Splitting a tournament into two subtournaments with given minimum outdegree

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Abstract

A (k_1, k_2)-*outdegree-splitting* of a digraph D is a partition (V_1, V_2) of its vertex set such that $D[V_1]$ and $D[V_2]$ have minimum outdegree at least k_1 and k_2, respectively. We show that there exists a minimum function f_T such that every tournament of minimum outdegree at least $f_T(k_1, k_2)$ has a (k_1, k_2)-outdegree-splitting, and $f_T(k_1, k_2) \le k_1^2/2+3k_1/2+k_2+1$. We also show a polynomial-time algorithm that finds a (k_1, k_2)-outdegree-splitting of a tournament if one exists, and returns 'no' otherwise. We give better bound on f_T and faster algorithms when $k_1 = 1$.

1 Introduction

Let D be a digraph. For a vertex $v \in V(D)$ the *outdegree* of v, denoted by $d_D^+(v)$, is the number of arcs directed away from v. The minimum outdegree over all vertices of D is denoted by $\delta^+(D)$. We drop D in $d_D^+(v)$ and $\delta^+(D)$ if it is clear from the context.

A (k_1, k_2)-*outdegree-splitting* of a digraph D is a partition (V_1, V_2) of its vertex set such that $D[V_1]$ and $D[V_2]$ have minimum outdegree at least k_1 and k_2, respectively. A digraph admitting a (k_1, k_2)-outdegree-splitting is said to be (k_1, k_2)-*outdegree-splittable*.

Problem 1 (Alon [1]). Is there a function f such that every digraph with minimum outdegree $f(k_1, k_2)$ has a (k_1, k_2)-outdegree-splitting?

The existence of the corresponding function f for the undirected analogue is easy and has been observed by many authors. Stiebitz [12] even proved the following tight result: if the minimum degree of an undirected graph G is $d_1 + d_2 + \cdots + d_k$, where each d_i is a non-negative integer, then the vertex set of G can be partitioned into k pairwise disjoint sets $V_1, \dots, V_k$, so that for all i, the induced subgraph on V_i has minimum degree at least d_i. This is clearly tight, as shown by an appropriate complete graph.

Problem 1 is equivalent to the following:

Problem 2. Is there a function $f'(k_1, k_2)$ such that every digraph with minimum outdegree $f'(k_1, k_2)$ has two disjoint (induced) subdigraphs, one of them with minimum outdegree k_1 and the other with minimum outdegree k_2?

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This follows from the following proposition.

Proposition 3. *Let D be a digraph with minimum outdegree at least $k_1 + k_2 - 1$. If D contains two disjoint subdigraphs D_1 and D_2 such that $\delta^+(D_1) = k_1$ and $\delta^+(D_2) = k_2$, then D has a (k_1, k_2)-outdegree-splitting.*

Proof. Consider two disjoint digraphs D_1 and D_2 with $\delta^+(D_1) = k_1$ and $\delta^+(D_2) = k_2$ such that $V(D_1) \cup V(D_2)$ is maximum. Suppose for a contradiction that $S = V(D) \setminus (V(D_1) \cup V(D_2))$ is not empty. Then every vertex $s \in S$ has at most $k_1 - 1$ outneighbours in D_1 otherwise $D_1 + s$ and D_2 contradict the maximality of D_1 and D_2. Hence every vertex of S has at least $k_1 + k_2 - 1 - (k_1 - 1) = k_2$ outneighbours in $D - D_1$. It follows that $D - D_1$ has minimum degree k_2. So D_1 and $D - D_1$ contradicts the maximality of D_1 and D_2. $\square$

Corollary 4. $f(k_1, k_2) \leq \max\{f'(k_1, k_2), k_1 + k_2 - 1\}$.

This implies in particular that $f(1, 1) = f'(1, 1) = 3$. Indeed Thomassen [13] showed that every digraph of minimum outdegree at least 3 has two disjoint cycles. (In this paper, paths and cycles are always directed.)

This is a special case of Bermond-Thomassen Conjecture [3]:

Conjecture 5 (Bermond and Thomassen [3])**.** *Every digraph with $\delta^+ \geq 2k - 1$ contains k disjoint cycles.*

Note that Alon [1] proved that if $\delta^+ \geq 64k$ there are k disjoint cycles.

A *tournament* is a digraph such that for every two distinct vertices u, v there is exactly one arc with ends $\{u, v\}$ (so, either the arc uv or the arc vu but not both).

In this paper, we settle Problem 1 for tournaments.

Theorem 6. *Every tournament of minimum outdegree at least $k_1^2/2 + 3k_1/2 + k_2 + 1$ has a (k_1, k_2)-outdegree-splitting.*

To prove Theorem 6, we shall prove the following theorem.

Theorem 7. *Every tournament with minimum outdegree at least k has a subtournament with minimum outdegree k and order at most $k^2/2 + 3k/2 + 1$.*

We can then easily derive Theorem 6.

Proof of Theorem 6. Let T be a tournament of minimum outdegree at least $k_1^2/2 + 3k_1/2 + k_2 + 1$. By Theorem 7, there exists a subtournament T_1 with minimum outdegree at least k_1 and order at most $k_1^2/2 + 3k_1/2 + 1$. Let $T_2 = T - T_1$. Then $\delta^+(T_2) \geq \delta^+(T) - |V(T_1)| \geq k_2$. Hence $(V(T_1), V(T_2))$ is a (k_1, k_2)-outdegree-splitting. $\square$

In fact, we prove a more general statement than Theorem 7 (Theorem 17). This enables us to prove the following generalization of Theorem 6.

Theorem 8. *Let $m = \max\{k_1^2/2 + 3k_1/2 + k_2 + u_1 + 1, k_1 + u_2\}$, let T be a tournament of minimum outdegree at least m and let U_1 and U_2 be two disjoints subsets of $V(T)$ of cardinality u_1 and u_2 respectively. Then there is a (k_1, k_2)-outdegree-splitting (V_1, V_2) of T such that $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$.*

The bound of Theorem 6 is certainly not tight. Theorem 6 asserts that every tournament with minimum outdegree 4 has a $(1, 1)$-outdegree-splitting, but we know that having outdegree 3 is sufficient.

Problem 9. What is the minimum integer $f_T(k_1, k_2)$ such that every tournament with minimum outdegree at least $f_T(k_1, k_2)$ has a (k_1, k_2)-outdegree-splitting?

