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Faculty of Mathematics and Physics

DOCTORAL THESIS

Bernard Lidický

Graph coloring problems

Department of Applied Mathematics

Supervisor of the doctoral thesis: doc. RNDr. Jiří Fiala, Ph.D.

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Všem lakýrníkům

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Varianty problému obarvení

Autor: Bernard Lidický

Katedra: Katedra aplikované matematiky

Vedoucí disertační práce: doc. RNDr. Jiří Fiala, Ph.D.

Abstrakt:

V této práci studujeme barvení grafů. Práce je rozdělena do tří částí, kde každá část se zabývá jinou variantou barvení. V první části se zabýváme 6-kritickými grafy na plochách a 6-kritickými grafy s malým počtem křížení. Hlavními výsledky jsou kompletní seznam 6-kritických grafů na Kleinově láhvi a 6-kritických grafů s křížícím číslem nejvýše čtyři.

Druhá část je věnována vybíravosti rovinných grafů bez krátkých cyklů. Ukazujeme, že rovinné grafy bez 3-,7- a 8-cyklů jsou 3-vybíravé a že rovinné grafy bez trojúhelníků a s jistým omezením na 4-cykly jsou též 3-vybíravé.

V poslední části se zaměřujeme na novější variantu barvení - vlnové barvení. Jde o koncept, který je motivovaný přiřazováním frekvencí a má zohlednit, že různé frekvence mají různý dosah. Zabýváme se pak zlepšováním odhadů nutného počtu barev k obarvení čtvercové a šestiúhelníkové mřížky.

Klíčová slova: kritické grafy, seznamové barvení, vlnové barvení, rovinné grafy, krátké cykly

Title: Graph coloring problems

Author: Bernard Lidický

Department: Department of Applied Mathematics

Supervisor: doc. RNDr. Jiří Fiala, Ph.D.

Abstract:

As the title suggests, the central topic of this thesis is graph coloring. The thesis is divided into three parts where each part focuses on a different kind of coloring. The first part is about 6-critical graphs on surfaces and 6-critical graphs with small crossing number. We give a complete list of all 6-critical graphs on the Klein bottle and complete list of all 6-critical graphs with crossing number at most four.

The second part is devoted to list coloring of planar graphs without short cycles. We give a proof that planar graphs without 3-,6-, and 7- cycles are 3-choosable and that planar graphs without triangles and some constraints on 4-cycles are also 3-choosable.

In the last part, we focus on a recent concept called packing coloring. It is motivated by a frequency assignment problem where some frequencies must be used more sparsely that others. We improve bounds on the packing chromatic number of the infinite square and hexagonal lattices.

Keywords: critical graphs, list coloring, packing coloring, planar graphs, short cycles

Contents

Chapter 1

Introduction

Graph coloring is a popular topic of discrete mathematics. It has roots in the Four color problem which was the central problem of graph coloring in the last century. The Four color problem asks if it is possible to color every planar map by four colors. Although it is claimed to the Four color theorem has its roots in practice there is no evidence that the question was raised by map-makers [50]. It seems that mathematicians just asked themselves.

Despite the theoretical origin the graph coloring has found many applications in practice like scheduling, frequency assignment problems, segmentation etc. A popular application in recent years is Sudoku.

Many variants and generalizations of the graph coloring have been proposed since the Four color theorem. In this thesis, we present new results on graph coloring, list coloring and packing coloring.

The next chapter contains a quick review of definitions used in this thesis. It is not a completely exhaustive list but it is covering the essential ones. Chapter 3 describes basic graph coloring methods which are demonstrated on simple theorems. It should help readers to follow proofs in the subsequent chapters.

The first chapter with new results is dedicated to ordinary coloring and critical graphs. We present theorems enumerating all 6-critical graphs on the Klein bottle and all 6-critical graphs with crossing number at most four. The chapter is based on papers *6-critical graphs on the Klein bottle* by Kawarabayashi, Král', Kynčl and Lidický [31] (published in SIAM Journal on Discrete Mathematics) and *5-colouring graphs with 4 crossings* by Erman, Havet, Lidický and Pangrác [17] (published in SIAM Journal on Discrete Mathematics).

In Chapter 5 we study list coloring which is a generalization of coloring where every vertex has its own list of colors. We focus on 3-choosability of planar graphs and give two different conditions implying that a planar graph is 3-choosable. The chapter is based on papers *On 3-choosability of plane graphs without 3-, 7- and 8-cycles* [11] (published in Discrete Mathematics) and *3-choosability of triangle-free planar graphs with constraints on 4-cycles* [12] (published in SIAM Journal on Discrete Mathematics) by Dvořák, Lidický and Škrekovski.

The last chapter deals with a recent concept called packing coloring. The concept is inspired by frequency assignment where some frequencies must be used more sparsely than the others. We improve bounds for the square grid and hexagonal grid. The chapter is based on a paper *The packing chromatic number of infinite product graphs* by Fiala, Klavžar and Lidický [19] (published in European Journal of Combinatorics) and a manuscript *The packing chromatic number of the square lattice is at least 12* Ekstein, Fiala, Holub and Lidický [13].

