B$ if $\varrho(p_j)$ is true and $f(A_j) = w_A$ otherwise. This can be done as VG_j's are disjoint. Observe that $Z_i = \{w_A, w_B\}$. Hence f can be also defined on CL_i for all $i \in [m]$. □

Lemma 9. *If at least one of decompositions $w_A \sim k - y$ and $w_B \sim k \sim y$ does not exist, then H-LIHom is $\mathcal{NP}$-hard.*

Proof. Let ϱ be an instance of 1-in-3-SAT.

If there exists a LI-homomorphism $f : G_\varrho \to H$, then we evaluate p_j true if $f(A_j) = w_B$ and false otherwise for all $j \in [n]$. The assumptions imply that

$f(z_4^i) = x$ for all $i \in [n]$, hence exactly one vertex of $f(z_1^i), f(z_2^i)$ and $f(z_3^i)$ is w_B and so every clause has exactly one literal evaluated as true.

On the other hand let φ be an 1-IN-3 evaluation of ϱ. We predefine a LI-homomorphism $f : G_\varrho \to H$ by mapping VG_j while requiring $f(A_j) = w_B$ if $\varrho(p_j)$ is true and $f(A_j) = w_A$ otherwise. This can be done as VG_j's are disjoint. Next we map $f(z_4^i)$ to x. As exactly one literal in every clause is true, exactly one of $f(z_1^i), f(z_2^i)$ and $f(z_3^i)$ is w_B. Hence f can be extended to a locally injective homomorphism to H. □

Recall that predefined mappings from proofs of Lemmas 8 and 9 can be easily extended to edges and so to LI-homomorphisms (similarly as in the proof of Theorem 5).

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Appendix

This Appendix contains the missing proofs and figures. We repeat the statements for easier reading.

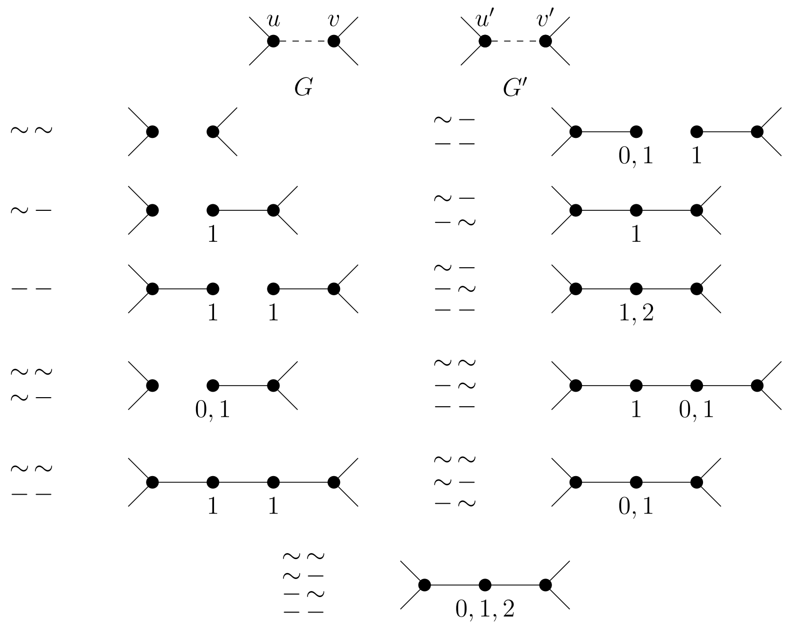

Fig. 1. Gadget joining vertices u' and v' in G'. The right gadget is chosen according to possible decompositions of a simple path joining big vertices u and v in G. The numbers below the vertices indicate the intervals given by f_l and f_u.

Corollary 1. *If there exist $x, y \in \mathbb{N}$ such that $c = ax = by$, then H-LIHOM is $\mathcal{NP}$-complete.*

Proof. Let G be an instance of $\mathcal{W}(1,1,1)$-LIHOM problem where we ask for existence of a locally injective homomorphism f from G to $\mathcal{W}(1,1,1)$. We denote the big vertices of $\mathcal{W}(1,1,1)$ by $u_{\mathcal{W}}$ and $v_{\mathcal{W}}$.

Let G' be a graph obtained from G by making every vertex of G big by adding pendant leaves to vertices of low degree and then subdividing every edge $c-1$ times.

It holds that $\boldsymbol{c^{H}_{\{a,b,c\}}}$ **:** $a -_k a, b -_k b, c -_c c$, since c can be used at most once in every decomposition of SP_c and $c = ax = by$.