Theorem 6 implies that $f_T(1, k) \leq k + 3$. We describe examples implying $f_T(1, k) \geq k + 2$, and we conjecture that this lower bound is the exact value.

Conjecture 10. *For any positive integer k, $f_T(1, k) = k + 2$.*

In Section 4, we establish this conjecture for $k \in \{2, 3, 4\}$, that is, we prove $f_T(1, 2) = 4$, $f_T(1, 3) = 5$, and $f_T(1, 4) = 6$.

Next we consider problems of deciding whether a digraph admits a $(k_1, k_2)-$outdegree-splitting.
OUTDEGREE SPLITTING
Input: A digraph D and two positive integers k_1 and k_2.
Question: Does D admit a (k_1, k_2)-outdegree-splitting?

Particular cases of this problem are when k_1 and k_2 are fixed integers and not part of the input. Hence for every fixed k_1, k_2, we have the following problem.
(k_1, k_2)-OUTDEGREE-SPLITTING
Input: A digraph D.
Question: Does D admit a (k_1, k_2)-outdegree-splitting?

Theorem 11. $(1, 1)$-OUTDEGREE-SPLITTING *is polynomial-time solvable.*

Proof. Let us describe a polynomial-time algorithm solving $(1, 1)$-OUTDEGREE SPLITTING.

If the input digraph D has a vertex with outdegree 0, then the answer is 'no' because this vertex has outdegree 0 in any subdigraph of D containing it. Henceforth we may assume that $\delta^+(D) \geq 1$.

It is well-known that a digraph with outdegree at least 1 contains a cycle. Therefore, Proposition 3 implies that a digraph with minimum outdegree at least 1 admits a $(1, 1)$-outdegree-splitting if and only if it contains two disjoint cycles. Thus it is enough to decide whether D contains two disjoint cycles.

But deciding whether a digraph contains two disjoint cycles can be done in polynomial time as shown by McCuaig [7]. (See also [9].) $\square$

In Section 5, we consider the restriction of these problems to tournaments.
TOURNAMENT OUTDEGREE-SPLITTING
Input: A tournament T and two positive integers k_1 and k_2.
Question: Does T admit a (k_1, k_2)-outdegree-splitting?

TOURNAMENT (k_1, k_2)-OUTDEGREE-SPLITTING
Input: A tournament T.
Question: Does T admit a (k_1, k_2)-outdegree-splitting?

TOURNAMENT $(1, 1)$-OUTDEGREE-SPLITTING is a particular case of $(1, 1)$-OUTDEGREE-SPLITTING, and thus is polynomial-time solvable. In Theorem 31, we show that, more generally, for any k_1, k_2, TOURNAMENT (k_1, k_2)-OUTDEGREE-SPLITTING can be solved in $O(n^{k^2/2+3k/2+3})$ time. We then describe a faster algorithm solving TOURNAMENT $(1, k_2)$-OUTDEGREE-SPLITTING. It runs in $O(n^3)$ time for $k_2 \geq 2$ and in $O(n^2)$ time for $k_2 = 1$. In view of these results, it is natural to ask the following.

Problem 12. Is TOURNAMENT OUTDEGREE-SPLITTING fixed-parameter tractable with (k_1, k_2) as a parameter? In other words, can we solve TOURNAMENT OUTDEGREE SPLITTING in $F(k_1, k_2)P(n)$ time, where F is an arbitrary computable function and P is a polynomial in the order n of the input tournament?

Finally, in Section 6, we present some possible directions for further research.

2 Definitions and folklore on tournaments

The *score sequence* of a tournament T, denoted by $s(T)$, is the non-decreasing sequence of outdegrees of its vertices. Landau [6] characterized the non-decreasing sequences of integers that are score sequences.

Theorem 13 (Landau [6]). *A non-decreasing sequence of non-negative integers $(s_1, s_2, \dots, s_n)$ is a score sequence if and only if :*

(i) $s_1 + s_2 + \dots + s_i \geq \binom{i}{2}$, *for* $i = 1, 2, \dots, n-1$, *and*

(ii) $s_1 + s_2 + \dots + s_n = \binom{n}{2}$.

Condition (ii) in the above theorem implies directly the following proposition.

Proposition 14. *Every tournament of order $2k$ has minimum degree less than k.*

Corollary 15. $f_T(k_1, k_2) \geq k_1 + k_2 + 1$.

Proof. Let T be a (k_1+k_2)-regular tournament of order $2k_1+2k_2+1$. In every bipartition (V_1, V_2) of $V(T)$, either $|V_1| \leq 2k_1$ or $|V_2| \leq 2k_2$. Thus, by Proposition 14, either $\delta^+(T[V_1]) < k_1$ or $\delta^+(T[V_2]) < k_2$. $\square$

An *ℓ-cycle* is a cycle of length ℓ. A tournament T is *transitive* if it contains no cycles. The score sequence of a transitive tournament of order n is $(0, 1, \dots, n-1)$.

We denote by $tt_3(T)$ the number of transitive subtournaments of order 3 in T and by $c_3(T)$ number of 3-cycles in T. Since a tournament of order 3 is either a transitive tournament or a 3-cycle, we have

$$tt_3(T) + c_3(T) = \binom{|T|}{3}.$$

Now if v is a vertex, the number of transitive subtournaments of order 3 with source v is $\binom{d^+(v)}{2}$. Hence

$$tt_3(T) = \sum_{v \in V(T)} \binom{d^+(v)}{2}.$$

A digraph D is *strongly connected* or *strong* if there is a path from u to v for every $u, v \in V(D)$. A digraph D is *k-strong* if $D - X$ is strong for every $X \subset V(D)$ where $|X| \leq k-1$. A (strong) *component* of D is a strong subdigraph of D which is maximal by inclusion.

Let T be a tournament. Let $T_1, T_2, \dots, T_m$ be the components of T. Then $(V(T_1), V(T_2), \dots, V(T_m))$ is a partition of $V(T)$ and without loss of generality, we may suppose that $T_i \to T_j$ whenever $i < j$. In this case we say that $T_1 \to T_2 \to \dots \to T_m$ is the *decomposition* of T. Component T_1 is said to be the *initial* component of T and T_m its *terminal* component.

A vertex is *pancyclic* in a digraph D if, for every $3 \leq \ell \leq |D|$, it is contained in an ℓ-cycle. To contain a pancyclic vertex, a tournament must contain a hamiltonian cycle. Therefore, it must be strong according to Camion's theorem [4]. Moon [8] showed that this condition is sufficient.