Chapter 2

Definitions

In this chapter we review definitions used in the following chapters. For more examples and details see books by Matoušek and Nešetřil [35], Bollobás [4] or Diestel [8].

2.1 Basics

We denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, the set of rational numbers by $\mathbb{Q}$ and the set of reals by $\mathbb{R}$. We abbreviate the set $\{1, 2, \dots n\}$ by $[n]$. For a set S we denote the set of all subsets of size k by $\binom{S}{k}$

2.2 Graphs

Graphs are the main objects studied in this thesis. A *graph* G is a pair (V, E), where V is a set of *vertices* of G and E is a set of *edges* of G. We consider only graphs where $E \subseteq \{\{u, v\} : u \in V, v \in V, u \neq v\}$. We also use notation $V(G)$ for vertices of G and $E(G)$ for the set of edges of G. These graphs are sometimes called *simple* graphs. We do not use and hence do not define hypergraphs, multigraphs or oriented graphs.

A *subgraph* of a graph $G = (V, E)$ is a graph $H = (V_H, E_H)$ where $V_H \subseteq V$ and $E_H \subseteq E$. If $G \neq H$ then H is called a *proper subgraph.* If $E_H = E \cap \binom{V}{2}$ then H is called an *induced* subgraph. A graph *induced* by $S \subseteq V$ is an induced subgraph W of G such that $V(W) = S$.

We call vertices u and v of a graph G *adjacent* if $\{u, v\} \in E(G)$. A vertex v is *incident* with an edge e if $v \in e$.

The set of all vertices adjacent to a vertex v is the *neighborhood* of v and it is denoted by $N(v)$. Size of the neighborhood is the *degree* of v, denoted by $\deg(v)$. A vertex v of degree k is a *k-vertex.* Analogously, we use terms $(\geq k)$-vertex, $(\leq k)$-vertex. The maximum degree among all vertices of a graph G is denoted by $\Delta(G)$ or simply by Δ if G is clear from the context.

A *complement* of a graph $G = (V, E)$, denoted by $\overline{G}$, is a graph $(V, \binom{V}{2} \setminus E)$.

Let $G = (V, E)$ be a graph and let S be a subset of V. We call S a *clique* if $\binom{S}{2}$ is a subset of E. The size of the largest clique is called the *clique number* and it is denoted by $\omega(G)$ or just by ω if G is clear from the context. We call S an *independent* set if S is a clique in $\overline{G}$. The size of the largest independent set, denoted by $\alpha(G)$ or just α, is called the *independence number.* We also use the notion of a *stable* set for an independent set.

Let us give a brief list of common graphs. A *path* on n vertices, denoted by P_n, is a graph $G = (V, E)$ with vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges $E = \{v_i v_{i+1} : 1 \leq i < n\}$, see Figure 2.1. Vertices v_1 and v_n are called the *ends* of the path. A *path of length* n (or an n*-path*) is a path on $n+1$ vertices. A *cycle* on $n \geq 3$ vertices, denoted by C_n, is a graph constructed from P_n by adding an extra edge joining ends of P_n, see Figure 2.1. A *complete graph* on n vertices, denoted by K_n, is a clique of size n, see Figure 2.1. Note that C_3 and K_3 are the same.

A graph $G = (V, E)$ is *bipartite* if $V = V_1 \cup V_2$ where both V_1 and V_2 are independent sets. The sets V_1 and V_2 are the *classes of the bipartition.* Note that a path is always bipartite, a cycle is bipartite if and only if it is on even number of vertices and a clique is bipartite if and only if it has at most two vertices. A *complete bipartite graph*, denoted by $K_{m,n}$, is a bipartite graph with classes of bipartition of sizes m and n and the maximum number of edges, see Figure 2.1.

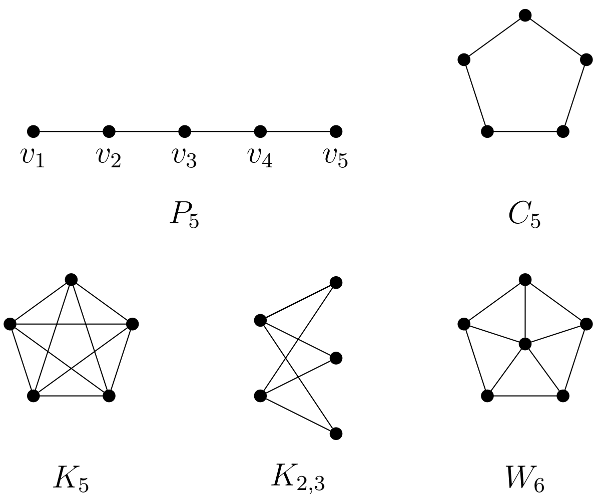


Figure 2.1: Examples of small graphs

A *join* of two graphs G_1 and G_2 is the graph obtained by adding all edges between vertices of G_1 and G_2. A *wheel* W_n on n vertices is the join of K_1 and C_{n-1}, see Figure 2.1. A k*-wheel* is W_{k+1}.