It is easy to see that if there exists LI-homomorphism $f : G \to \mathcal{W}(1,1,1)$ then there exists a LI-homomorphism f' from G' to H (since allowed decompositions of SP_c). To complete the proof we need to show that existence of $f' : G' \to H$ implies existence of $f : G \to \mathcal{W}(1,1,1)$.

Let f' exists. Recall that vertices of G are in one-to-one correspondence to big vertices of G'. For every $t \in V(G)$ let $t' \in V(G')$ be the corresponding vertex. We define $f(t)$ to be $u_{\mathcal{W}}$ if $f'(t') = w_A$ and $v_{\mathcal{W}}$ if $f'(t') = w_B$. It is not hard to prove that such predefined mapping f can be generalized to locally injective homomorphism from G to $\mathcal{W}(1,1,1)$. $\square$

Lemma 1. *Let $a, b, c \in \mathbb{N}$ such that $GCD(a, b, 2c) = 1$. Then exist $s, t, z \in \mathbb{N}$ such that $as = bt + 2cz + c$ and $t > z$.*

Proof. Let d be $GCD(b, 2c)$. Then by Proposition 2 there exist $i, j \in \mathbb{Z}$ such that $bi + (2c)j = d$. As $GCD(a, d) = 1$ we get that there exist $p, q \in \mathbb{Z}$ such that $pd + qa = 1$. Hence $pd \equiv 1 \mod a$ and thus exist $l \in \mathbb{N}$ such that $ld \equiv -c \mod a$. Therefore there exist $z, t \in \mathbb{N}$ such that $t > z$ and $bt + (2c)z \equiv -c \mod a$ which proves the lemma. $\square$

Lemma 2. *There do not exist mappings*

1. *both $w_A \sim k - x$ and $w_B \sim k - x$*
2. *both $w_A \sim k \sim x$ and $w_B \sim k \sim x$*

Proof. In the first case we observe that there exist mappings $f_1' = w_A \sim (k-c) \simeq y$ and $f_2' = w_B \sim (k-c) \simeq y$ contradicting the choice of k. In the second case we consider mappings $f_1'' = w_A \sim (k-d) \simeq x$ and $f_2'' = w_B \sim (k-d) \simeq x$, where d is the length of the loop at x in H. This also contradicts the choice of k. $\square$

Lemma 3. *There do not exist mappings*

1. *$w_A \sim k \sim y$*
2. *$w_B \sim k - y$*

Proof. We proceed case by case.

1. Suppose that there exists $f' = w_A \sim k \sim y$. Recall that there exist $f_1 = w_A \sim k \sim x$. As $\{x, y\} = \{w_A, w_B\}$, we can view w_A as x and x, y as w_A, w_B, which gives mappings $w_A \sim k \sim x$ and $w_B \sim k \sim x$. Their existence contradicts Lemma 2.
2. Suppose that there exists $f' = w_B \sim k - y$. Recall that there exist $f_2 = w_B \sim k - x$. As both mappings end with bridge edges, there exist mappings $f'' = w_B \sim (k-c) \sim x$ and $f_2' = w_B \sim (k-c) \sim y$. As $\{x, y\} = \{w_A, w_B\}$, we can view x, y as w_A, w_B and obtain a contradiction with the choice of k.

$\square$

Lemma 4. *For every $z \in \{w_A, w_B\}$ there exists an Z-gadget.*

Proof. The Z-gadget can be constructed in the following way. Take two triangles on vertices u_1, u_2, u_3 and v_1, v_2, v_3 (see Figure 2) and two paths with endpoints u_4, u_5 of length a and v_4, v_5 of length b. Subdivide each edge of the first triangle $a-1$ times and $b-1$ times for the second one. Then connect u_1 and v_1 by a path of length c, add a vertex of degree 1 to u_2, v_2, u_4 and v_4. Finally, connected u_4, v_3 and v_4, u_3 by paths of length c.

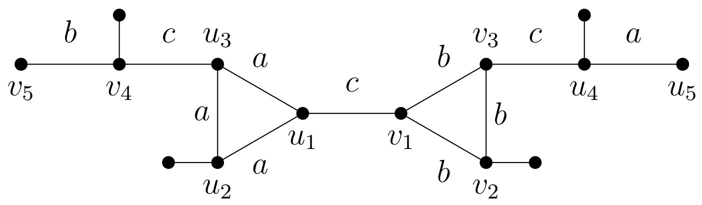

Fig. 2. Gadget X.