Theorem 16 (Moon [8]). *Every vertex of a strong tournament is pancyclic.*

We sometimes use the results of this section without referring to them.

3 Small subtournament of minimum outdegree k

We now prove Theorem 7. In fact, we prove a more general theorem whose particular case with $U = \emptyset$ is Theorem 7.

Theorem 17. *Let T be a tournament with minimum outdegree at least k and $U \subseteq V(T)$ be a subset of vertices. There is a subtournament T' of T with minimum outdegree k such that $U \subseteq V(T')$ and $|V(T')| \leq |U| + k^2/2 + 3k/2 + 1$.*

Proof. For every p, we prove the result for all sets U of size p by induction on $|V(T)|$, the result holding trivially if $|V(T)| \leq p + k^2/2 + 3k/2 + 1$.

Let T be a tournament of order at least $p + k^2/2 + 3k/2 + 2$ with minimum outdegree at least k and U a set of p vertices of T. Let S be the set of vertices of degree k in T. There are $k|S|$ arcs with their tail in S. Among them $|S|(|S| - 1)/2$ are in S and the remaining ones have their heads out of S. Hence $|N^+(S)| \leq |S| + k|S| - |S|(|S| - 1)/2$. Now the polynomial $P(x) = (k + 3/2)x - x^2/2$ increases on $[0, k + 3/2]$ and decreases on $[k + 3/2, +\infty[$. Moreover $P(k + 1) = P(k + 2) = k^2/2 + 3k/2 + 1$. Consequently, $|N^+(S)| \leq k^2/2 + 3k/2 + 1$.

Since $|V(T)| \geq p + k^2/2 + 3k/2 + 2$, there is a vertex v which is not in $N^+(S) \cup U$. Thus $T - v$ has minimum outdegree at least k and by induction $T - v$ (and thus also T) has a subtournament T' with minimum outdegree k such that $U \subseteq V(T')$ and $|V(T')| \leq |U| + k^2/2 + 3k/2 + 1$. □

The bound $k^2/2 + 3k/2 + 1$ is Theorem 7 is tight in the following sense.

Proposition 18. *For every non-negative integer k, and for every $n \geq k^2/2+3k/2+1$, there is a tournament $T(n, k)$ of order n and a set $W \subset V(T)$ of order $n - k^2/2 + 3k/2 + 1$ such that for every $U \subset W$, every subtournament T' with minimum outdegree k such that $U \subseteq V(T')$ has order at least $|U|+k^2/2+3k/2+1$.*

Proof. Consider the disjoint union of a strong tournament S of order $k + 1$ and a transitive tournament TT of order $k(k + 1)/2$. Set $V(S) = \{s_1, \dots, s_{k+1}\}$. Partition $V(TT)$ into $k + 1$ sets $A_1, \dots, A_{k+1}$ such that $|A_i| = k - d^+_S(s_i)$. This is possible since $\sum_{i=1}^{k+1} d^+_S(s_i) = k(k + 1)/2$, so $\sum_{i=1}^{k+1}(k - d^+_S(s_i)) = |V(TT)|$. Now for each i, add the arc $s_i a$ for all $a \in A_i$, and all the arcs bs_i for all $b \in V(TT) \setminus A_i$. The resulting tournament R has order $k^2/2 + 3k/2 + 1$ and minimum outdegree k.

Let R' be a subtournament of R with outdegree at least k.

It must contains a vertex of S, because all subtournaments of TT are transitive. But each element s of S has outdegree exactly k in R, so if $s \in V(R')$, then $N^+_R(s) \subset V(R')$. Since S is strong, it has a hamiltonian cycle by Camion's Theorem, and so $V(S) \subset V(R')$. But by construction, every vertex in R is dominated by a vertex in S, and thus must be in R'. Hence $R = R'$.

Set $p = n - k^2/2 + 3k/2 + 1$. Let $T(n, k)$ be a tournament obtained from the disjoint union of R and the transitive tournament TT_p of order p by adding all arcs from TT_p towards R. Then, for any set $U \subset V(TT_p)$, every subtournament of $T(n, k)$ with minimum outdegree k containing U must also contain $V(R)$ and thus has order at least $|U| + k^2/2 + 3k/2 + 1$. □

We can then easily derive Theorem 8 from Theorem 17.

Proof of Theorem 8. Let T be a tournament of minimum outdegree at least m. The tournament $T - U_2$ has minimum outdegree at least k_1 because $|U_2| = u_2$. Thus, by Theorem 17, there exists a subtournament T_1 of $T - U_2$ with minimum degree at least k_1 and order at most $k_1^2/2 + 3k_1/2 + u_1 + 1$ such that $U_1 \subseteq V(T_1)$. Set $V_1 = V(T_1)$, $T_2 = T - T_1$, and $V_2 = V(T_2)$. By definition, $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. Now $\delta^+(T_2) \geq \delta^+(T) - |V_1| \geq k_2$. Hence $(V(T_1), V(T_2))$ is a (k_1, k_2)-outdegree-splitting. □

3.1 Outdegree-critical tournaments

Theorem 7 can be rephrased in terms of k-outdegree-critical tournament. A tournament T is said to be k-*outdegree-critical* if it has minimum outdegree k and all its proper subtournaments have outdegree less than k. Theorem 7 implies that all k-outdegree-critical tournaments have bounded size. Hence a natural problem is the following.

Problem 19. *Describe the k-outdegree-critical tournaments.*

The unique 1-outdegree-critical tournament is the 3-cycle.

We now show that the 2-outdegree-critical tournaments are those depicted in Figure 1.