Two graphs G_1 and G_2 are *isomorphic* if there exists a bijective mapping called *isomorphism* $\varrho : G_1 \to G_2$ sending $V(G_1)$ to $V(G_2)$ and $E(G_1)$ to $E(G_2)$.

A *three* is a graph without cycles as induced subgraphs. Let G_1 and G_2 be graphs. A *cartesian product* of graphs G_1 and G_2, denoted by $G_1 \square G_2$ is a graph G wiht vertex set $V(G) := \{(u, v) : u \in V(G_1), v \in V(G_2)\}$ and edge set $E(G) := \{(u_1, v_1)(u_2, v_2) : u_1 u_2 \in E(G_1) \text{ or } v_1 v_2 \in E(G_2)\}$. For example $P_2 \square P_2$ is isomorphic to C_4.

A graph $G = (V, E)$ is *connected* if for every two vertices u and u of V it contains a path with ends u and v as a subgraph. A *connected component* of G is maximal induced connected subgraph of G.

Let C be a cycle in a graph G. An edge $e = uv \in E(G)$ is a *chord* of C if both $u, v \in V(C)$ but $e \notin E(C)$.

2.3 Graph drawing

It is often desired that a graph is not just an abstract structure but has also some drawing where vertices are represented by points and edges by simple curves. More precisely, a *drawing* $\tilde{G}$ of a graph $G = (V, E)$ on a surface S consists of a mapping D from $V \cup E$ into S such that

(i) $D(u) \neq D(v)$ whenever $u \neq v$ for every $u, v \in V$;

(ii) for any edge $e = uv$, the image of $D(e) = \tilde{e}$ is the image of a continuous injective mapping ϕ_e from $[0, 1]$ to S which is simple (i.e. does not intersect itself) such that $\phi_e(0) = D(u)$, $\phi_e(1) = D(v)$ and $\phi_e(]0, 1[) \cap \tilde{V} = \emptyset$;

(iii) every point in S is in at most two images of edges, unless it is an image of some vertex;

(iv) for two distinct edges e_1 and e_2 of E, the edges $\tilde{e_1}$ and $\tilde{e_2}$ intersect in a finite number of points.

We usually do not distinguish vertex and edge sets of a graph from their images in a drawing as it does not lead to any confusion and simplifies presentation.

Let G be drawn (embedded) without crossings. Connected regions of S after removal of edges and vertices of G are called *faces* of G. We denote the set of faces by $F(G)$. A *facial walk* is walk around the boundary of a face. It may happen that vertices or edges have mutiple occurrences along the walk. A face is a *k-face* if the walk encounters k vertices including repetitions. If G is drawn without crossings then it is also the number of encountered edges, again with repetitions.

Note that our definition allows edges to intersect in their interior points. We call such vertex, which belongs to images of two distinct edges and is not image of any vertex, *crossing*. Formally, it is a point of $\phi_{e_1}(]0, 1[) \cap \phi_{e_2}(]0, 1[)$ for some edges e_1 and e_2. A *portion* of an edge e is a subarc of $\phi_e[0, 1]$ between two consecutive endpoints or crossings on e. A portion from a to b is called an *(a, b)-portion*.

The minimum number of crossings among all drawings of G is the *crossing number* of G, denoted by $\mathrm{cr}(G)$. A subset of vertices C is a *crossing cover* if $\mathrm{cr}(G - C)$ is zero.

A graph is *planar* if it has a drawing in the plane without crossings. A *plane graph* is a planar graph together with a drawing without crossings. In a drawing in the plane there is exactly one unbounded face. It is called the *outer face*.

The following well known proposition is an easy consequence of Euler's Formula.

Proposition 2.1. *If $G = (V, E)$ is planar, then $|E| \leq 3|V| - 6$.*

If all faces of a drawing of a graph G are homeomorphic to an open disk we call it a *2-cell embedding*.

2.4 Graph coloring

Graph coloring is a popular topic in combinatorics. There are numerous variants of colorings and various related problems. We only briefly review concepts used in this thesis. An interested reader might want to look in books by Chartrand and Zhang [6], Wilson [50] or Jensen and Toft [29] to discover more about graph coloring.

Let G be a graph and C be a set of colors. A *coloring* is a mapping $\varphi : V(G) \to C$ such that for every $uv \in E(G)$ holds $\varphi(u) \neq \varphi(v)$. Sometimes, it is called a *proper* coloring. Usually, we are interested only in proper colorings and minimizing the number of colors used. If a graph G can be colored using k colors we say that G is k-colorable. The smallest k for which G is k-colorable is called the chromatic number and it is denoted by $\chi(G)$. A graph G is k-critical if $\chi(G) = k$ but every proper subgraph of G is $(k-1)$-colorable. See Figure 2.2(a) for an example of a 3-colored graph.