By considering the ordering of $\{a, b, c\}$ we get that u_i must be mapped to w_A and v_i to w_B for $1 \leq i \leq 5$. Then if z is w_A then define u_5 as z-vertex and otherwise define v_5 as z-vertex. Clearly the edge incident to z-vertex must be a loop edge. □

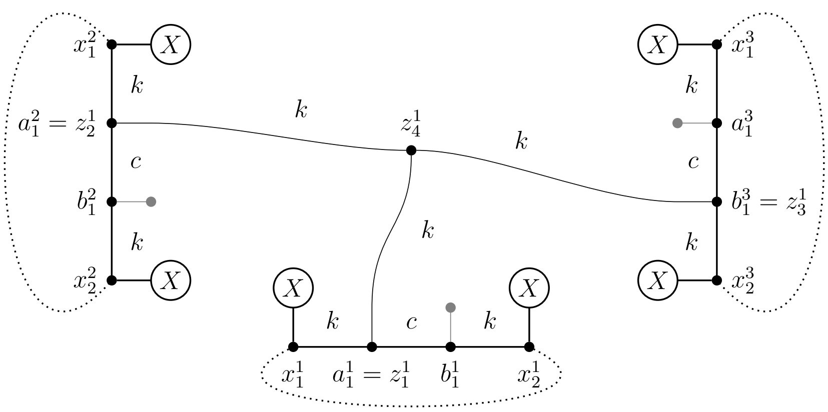

Fig. 3. Clause $C_1 = (p_1 \vee p_2 \vee \neg p_3)$.

Lemma 5. *If c is not divisible neither by a nor b then there exists a* variable gadget *$VG(i)$ for all $i \in \mathbb{N}$.*

Proof. We first describe a construction of $VG(i)$ and then argue that it is indeed a variable gadget.

We start the construction by taking i copies of the X-gadget $X_1, \ldots, X_i$ with x-vertices $x_1, \ldots, x_i$ (see Figure 4). We continue by adding paths $x_j a_j b_j x_{j+1}$ of length three for all $j \in [i]$ where x_{i+1} is x_1. Finally, we subdivide both edges incident to x_j that are not contained in gadget X_j $k-1$ times and the edge $a_j b_j$ $c-1$ times for all $j \in [i]$. The resulting graph is $VG(i)$ and $A = \{a_j : 1 \leq j \leq i\}$ and $B = \{b_j : 1 \leq j \leq i\}$.

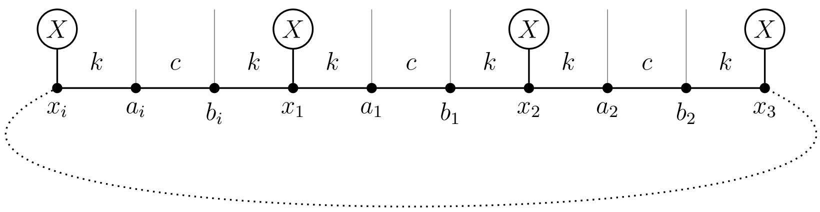

Fig. 4. Gadget $VG(i)$ from Lemma 5. Light lines indicate edges which will be present in a graph containing the gadget.

Let G be a graph containing a copy of $VG(i)$ where $A \cup B \subseteq B(G)$ and $f : G \to H$ be LI-homomorphism. Observe that $f(x_j) = x$ because it belongs to a copy of X-gadget and one of the two edges incident to x_j not contained in the X-gadget is mapped to a loop edge and the other one is mapped to a bridge edge for all $j \in [i]$. Since neither a nor b divides c, the only decomposition of a simple path of length c is $c -_c c$ and so $g_f(a_j, b_j) = c = g_f(b_j, a_j)$ and $f(a_j) \neq f(b_j)$ for every $j \in [i]$.

Suppose that $st(x_1, a_1)$ is a loop edge. We know that $st(a_1, x_1)$ is a loop edge (because $st(a_1, b_1)$ is a bridge edge) and so $f(a_1) = w_A$ and $g_f(a_1, x_1) = a$ (because $x \sim k \sim w_A$ and Lemma 2). It means that $f(b_1) = w_B$ and since $st(b_1, x_2)$ is a loop edge and $f(x_2) = x$ we have that $st(x_2, b_1)$ is a bridge edge and consequently $st(x_2, a_2)$ is a loop edge. Now we can continue in the same way in gadget $VG(i)$ and we get that $st(a_j, b_j)$ as well as $st(b_j, a_j)$ is a bridge edge, $f(a_j) = w_A$ and $f(b_j) = w_B$ for all $j \in [i]$, what proves that $VG(i)$ is a variable gadget.