Theorem 20. *Every tournament with minimum outdegree 2 has a subtournament isomorphic to one of those depicted in Figure 1.*

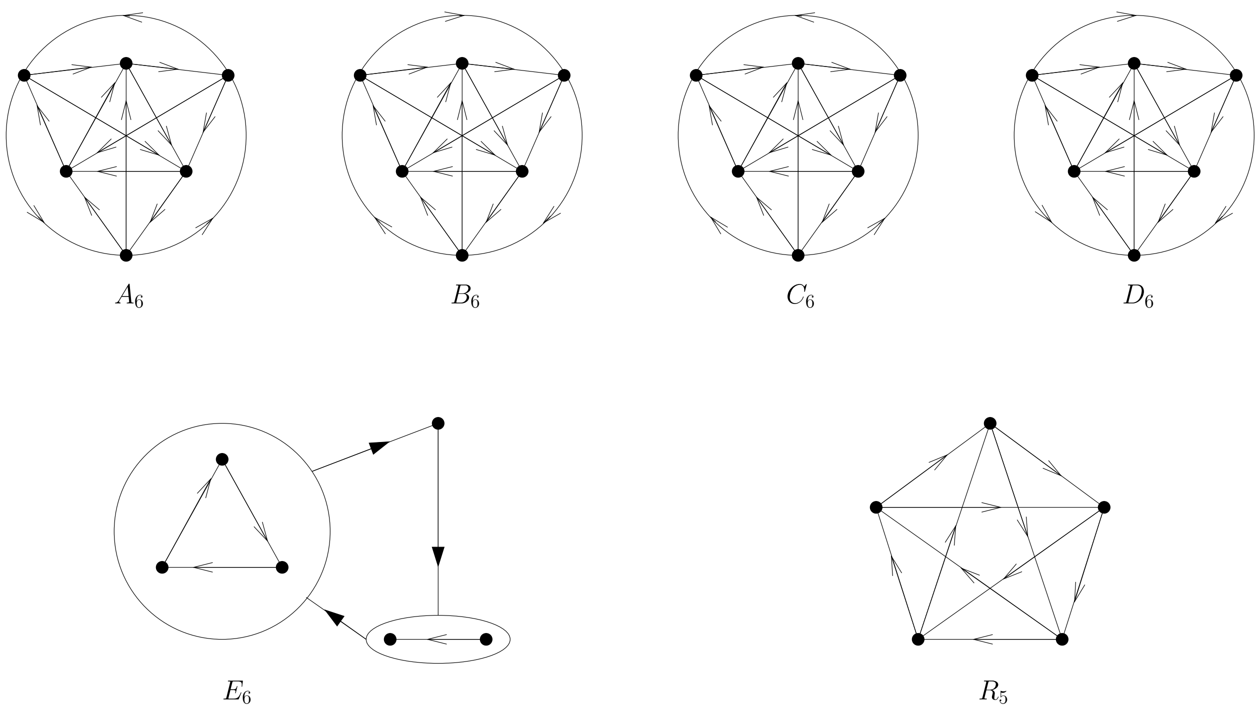

Figure 1: The 2-outdegree-critical tournaments.

Proof. By induction on $|V(T)|$, the result holding trivially when $|V(T)| < 5$.

Let T be a tournament of order at least 5 with minimum outdegree 2. Every vertex v has an inneighbour u such that $d^+(u) = 2$, for otherwise $T - v$ has minimum outdegree at least 2 and by induction $T - u$ (and thus also T) has a subtournament with minimum outdegree 2 and with order 5 or 6.

Let S be the set of vertices of outdegree 2 in T. By the previous remark, S is not empty and $T[S]$ has minimum indegree at least 1. Hence $T[S]$ contains a 3-cycle $C = (x_1, x_2, x_3)$. For $i = 1, 2, 3$, let y_i be the outneighbour of x_i in $V(T) \setminus \{x_1, x_2, x_3\}$. If all y_i are distinct, then each y_i dominates $\{x_1, x_2, x_3\} \setminus \{x_i\}$, and so $T[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is one of the tournaments A_6, B_6, C_6 and D_6. If $y_1 = y_2 = y_3$, let z_1 and z_2 be the two outneighbours of y_1. These two vertices dominate $\{x_1, x_2, x_3\}$, so $T[\{x_1, x_2, x_3, y_1, z_1, z_2\}]$ is isomorphic to E_6 . If $y_1 = y_2 \neq y_3$, then y_3 dominates x_1 and x_2, and y_1 dominates x_3. If y_1 dominates y_3, then $T[\{x_1, x_2, x_3, y_1, y_3\}]$ is isomorphic to R_5. If y_1 is dominated by y_3, let z be an outneighbour of y_1 distinct from x_3. The vertex z dominates $\{x_1, x_2\}$, so $T[\{x_1, x_2, x_3, y_1, y_3, z\}]$ is isomorphic to C_6 or D_6.

□

4 $(1, k)$-outdegree-splitting of tournaments

4.1 Improved upper bound for $f_T(1, k)$

A 3-cycle C in a tournament T is said to be *k-good* if $\delta^+(T - C) \geq k$. Clearly, if C is a k-good 3-cycle, then $(V(C), V(T - C))$ is a $(1, k)$-splitting of T.

Lemma 21. *Let k be an integer and let T be a strong tournament with minimum outdegree at least $k + 2$. Let S be the set of vertices with outdegree $k + 2$ in T. If T has no k-good 3-cycle, then the following hold.*

(i) Every arc is dominated by a vertex in S.

(ii) For every vertex v, the subtournament $T[N^-(v) \cap S]$ has minimum indegree 1 and at least five vertices.

(iii) $|V(T)| \leq \frac{1}{10}(k + 7)(k + 8)$.

Proof. Suppose that T contains no k-good 3-cycle. A 3-cycle C in T is *S-dominated* if there is a vertex $x \in S$ dominating C. Clearly, a 3-cycle in T is k-good if and only if it is not S-dominated. Hence all 3-cycles are S-dominated.

(i) Let uv be an arc. Since T is strong, there is a 3-cycle C containing u by Theorem 16. This cycle is dominated by a vertex $s \in S$. If s dominates v, then s dominates the arc uv. If not, then uvs is a 3-cycle. This cycle is dominated by a vertex in s' in S, which thus dominates uv.

(ii) Let v be a vertex of T. By (i), v is dominated by a vertex in S, so $N^-(v) \cap S$ is not empty. For any vertex $s \in N^-(v) \cap S$, the arc sv is dominated by a vertex $s' \in S$, which is distinct from s. Hence $T[N^-(v) \cap S]$ has indegree at least 1 and thus contains a 3-cycle $s_1s_2s_3$. This 3-cycle is dominated by a vertex $s \in S$.

Assume fist $s \to v$. By (i) the arc sv is dominated by a vertex s' of S. Clearly $s' \notin \{s_1, s_2, s_3\}$, because dominates $s_1s_2s_3$. Hence s_1, s_2, s_3, s, s' are five vertices in $N^-(v) \cap S$.