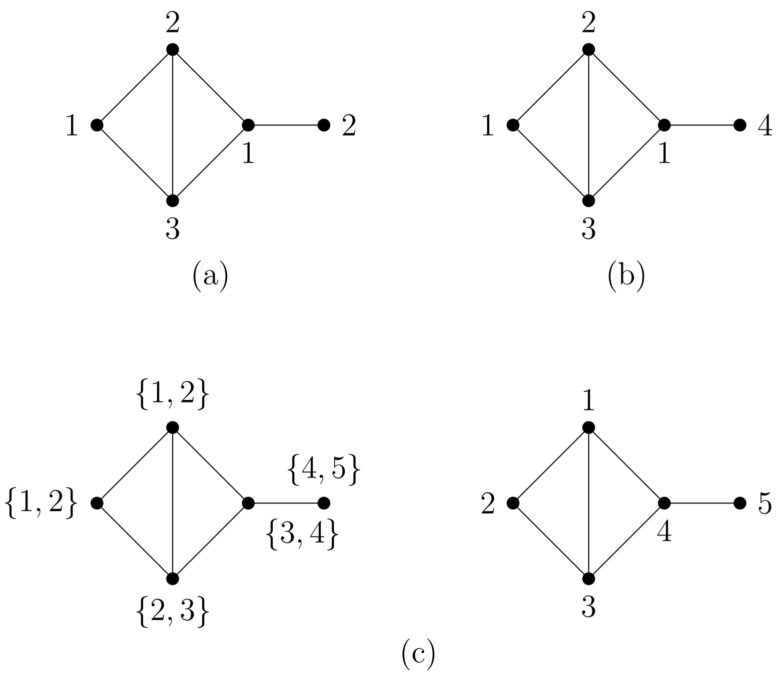


Figure 2.2: Examples of (a) a coloring, (b) a packing coloring and (c) a list coloring.

2.4.1 List coloring

List coloring is a generalization of coloring where each vertex has its own list of available colors. The concept of list colorings and choosability was introduced by Vizing [47] and independently by Erdős et al. [16].

A *list assignment* of G is a function L that assigns each vertex $v \in V(G)$ a list $L(v)$ of colors. An *L-coloring* is a mapping $\varphi : V(G) \to \bigcup_v L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent vertices of G. If G admits an L-coloring then it is *L-colorable.* A graph G is *k-choosable* if, for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, there exists an L-coloring of G. The *choosability* of a graph G denoted by $\mathrm{ch}(G)$ is the smallest k such that G is k-choosable. See Figure 2.2(c) for an example of a graph with a list L assignment and a possible L-coloring.

2.4.2 Packing coloring

The packing coloring is a recent concept coming from frequency assignment. It is based on the fact that waves with lower frequency have longer reach.

A graph coloring can be viewed as a partitioning of the vertex set of a graph G into disjoint classes $X_1, \ldots, X_k$ (representing one color each) such that X_i is an independent set.

In this model, we also ask for a partition of the vertex set of a graph G into disjoint classes $X_1, \ldots, X_k$ but the classes are bit more restricted. Each color class X_i should be an *i-packing*, that is, a set of vertices with the property that any distinct pair $u, v \in X_i$ satisfies $\mathrm{dist}(u, v) > i$. Here $\mathrm{dist}(u, v)$ denotes the usual shortest path distance between u and v.

See Figure 2.2(b) for an example of a graph with packing chromatic number four.

Such partition is called a *packing k-coloring*, even though it is allowed that some sets X_i may be empty. The smallest integer k for which there exists a packing k-coloring of G is called the *packing chromatic number* of G and it is denoted by $\chi_\rho(G)$. This concept was introduced by Goddard et al. [25] under the name *broadcast chromatic number.* The term packing chromatic number was later (even if the corresponding paper was published earlier) proposed by Brešar et al. [5].

Sloper [41] followed with a closely related concept, the eccentric coloring. An *eccentric coloring* of a graph is a packing coloring in which a vertex v is colored with a color not exceeding the eccentricity of v.

Chapter 3

Graph coloring techniques

In this chapter we describe a few basic methods of graph coloring. They are used in the following chapters and it might help the reader to see them in action on simple problems first.

3.1 The Four color theorem

The Four color theorem states that every planar graph can be colored by four colors. It is the most famous result of graph coloring and also the motivation for the whole area of graph coloring. It can be used to derive numerous other results.

As an example we show that outerplanar graphs are 3-colorable. A graph is *outerplanar* if it has an embedding in the plane such that all its vertices are incident with the outer face.

Theorem 3.1. *Let G be an outerplanar graph. Then $\chi(G) \leq 3$.*

Proof. Let G' be obtained from G by adding a new v vertex adjacent to every vertex of G. Such vertex adjacent to all other vertices is called an *apex*. Observe that G' is planar as v might be added to the face incident with all vertices of G, see Figure 3.1.