If $st(x_1, a_1)$ is a bridge edge then $st(x_1, b_i)$ is a loop edge and we can continue in a similar way (but in an opposite direction) as in the previous paragraph. We get that $st(a_j, b_j)$ as well as $st(b_j, a_j)$ is a bridge edge, $f(a_j) = w_B$ and $f(b_j) = w_A$ for all $j \in [i]$, what proves that $VG(i)$ is a variable gadget. $\square$

Lemma 6. *If c is divisible by exactly one of the numbers a or b then there exists a* variable gadget *$VG(i)$ for all $i \in \mathbb{N}$.*

Proof. Without loss of generality we can suppose that c is divisible by a and not divisible by b (otherwise we simply change a, resp. A-gadget by b, resp. B-gadget in the proof). We prove this lemma in a similar way as Lemma 5. We first describe a construction of $VG(i)$ and then argue that it is indeed a variable gadget.

We start the construction by taking i copies of the A-gadget $A_1, \ldots, A_i$ with a-vertices $v_1, \ldots, v_i$ (see Figure 5). We continue by adding paths $v_j a_j b_j v_{j+1}$ of length three for all $j \in [i]$ where v_{i+1} is v_1. Finally, we subdivide both edges incident to v_j that are not contained in gadget A_j $c-1$ times and the edge $a_j b_j$ $b+c-1$ times for all $j \in [i]$. The resulting graph is $VG(i)$ and $A = \{a_j : 1 \leq j \leq i\}$ and $B = \{b_j : 1 \leq j \leq i\}$.

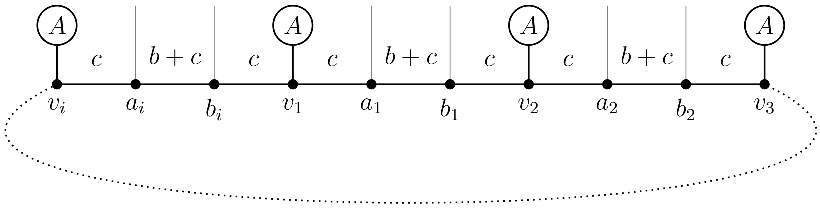

Fig. 5. Gadget $VG(i)$ from Lemma 6. Light lines indicate edges which will be present in a graph containing the gadget.

Since c is divisible by a and not divisible by b, the only possible decompositions of simple paths of length c and $b+c$ are:

$$\boldsymbol{(c)^H_{\{a,b,c\}}} : \; a -_k a, c -_c c$$

$$\boldsymbol{(b+c)^H_{\{a,b,c\}}} : \; b -_c c, (a -_k a), (a -_k c)$$

Recall that decompositions $a -_k a$ and $a -_k c$ of SP_{b+c} are possible only if $a = 1$, because otherwise b is divisible by a and we have $GCD(a, b, c) = a > 1$.

Let G be a graph containing a copy of $VG(i)$ where $A \cup B \subseteq B(G)$ and $f : G \to H$ be LI-homomorphism. Observe that $f(v_j) = w_A$ because it belongs to a copy of A-gadget and one of the two edges incident to v_j not contained in the A-gadget is mapped to a loop edge and the other one is mapped to a bridge edge for all $j \in [i]$.

Suppose that $st(v_1, a_1)$ is a bridge edge. Then we have $g_f(v_1, a_1) = c = g_f(a_1, v_1)$ and $f(a_1) = w_B$. Then necessarily $g_f(a_1, b_1) = b$, $g_f(b_1, a_1) = c$ and $f(b_1) = w_A$, what imply $g_f(b_1, v_2) = a = g_f(v_2, b_1)$ and $g_f(v_2, a_2) = c$. We can continue in the same way in gadget $VG(i)$ and we get that $st(a_j, v_j)$ as well as $st(b_j, a_j)$ is a bridge edge, $f(a_j) = w_B$ and $f(b_j) = w_A$ for all $j \in [i]$, what proves that $VG(i)$ is a variable gadget.

If $st(v_1, a_1)$ is a loop edge then $st(v_1, b_i)$ is a bridge edge and we can continue in the similar way (but in the opposite direction) as in the previous paragraph. We get that $st(a_j, b_j)$ as well as $st(b_j, v_{j+1})$ is a bridge edge, $f(a_j) = w_A$ and $f(b_j) = w_B$ for all $j \in [i]$, what proves that $VG(i)$ is a variable gadget. $\square$