Assume now that $v \to s$. Then ss_1v is a 3-cycle which is dominated by a vertex s'. This vertex is in $N^-(v) \cap S$ and is distinct from s_2, s_3 because it dominates s. Furthermore, by (i) there is a vertex t of S dominating $s'v$. If $t \notin \{s_1, s_2, s_3\}$, then s_1, s_2, s_3, s', t are five vertices in $N^-(v) \cap S$. So we may assume that $t \in \{s_1, s_2, s_3\}$ and , without loss of generality, $t = s_2$. Now, there is a vertex s'' dominating the 3-cycle ss_2v. This vertex is distinct from s_1, s_3 because it dominates s, and is distinct from s' because it dominates s_2. Hence, s_1, s_2, s_3, s', s'' are five vertices in $N^-(v) \cap S$.

(iii) By (ii), every vertex has at least four inneighbours in S. Thus

$$|V(T)| = |N^+(S)| \leq |S| + \frac{1}{5}\left((k+2)|S| - \binom{|S|}{2}\right).$$

But the polynomial $Q(x) = x + \frac{1}{5}\left((k+2)x - \binom{x}{2}\right) = \frac{1}{10}x(2k+15-x)$ increases on $[0, k+15/2]$ and decreases on $[k+15/2, +\infty[$ and $Q(k+7) = Q(k+8) = \frac{1}{10}(k+7)(k+8)$. Consequently, $|V(T)| \leq \frac{1}{10}(k+7)(k+8)$. □

Theorem 22. *Let k be an integer in $\{1,2,3,4\}$. If T is a tournament with minimum outdegree at least $k+2$, then T contains a k-good 3-cycle.*

Proof. It is sufficient to prove the result for strong tournaments. Indeed if T is not strong, then its terminal component T' has also outdegree at least $k+2$. Moreover, every 3-cycle that is k-good in T' is also k-good in T.

Henceforth, we may assume that T is strong. Let S be the set of vertices with outdegree $k+2$ in T.

- Assume $k \in \{1,2\}$. Then every vertex of S has outdegree at most 4 in $T[S]$, so $T[S]$ has a vertex with indegree at most 4. Thus, by Lemma 21-(ii), T has a 1-good 3-cycle.

- Assume $k = 3$. Since $\delta^+(T) \geq 5$, then $|V(T)| \geq 11$. By Lemma 21-(iii), we have the result if $|V(T)| > 11$. Henceforth we may assume $|V(T)| = 11$, so T is 5-regular. Hence $tt_3(T) = \sum_{v \in V(T)} \binom{5}{2} = 110$. Thus $c_3(T) = \binom{11}{3} - 110 = 55$. Now a tournament of order 5 contains at most five 3-cycles, and it contains exactly five if and only if it is R_5 the 2-regular tournament on 5-vertices. If all the 3-cycles are dominated, the outneighbourhood of every vertex induces an R_5. But then a vertex u dominates at most two inneighbours of any other vertex v. Now if T had no k-good 3-cycles, then by Lemma 21-(ii), for every vertex v the subtournament $T[N^-(v)]$ would have a 3-cycle, which cannot be dominated and thus is k-good, a contradiction.

- Assume $k = 4$. Since $\delta^+(T) \geq 6$, then $|V(T)| \geq 13$. By Lemma 21-(iii), we have the result if $|V(T)| > 13$. Henceforth we may assume $|V(T)| = 13$, so T is 6-regular. It is possible to test all 6-regular graphs on 13 vertices using a simple computer program and verify that each of them has at least one good 3-cycle.

□

Corollary 23. *For $k \in \{1,2,3,4\}$, $f_T(1,2) = k+2$.*

Proof. Let $k \in \{1,2,3,4\}$. Theorem 22 implies $f_T(1,k) \leq k+2$ and Corollary 15 yields $f_T(1,k) \geq k+2$. □

We believe that Theorem 22 can be extended to all values of k.

Conjecture 24. *Let k be a positive integer. If T is a tournament with minimum outdegree at least $k+2$, then T contains a k-good 3-cycle.*

A first step to prove this conjecture is the following.

Conjecture 25. *Let k be a positive integer. If T is a $(k+2)$-regular tournament, then T contains a k-good 3-cycle.*

If true Conjecture 24 would be best possible.

Proposition 26. *Let k be a positive integer. For any $n \geq 3k+3$, there is a tournament of order n with minimum outdegree $k+1$ that does not admit any $(1,k)$-outdegree-splitting.*

Proof. Let $n \geq 3k+3$. Let T be a tournament of order n whose vertex set can be partitioned into $(X_1, X_2, \{x\})$ such that $X_1 \to X_2$, $X_2 \to x$, $x \to X_1$, $T[X_1]$ is a transitive tournament of order $n-2k-2$, and $T[X_2]$ is a k-regular tournament.

Clearly, $\delta^+(T) = k+1$. Let us now prove that T has no $(1,k)$-outdegree-splitting.

Suppose for a contradiction that T admits a $(1,k)$-outdegree-splitting (V_1, V_2). The set V_2 must contain a vertex in X_2 because $T[X_1 \cup \{x\}]$ is transitive. The subtournament $T[V_1]$ contains a 3-cycle C. This cycle either contains x or is contained in C_1.

- If C contains x, then $C = xx_1x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. But $T[X_2]$ is k-regular, so it is strong. Thus there is a vertex u of $V_2 \cap X_2$ dominating a vertex in $V_1 \cap X_2$. Thus u has outdegree at most $k-1$ in $T[V_2]$, a contradiction.

- If C is contained in C_1, then $|V_2 \cap X_2| \leq 2k-2$. Therefore $T[V_2 \cap X_2]$ has a vertex u with outdegree less than $k-1$. This vertex u has outdegree less than k in $T[V_2]$, a contradiction.

□

4.2 Existence of k-good 3-cycles

A result of Song [11] states that every 2-strong tournament of order at least 6 can be split into a 3-cycle and a strong subtournament unless it is P_7, the Paley tournament of order 7. Since P_7 is 3-regular, it has a 1-good 3-cycle by Theorem 22. Therefore we obtain the following.

Theorem 27. *Every 2-strong tournament of order at least 6 has a 1-good 3-cycle and thus admits a $(1,1)$-outdegree-splitting.*

In fact, having a 1-good 3-cycle is equivalent to having a $(1,1)$-outdegree-splitting.

Proposition 28. *Let T be a tournament. Then T has a $(1,1)$-outdegree-splitting if and only if it has a 1-good 3-cycle C.*

Proof. As we already observed, if C is a 1-good 3-cycle, then T has a $(1,1)$-outdegree-splitting.