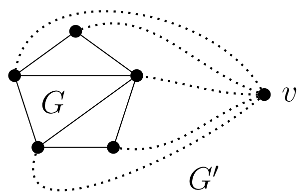


Figure 3.1: Adding an apex vertex v.

Now we use the four color theorem and obtain a coloring $\varphi : V(G') \rightarrow \{1, 2, 3, 4\}$. Assume without loss of generality that $\varphi(v) = 4$. As v is adjacent to all other vertices φ restricted to $V(G)$ uses only colors $\{1, 2, 3\}$. Hence φ is the desired coloring of G using three colors. □

3.2 Degeneracy

Degeneracy is a popular way of attacking not only coloring problems. A graph G is *k-degenerated* if every subgraph of G contains a vertex of degree at most k. In particular, G itself contains a vertex of degree at most k.

An example of k-degenerated graphs are planar graphs, which are 5-degenerated, trees, which are 1-degenerated or outerplanar graphs, which are 2-degenerated.

Using degeneracy we show that outer planar graphs are 3-choosable.

Theorem 3.2. *Let G be an outerplanar graph. Then* $\mathrm{ch}(G) \leq 3$.

Proof. Suppose, by way of contradiction, that G is an outerplanar graph and L is a list assignment assigning at least three colors to each vertex of G and G is not L colorable. Moreover, suppose that G is minimal such graph in number of vertices.

As outerplanar graphs are 2-degenerated, G contains a 2-vertex v. Let H be an induced subgraph on $V(G) \setminus v$. There exists an L-coloring $\varphi : u \to L(u)$ for every vertex $u \in V(H)$ since G was a minimal counterexample.

Let $v_1, v_2 \in V(G)$ be the two distinct neighbors of v. We extend φ to v by assigning $\varphi(v) := L(v) \setminus \{\varphi(v_1), \varphi(v_1)\}$. Such assignment exists as $|L(v)| \geq 3$. The extended φ contradicts that G is no L-colorable. □

3.3 Kempe chains

Kempe chains were discovered as an attempt to prove the Four color theorem. Let G be a properly colored graph and let H be some maximal connected 2-colored subgraph. The key observation is that it is possible to exchange colors of vertices of H and the resulting coloring of G is again proper. More formally, let G be a graph, φ its proper coloring and c_1 and c_2 two colors. Let H be a maximal (in inclusion) connected subgraph of G such that for every $v \in V(H)$ holds $\varphi(v) \in \{c_1, c_2\}$. Define a coloring ϱ for every vertex $v \in V(G)$ in the following way:

$$\varrho(v) := \begin{cases} c_1 & \text{if } v \in V(H) \text{ and } \varphi(v) = c_2\,, \\ c_2 & \text{if } v \in V(H) \text{ and } \varphi(v) = c_1\,, \\ \varphi(v) & \text{otherwise.} \end{cases}$$

Due to maximality of H, ϱ is a proper coloring. We call H a *Kempe chain*. See Figure 3.2 for an example.

A classical example for Kempe chains is that planar graphs are 5-colorable. First, we prove a simple lemma which will help even in other examples.

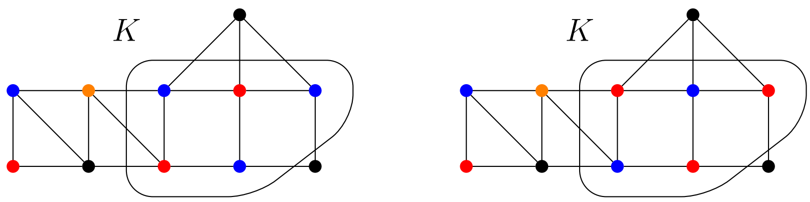


Figure 3.2: Using Kempe K of red and blue vertices.

Lemma 3.3. *Let G be a graph and v its vertex of degree at most four. If the graph induced on $V(G) \setminus s$ is 5-colorable then so is G.*

Proof. Let H be an induced subgraph of G on vertices $V(G) \setminus \{v\}$. As H is 5-colorable there exists a proper coloring $\varphi : V(H) \to C$ where C is a set of five colors.

Let $c \in C \setminus \bigcup_{u \in N(v)} \varphi(u)$. Note that c exists as v has at most four neighbors. Extending φ by defining $\varphi(v) := c$ yields a proper coloring of G. □

Theorem 3.4. *Let G be a planar graph. Then $\chi(G) \leq 5$.*

Proof. Suppose for contradiction that there exists a counterexample and G is the smallest one in the number of vertices.

Let $\tilde{G}$ be an embedding of G in the plane and let v be a (≤ 5)-vertex of G. Recall that v exists as planar graphs are 5-degenerated. Moreover, Lemma 3.3 implies that v is a 5-vertex.

Let H be an induced subgraph of G on $V(G) \setminus \{v\}$. As H is smaller than G, there exists a proper coloring $\varphi : V(H) \to C$ where $C = \{1, 2, 3, 4, 5\}$.