Conversely, suppose that T admits a $(1,1)$-outdegree-splitting (V_1, V_2). Then for $i = 1,2$, $T[V_i]$ contains a 3-cycle C_i. Let S_2 be the largest set such that $V_2 \subseteq S_2 \subseteq V(T - C_1)$ and $\delta^+(T[S_2]) \geq 1$. If $S_2 = V(T - C_1)$, then C_1 is k-good. If not, then let $R = V(T - C_1) \setminus S_2$. By definition, $S_2 \to R$. Thus $\delta^+(T - C_2) \geq 1$, and C_2 is k-good. □

Unfortunately, Proposition 28 cannot be generalized for larger value of k in the sense that there are tournaments with a $(1,k)$-splitting and no k-good 3-cycles. Furthermore, there are such tournaments with minimum outdegree $k+1$; this shows that the condition of having minimum outdegree $k+2$ in Conjecture 24 is best possible.

Proposition 29. *Let k be an integer greater than 1. There exists a tournament of order at $3k+3$ with minimum outdegree $k+1$ such that T has a $(1,k)$-splitting but no k-good 3-cycles.*

Proof. Let T be a tournament whose vertex set can be partitioned into $(X_1, X_2, X_3, \{x\})$ such that $X_1 \to X_2$, $X_1 \cup X_2 \to X_3$, $X_3 \to x$, $x \to X_1 \cup X_2$, $T[X_1]$ is a transitive tournament of order $k-2$, and $T[X_2]$ is a 3-cycle and $T[X_3]$ is a k-regular tournament.

Clearly, $(X_1 \cup X_2 \cup \{x\}, X_3)$ is a $(1,k)$-splitting of T.

Let us now prove that no 3-cycle is k-good. There are three kinds of 3-cycles: $T[X_2]$, 3-cycles contained in $T[X_3]$, and 3-cycles of the form xyz with $y \in X_1 \cup X_2$ and $z \in X_3$.

- $T[X_2]$ is not k-good, because x has outdegree less than k in $T - X_2$.
- If C is a 3-cycle in $T[X_3]$, then $T[X_3] - C$ has at most $2k-2$ vertices and thus contains a vertex v of outdegree less than $k-1$. Therefore v has outdegree less than k in $T - C$. So C is not k-good.
- If C is a 3-cycle of the form xyz with $y \in X_1 \cup X_2$ and $z \in X_3$, then every inneighbour v of z in $T[X_3]$ has outdegree less than k in $T - C$. So C is not k-good.

□

Problem 30. *For any fixed $k \geq 2$, are there infinitely many strong tournaments with minimum outdegree $k+1$ that have a $(1,k)$-splitting but no k-good 3-cycles ?*

5 Finding outdegree splittings in tournaments

Theorem 31. *For every positive integers k_1 and k_2,* TOURNAMENT (k_1, k_2)-OUTDEGREE-SPLITTING *is polynomial-time solvable.*

Proof. Let $g(k) = k^2/2 + 3k/2 + 1$. Let T be a tournament of order n. If T has a (k_1, k_2)-outdegree-splitting (V_1, V_2), then V_1 contains a subset S_1 of size at most $g(k_1)$ such that $\delta^+(T[S_1]) \geq k_1$.

The algorithm considers all subsets S_1 of order at most $g(k_1)$. For each of them, we first check if $\delta^+(T[S_1]) \geq k_1$. If no, we proceed to the next subtournament. If yes, we check if there is a (k_1, k_2)-outdegree-splitting (V_1, V_2) such that $S_1 \subseteq V_1$ using a procedure $\texttt{extend}(S_1)$. If this procedure, returns 'yes', then we also return 'yes'. If not we proceed to the next subtournament.

The procedure $\texttt{extend}(S_1)$ proceeds as follows. If $S_1 = V(T)$, return 'no'. If $T - S_1$ has minimum outdegree at least k_2, we return $(S_1, V(T) \setminus S_1)$. Otherwise, pick a vertex x of $V(T) \setminus S_1$ having outdegree less than k_2 in $T - S_1$ and return $\texttt{extend}(S_1 \cup \{x\})$.

The procedure extend runs in $O(n^2)$-time. (We only need to make $O(n)$ updates on the score sequence). At worse, we run it for each subset S_1 of size at most $g(k_1)$. There are $O\left(n^{g(k_1)}\right)$ such subsets. Hence the algorithm runs in $O\left(n^{g(k_1)+2}\right)$ time. □

The running time of the algorithm given in the proof of Theorem 31 is certainly not optimal. When $k_1 = 1$, running time is $O(n^5)$. We now give a faster algorithm, that runs in $O(n^3)$ time for $k_1 = 1$ and $k_2 \geq 2$ and in $O(n^2)$ time for $k_1 = k_2 = 1$. This algorithm is also faster that the one described in Theorem 11.

The key ingredients are the following three statements. The first one is an immediate extension of Proposition 3 with an identical proof, which translates into a $O(n^2)$-time algorithm.

Proposition 32. *Let D be a digraph of order n. If D contains two disjoint digraphs D_1, D_2 such that $\delta^+(D_i) = k_i$ for $i = 1,2$ and $d^+_D(v) \geq k_1 + k_2 - 1$ for all $v \in V(D - (D_1 \cup D_2))$, then D admits a (k_1, k_2)-outdegree-splitting. Moreover such a (k_1, k_2)-outdegree-splitting can be found in $O(n^2)$ time.*

The second one is an algorithmic version of Theorem 17.

Proposition 33. *Let T be a tournament with minimum outdegree at least k. One can find in $O(n^3)$ time a subtournament T' of T with minimum outdegree k such that $|V(T')| \leq k^2/2 + 3k/2 + 1$.*

Proof. By the proof of Theorem 17, if $|V(T)| > k^2/2 + 3k/2 + 1$, then it contains a vertex x such that $T - x$ has minimum outdegree at least k. Such a vertex can be found in $O(n^2)$ time, by finding the set S of vertices with outdegree k, and taking x not in $S \cup N^+(S)$. We then recursively apply the procedure to $T - x$. As we reduce the order of the tournament at most n times, we find the desired subtournament T' in $O(n^3)$ time. □

Lemma 34. *Let T be a tournament and v a vertex of T. If T has a $(1, k)$-outdegree-splitting (V_1, V_2) with $v_1 \in V_1$, then there is a 3-cycle C_1 in $T[V_1]$ such that $v \in V(C_1)$ or $V(C_1) \subseteq N^+(v)$.*

Proof. Let $N_1 = N^+(v) \cap V_1$. If $T[N_1]$ has a cycle, then it is the desired 3-cycle. Otherwise, $T[N_1]$ is a transitive tournament. Now the sink w of $T[N_1]$ has an outneighbour u in $T[V_1]$, which is necessarily an inneighbour of v, by definition of N_1. Therefore uvw is the desired 3-cycle. □

Theorem 35. *(i)* TOURNAMENT $(1, 1)$-OUTDEGREE-SPLITTING *can be solved in $O(n^2)$ time;*

(ii) for all $k \geq 2$, TOURNAMENT $(1, k)$-OUTDEGREE-SPLITTING *can be solved in $O(n^3)$ time.*

Proof. (i) Let us describe a procedure $(1, 1)$−`split`(T) that given a tournament T returns 'yes' if it admits a $(1, 1)$-outdegree-splitting, and returns 'no', otherwise.