Our goal is to modify φ and extend it to a proper coloring of G. Let C_v be the set of colors assigned to neighbors of v, formally $C_v = \{c : \exists u \in N(v)\ \varphi(u) = c\}$. If there exists $c \in C \setminus C_v$ then assigning $\varphi(v) := c$ extends φ to a proper coloring of G. Hence all five neighbors of v have different colors. Assume without loss of generality that v_1, v_2, v_3, v_4 and v_5 are the neighbors of v in the clockwise order from v in $\tilde{G}$. Assume also that $\varphi(v_i) = i$ for $i \in \{1, 2, 3, 4, 5\}$. See Figure 3.3.

Finally, Kempe chains come into the scene. Let K be a Kempe chain of colors 2 and 5 containing vertex v_2. If K does not contain v_5 we exchange colors on K and the resulting coloring ϱ can be extended to a coloring of G by assigning $\varrho(v) := 2$. Note that $\varrho(v_2) = 5$ and $\varrho(v_i) = \varphi(v_i)$ for $i \in \{1, 3, 4, 5\}$. Hence ϱ is a proper coloring of G.

So we assume that K contains both v_2 and v_5. Switching colors on K does not help in this case as the neighborhood of v would still contain all five colors. The existence of K implies that there exists a path of vertices colored 2 and 5 connecting v_2 and v_5. Together with planarity it implies that there does not exist a path of vertices colored 1 and 4 connecting v_1 and v_4. See Figure 3.3. Hence we take a Kempe chain L on colores 1 and 4 containing v_1 and flip colors on L. The resulting coloring ϱ can be extended to a proper coloring of G by assigning $\varrho(v) := 1$. □

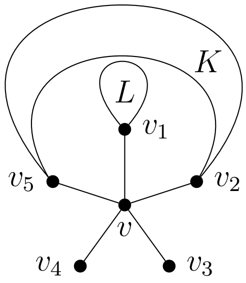


Figure 3.3: Configuration around v in Theorem 3.4.

3.4 Identification

Sometimes it might be helpful if two vertices are colored the same. Identification is a way of achieving this. It might also be called a contraction.

We demonstrate the technique on an alternative proof of Theorem 3.4.

Proof. Suppose for contradiction that there exists a counterexample and G is the smallest one in the number of vertices. Let $C = \{1, 2, 3, 4, 5\}$ be colors.

Recall that planar graphs are 5-degenerated. Hence G contains a vertex v of degree at most five. Lemma 3.3 implies that v is a 5-vertex.

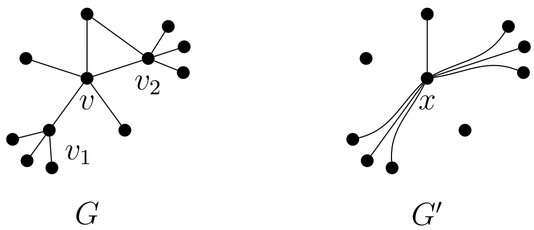


Figure 3.4: Removing vertex v and identifying v_1 and v_2.

Observe that $N(v)$ does not induce a clique as K_5 is not a planar graph. Hence there exist nonadjacent $v_1, v_2 \in N(v)$. Let G' be obtained from G by removing vertices v, v_1 and v_2 and by adding a new vertex x adjacent to $N(v_1) \cup N(v_2) \setminus \{v, v_1, v_2\}$, see Figure 3.4. Note that G' is planar as it can alternatively be obtained from G by contracting edges v_1v and v_2v and maybe removing some other edges and contracting and removing edges preserves planarity.

Let $\varphi : V(G') \to C$ be a coloring of G'. It exists as G has the minimum number of vertices. Let $C' = \bigcup_{u \in N(v) \cup \{x\} \setminus \{v_1, v_2\}} \varphi(u)$. Note that $|C'| \leq 4$ as the union is over four vertices. Let $c \in C \setminus C'$. Let $\varrho : V(G) \to C$ be defined in the following way:

$$\varrho(u) := \begin{cases} \varphi(x) & \text{if } u \in \{v_1, v_2\}\,, \\ c & \text{if } u = v\,, \\ \varphi(u) & \text{otherwise.} \end{cases}$$

Observe that ϱ is a proper coloring of G contradicting that G is not 5-colorable.

□

3.5 Discharging

Discharging is a very powerful technique for proving various theorems about planar graphs. It was successfully applied on various problems. Even the Four color theorem was proved by using discharging. A discharging proof might be very ugly but you will get the result. Especially theorems with some local constraints are likely to be proved using discharging.

Discharging proofs are usually by contradiction. We start with a minimum counterexample G. First, we study some *reducible* configurations which cannot occur in a minimum counterexample. Then *initial charges* are assigned to vertices and faces of G. A crucial property of the charges should be that a planar graph has sum of all the charges negative. Then we apply some rules for shifting the charges between vertices and faces such that the sum of all charges is preserved. Finally, we verify that the final charges of every face and every vertex are nonnegative. Hence G is not a planar graph, which is a contradiction.