0. We first compute the outdegree of every vertex and we determine $\delta^+(T)$. This can be done in $O(n^2)$ time.

1. If $\delta^+(T) = 0$, then the tournament T has no $(1, 1)$-outdegree-splitting, and we return 'no'.

2. If $\delta^+(T) \geq 3$, the answer is 'yes', by Corollary 23.

3. If $\delta^+(T) \in \{1, 2, \}$, let v be a vertex of degree 1 or 2 in T. Without loss of generality, one may look for a $(1, 1)$-outdegree-splitting (V_1, V_2) of T such that $v \in V_1$. For every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check whether $T - \{u, v, w\}$ contains a 3-cycle. If yes for at least one choice of $\{u, v, w\}$, the answer is 'yes' by Proposition 3 since $\delta^+(T) \geq k$. If not, then return 'no' . This is valid by Lemma 34.

Given its score sequence, checking if a tournament of order n contains a 3-cycle can be done in $O(n)$ by checking whether the score sequence is distinct from $(0, 1, 2, \ldots, n - 1)$, the score sequence of the transitive tournament. Since the score sequence of $T - \{u, v, w\}$ can be obtained in linear time from the list of outdegrees of T, checking if $T - \{u, v, w\}$ contains a cycle can be done in $O(n)$ time.

Now since v has degree at most 2, the procedure considers at most $2(n - 1)$ subtournaments $T - \{u, v, w\}$. Therefore $(1, 1)$−`split` runs in $O(n^2)$ time.

(ii) Let us describe a procedure $(1, k)$−`split`(T) that given a tournament T returns 'yes' if T if it admits a $(1, k)$-outdegree-splitting, and return 'no', otherwise.

0. We first compute the outdegree of every vertex and we determine $\delta^+(T)$. This can be done in $O(n^2)$ time.

1. If $\delta^+(T) = 0$, then the tournament T has no $(1, k)$-outdegree-splitting, and we return 'no'.

2. If $1 \le \delta^+(T) \le k - 1$, let U_1 be the set of vertices of degree less than k in T. Clearly, for any $(1, 2)$-outdegree-splitting (V_1, V_2) of T, $U_1 \subseteq V_1$. Let v be a vertex of U_1. For every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check whether $T - (U_1 \cup \{u, v, w\})$ contains a subtournament of minimum outdegree k using the procedure `Outdegree-`k`-Subtournament` described below. If yes for at least one choice of $\{u, v, w\}$, the answer is 'yes' by Proposition 32 since all vertices of $V(T) \setminus U_1$ have outdegree at least k in T. If not, then return 'no' . This is valid by Lemma 34.

3. If $\delta^+(T) \ge k$, then we first find a subtournament T' of T with $\delta(T') \ge k$ and $|V(T')| \le k^2/2 + 3k/2 + 1$. If $T - T'$ contains a 3-cycle, then T admits a $(1, k)$-outdegree-splitting by Proposition 3, and so we return 'yes'. If not then $T - T'$ is a transitive tournament and all 3-cycles of T intersect T' and therefore there are at most $(k^2/2 + 3k/2 + 1)n^2$ of them. For each 3-cycle C, we check with `Outdegree-`k`-Subtournament` whether $T - C$ contains a subtournament of minimum outdegree k. If yes, for one of them, then we return 'yes' because there is a $(1, k)$-outdegree-splitting by Proposition 3. If not, then we return 'no'.

Remark 36. In the above procedure, one can shorten Step 3 if $\delta^+(T) \ge k+2$. In this case, by Corollary 23, we can directly return 'yes'.

The procedure `Outdegree-`k`-Subtournament`(T) takes as an input the tournament T as well as its list of outdegrees and a list L of vertices in the transitive tournament $T - T'$ ordered in increasing order of their outdegrees. Observe that the list of outdegrees is already computed when `degree-`$(1,k)$`-split` call this procedure and the order of $T - T'$ can be computed just once after computing T'. First, we alter the list of outdegrees by keeping the outdegrees for vertices in T' but for vertices in $T - T'$ we count only outneighbours in T'. At each step, `Outdegree-`k`-Subtournament` first checks $V(T)$ and returns 'no' if $V(T) = \emptyset$, otherwise it tries to finds a vertex v with $d^+(v) < k$. Notice that possible candidates for v are only vertices in T' and the first k vertices in L. If there is no such vertex v, it returns 'yes'. Otherwise it removes v and tries again. If $v \in V(T')$, then it decreases the outdegree of all inneighbours of v and if $v \notin V(T')$, then it decreases outdegrees only for inneighbours from $V(T')$. The total time spent on a vertex $v \in V(T')$ is $O(n)$, which gives $O(V(T')n) = O(n)$ in total. The total time spent on a vertex $v \notin V(T')$ is $O(1)$, which gives $O(n)$ in total. Therefore, `Outdegree-`k`-Subtournament` runs in $O(n)$ time.

Now Step 1 runs in constant time. In Step 2, there are at most $k + 1$ candidates for w, and thus `Outdegree-`k`-Subtournament` is called less than $(k + 1)n$ times. Therefore Step 2 runs in $O(n^3)$ time. Step 3 first finds a small subtournament T' with outdegree k, which can be done in $O(n^3)$ time by Proposition 33. Then it runs $O(n^2)$ times `Outdegree-`k`-Subtournament`. Therefore Step 3 runs in $O(n^3)$ time.

Overall $(1, k)$`-split` runs in $O(n^3)$ time. □

The procedure $(1, k)$`-split`(T) can be modified to find a $(1, k)$-outdegree-splitting if it exists, using Proposition 32 instead of Proposition 3.