Sometimes, the set of reducible configurations is called a set of *unavoidable configurations* as every graph satisfying the assumptions of a proven theorem must contain a reducible configuration.

We demonstrate the technique on yet another proof of Theorem 3.4.

Proof. Let G be a counterexample with the minimum number of vertices. We may assume that G is connected as otherwise we may color each connected component separately.

First we argue that G does not contain any separating triangle. Let T be a triangle and let G_1 and G_2 be induced subgraphs of G such that $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, $V(T) = V(G_1) \cap V(G_2)$, $|V(G_1)| > 3$ and $|V(G_2)| > 3$. By the minimality of G let φ_i be a 5-coloring of G_i for $i \in \{1, 2\}$. By permuting the colors we may assume that $\varphi_1 = \varphi_2$ on vertices of T. Finally, we define a 5-coloring φ of G:

$$\varphi(u) := \begin{cases} \varphi_1(u) & \text{if } u \in V(G_1)\,, \\ \varphi_2(u) & \text{if } u \in V(G_2). \end{cases}$$

Reducible configurations

Simple reducible configurations are vertices of degree at most four. They are reducible due to Lemma 3.3.

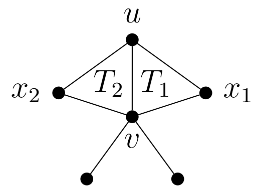


Figure 3.5: A reducible configuration from Section 3.5.

Next reducible configuration we use is a 5-vertex v incident with two triangles T_1, T_2 sharing an edge containing v, see Figure 3.5. To reduce this configuration we use the identification trick from Section 3.4. Let u be the other vertex shared by the triangles and x_i be the third vertex of T_i for $i \in \{1, 2\}$. Note that x_1 is not adjacent to x_2 as otherwise x_1x_2v would be a separating triangle. Let G' be obtained from G by removing v and identification of x_1 and x_2 into a new vertex x. As G has the minimum number of vertices there exists a proper 5-coloring $\varphi' : V(G') \to C$. It may be transformed into a 5-coloring φ of G in the same way as in the proof in Section 3.4.

Initial charges

Let $\tilde{G}$ be an embedding of G in the plane and let f be a face of $\tilde{G}$. By $\ell(f)$ we define the length of a facial cycle around f where bridge edges are counted twice. We define the initial charges ch for a vertex v and a face f in the following way:

$$\mathrm{ch}(v) := \deg(v) - 6 \text{ and } \mathrm{ch}(f) := 2\ell(f) - 6.$$

Let us verify that the sum of all charges is negative by Euler's formula:

$$\begin{aligned}
\sum_{v \in V(G)} \mathrm{ch}(v) + \sum_{f \in F(G)} \mathrm{ch}(f) &= \sum_{v \in V(G)} (\deg(v) - 6) + \sum_{f \in F(G)} (2\ell(f) - 6) \\
&= (2|E(G)| - 6|V(G)|) + (4|E(G)| - 6|F(G)|) \\
&= 6(|E(G)| - |V(G)| - |F(G)|) \\
&= -12.
\end{aligned}$$

Discharging rules

We use only one discharging rule to redistribute the initial charge to increase charges of 5-vertices.

Rule 1. *A* (≥ 4)*-face sends charge* $\frac{1}{2}$ *to every adjacent 5-vertex.*

Final charges

We use $\mathrm{ch}^*(x)$ to denote the final charge of a vertex or face x. Now we must show that $\mathrm{ch}^*(x) \geq 0$ for every vertex and face x.

Let v be a vertex of $\tilde{G}$. If $\deg(v) \geq 6$ then $\mathrm{ch}(v) = \mathrm{ch}^*(v) \geq 0$. If $\deg(v) = 5$, then v is adjacent to at most two triangular faces. Otherwise we have a reducible configuration. Hence v is adjacent to at least three (≥ 4)-faces that are each sending charge $\frac{1}{2}$ to v according to Rule 1. Thus $\mathrm{ch}^*(v) = \mathrm{ch}(v) + \frac{3}{2} \geq \frac{1}{2}$.

Let f be a face of $\tilde{G}$. If f is a triangle then $\mathrm{ch}(f) = \mathrm{ch}^*(f) = 0$. Otherwise, Rule 1 may be applied to every vertex of f. Hence $\mathrm{ch}^*(v) = \mathrm{ch}(f) + \ell(f)/2 = \frac{3}{2}\ell(f) - 6 \geq 0$.

As the resulting charges are nonnegative for every vertex and every face, we get a contradiction with the assumption that G is planar.

□

3.6 Computer search

Using computer might save a lot of tedious case analysis when dealing with a large amount of cases. The disadvantage of computer assisted proofs is that computer programs depend on many sources of possible mistakes. Mistakes might come from source codes, compilers, operating systems and even hardware. A common way of handling this is to write two different programs in two different languages, let them run in different environments and verify that they were computing the same thing. Of course, it does not mean that the result is correct but lowers the probability of a mistake.