In contrast, the procedure $(1, 1)$`-split`(T) cannot be instantly modified into a procedure that finds a $(1, 1)$-outdegree-splitting if it exists. However, using a similar approach, we now describe such a procedure.

Theorem 37. *One can find a $(1, 1)$-outdegree-splitting of a tournament in $O(n^2)$ time.*

Proof. Let us describe a procedure $(1,1)\texttt{-findsplit}(T)$ that returns a $(1,1)$-outdegree-splitting of the tournament T if it admits one, and return 'no', otherwise.

We first compute the outdegree of every vertex and we determine $\delta^+(T)$.

If T contains a vertex of outdegree 0, then we return 'no'. If $\delta^+(T) \geq 4$, then we pick a vertex x and find a 3-cycle C containing x. Such a cycle can be found in $O(n^2)$ by testing if there is an arc from $N^+(x)$ to $N^-(x)$. We return $(V(C), V(T-C))$. This is valid since $\delta^+(T-C) \geq \delta^+(T) - |V(C)| \geq 1$.

If $\delta^+(T) \leq 3$, we choose a vertex v such that $d^+(v) \in \{1,2,3\}$. If $T[N^+(v)]$ induces a 3-cycle, then we check whether $T - N^+(v)$ contains a cycle C. If yes, we extend $(T[N^+(v)], C)$ into a $(1,1)$-outdegree-splitting by Proposition 32. If not, for every $w \in N^+(v)$ and $u \in N^+(w) \setminus N^+(v)$, we check if $T - \{u, v, w\}$ contains a cycle $C(uvw)$. If yes for at least one choice of $\{u, w\}$, then we extend $(uvw, C(uvw))$ into a $(1,1)$-outdegree-splitting by Proposition 32 and we return 'no' otherwise. This is valid by Lemma 34.

Since there are at most three candidates for w, there are $O(n)$ cases to check. Therefore $(1,1)\texttt{-findsplit}$ runs in $O(n^2)$ time. □

Remark 38. The proof of Proposition 28 yields a $O(n^2)$-time procedure to find a 1-good 3-cycle given a $(1,1)$-outdegree-splitting. Combining this procedure with $(1,1)\texttt{-findsplit}$, we obtain a $O(n^2)$-time algorithm that finds a 1-good 3-cycle in a tournament if it exists, and returns 'no' otherwise.

6 Further research

6.1 Splittable score sequences

Being $(1,1)$-outdegree-splittable is not determined by the score sequence. For example, the two tournaments depicted Figure 2 have score sequences $(2,2,2,2,3,4)$ but the one to the left has no $(1,1)$-outdegree-splitting (See Proposition 26) while the one to the right admits the $(1,1)$-outdegree-splitting $(\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\})$.

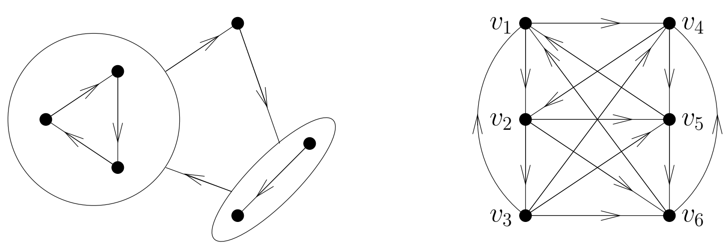

Figure 2: Non-$(1,1)$-outdegree-splittable and $(1,1)$-outdegree-splittable tournaments with the same score sequence

However there are score sequences s such that all tournaments with score sequence s are $(1,1)$-outdegree-splittable. Such score sequences are said to be $(1,1)$-*outdegree-splittable*. For example, Theorem 22 implies that $(s_1, \ldots, s_n)$ is $(1,1)$-outdegree-splittable.

Problem 39. *Which score sequences are* $(1,1)$*-outdegree-splittable?*

6.2 Erdős-Posa property for digraphs with minimum outdegree k

McCuaig's algorithm [7] relies on the theorem stating that a digraph D has either two disjoint cycles or a set S of at most three vertices such that $D - S$ is acyclic. More generally, Reed et al. [9] showed that cycles in digraphs have the Erdös-Posa property.

Theorem 40 (Reed et al. [9])**.** *For every positive integer n, there exists an integer $t(n)$ such that for every digraph D, either D has a n pairwise-disjoint cycles, or there exists a set T of at most $t(n)$ vertices such that $D - T$ is acyclic.*

It is then natural to ask whether digraphs with maximum outdegree k have the the Erdös-Posa property.

Problem 41. *Let k be a fixed integer. For every positive integer n, does there exist an integer $t_k(n)$ such that for every digraph D, either D has a n pairwise-disjoint subdigraphs with minimum outdegree k, or there exists a set T of at most $t_k(n)$ vertices such that $\delta^+(D - T) < k$?*

6.3 Strong connectivity and outdegree-splitting with prescribed vertices

Any $f_T(k_1, k_2)$-strong tournament has minimum outdegree at least $f_T(k_1, k_2)$ and thus admits a (k_1, k_2)-outdegree-splitting. Therefore, it is natural to ask the following.

Problem 42. *What is the minimum integer $h_T(k_1, k_2)$ such that every $h_T(k_1, k_2)$-strong tournament T of order at least $2k_1 + 2k_2 + 2$ contains a (k_1, k_2)-outdegree-splitting?*

The condition $|V(T)| \geq 2k_1 + 2k_2 + 2$ is the above problem is just to avoid the small tournaments that cannot have any (k_1, k_2)-outdegree-splitting for cardinality reasons. Clearly, $h_T(k_1, k_2) \leq f_T(k_1, k_2)$. But it is very likely that $h_T(k_1, k_2)$ is smaller than $f_T(k_1, k_2)$. As mentionned in the beginning of Subsection 4.2, a result of Song [11] implies that $h_T(1, 1) \leq 2$ (In fact $h_T(1, 1) = 2$ because a 1-strong tournament T with a vertex v such that $T - v$ is a transitive tournament has clearly no $(1, 1)$-outdegree-splitting.) whereas $f_T(1, 1) = 3$.

One might also ask similar questions for outdegree-splitting with prescribed vertices (as in Theorem 8). Bang-Jensen et al. [2] proved that if T is a tournament of order 8 and xy an arc in T such that $T \setminus xy$ is 2-strong, then T contains an outdegree-1-splitting (V_x, V_y) with $x \in V_x$ and $y \in V_y$.

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