Apart from checking many cases, computer programs might be used to search through small configurations and it might give a hint what could be true.

As an example we use a simple result that the packing chromatic number of a two-way infinite path is four.

Theorem 3.5. *Let P_∞ be a two-way infinite path. Then $\chi_\rho(P_\infty) = 3$.*

Proof. First, we need to show that $\chi_\rho(P_\infty) > 2$. We may write a program for generating all colorings of P_4 by two colors and checking if they are proper packing colorings. After checking all 16 configurations, the program will output that P_4 cannot be packing colored by two colors. It implies that $\chi_\rho(P_\infty) > 2$.

A computer program might also be used for proving $\chi_\rho(P_\infty) \leq 3$. First possibility is to use a greedy kind of an algorithm for coloring a long path, say P_{30}. A human may easily find a repeating pattern in the result and prove that such pattern might be used for coloring the infinite path. Program may be used also for generating the pattern by trying to color a cycle. In this case C_4 is sufficient. Even the correct length of the cycle and hence the period of the pattern might be found by a computer. Just for completeness we present also a repeating pattern for coloring P_∞ in Figure 3.6.

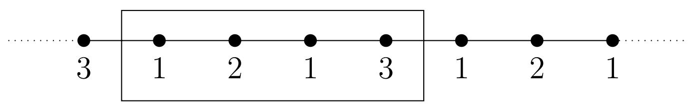


Figure 3.6: Pattern for coloring a two-way infinite path.

□

3.7 Thomassen's technique

The last technique we describe in this chapter was developed by Thomassen. It was used for proving a beautiful theorem that every planar graph is 5-choosable which we present here.

The main idea is to use induction on a slightly stronger statement where vertices in the outer face have less colors in their lists and peel vertices from the outer face.

Theorem 3.6. *Let G be a plane graph and let L be a list assignment. If there are at most two adjacent vertices a and b in the outer face with lists of size at least one such that $|L(a) \cup L(b)| \geq 2$, all other vertices in the outer face have lists of size at least three and all the other vertices have lists of size five then G is L-colorable.*

Proof. Suppose for contradiction that G is a counterexample with the minimum number of vertices and L is the corresponding list assignment.

Let us first note that a and b are called *precolored* vertices as there is no freedom in coloring them.

If G is not connected we may color each component separately by induction. Hence we may assume that G is connected.

Suppose that G contains a cut-vertex v. We split G into two proper induced subgraphs G_1 and G_2 of G such that $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = \{v\}$ and $E(G_1) \cup E(G_2) = E(G)$. Without loss of generality let $a, b \in V(G_1)$. First, we get from induction an L-coloring φ_1 of G_1. Let L' be obtained by restricting L to $V(G_2)$ and assigning $L'(v) := \varphi_1(v)$. Note that G_2 and L' satisfy the assumptions of the theorem and hence we can use induction to obtain L'-coloring φ_2 of G_2. As both of the colorings agree on v we define the final L-coloring φ by combining φ_1 and φ_2. Hence G is 2-connected and the outer face of G is a cycle.

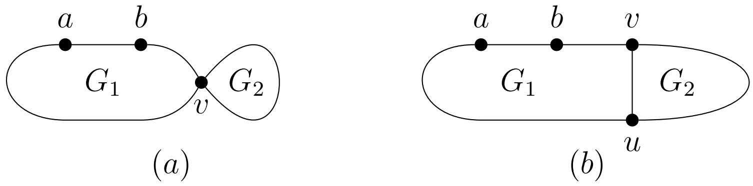


Figure 3.7: A cut and a chord from Theorem 3.6.

Next we show that the outer face of G does not contain any chord. The argument is analogous to the one in the previous paragraph. Suppose that there exists a chord $uv \in E(G)$ where both u, v are in the outer face but the edge uv is not. Then $X = \{u, v\}$ is a 2-cut. Let G_1 and G_2 be proper induced subgraphs such that $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = X$ and $E(G_1) \cup E(G_2) = E(G)$. Assume without loss of generality that $a, b \in V(G_1)$. First, we get from induction an L-coloring φ_1 of G_1. Let L' be obtained by restricting L to $V(G_2)$ and assigning $L'(u) := \varphi_1(u)$ and $L'(v) := \varphi_1(v)$. Note that G_2 and L' satisfy the assumptions of the theorem and hence we can use induction to obtain L'-coloring φ_2 of G_2. As both of the colorings agree on u and v we define the final L-coloring φ by combining φ_1 and φ_2. Hence the outer face of G has no chords.

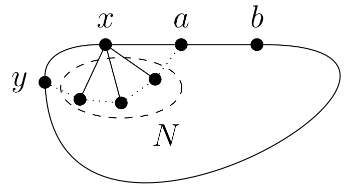


Figure 3.8: A 2-connected case in Theorem 3.6.

Assume without loss of generality that $|L(a)| = |L(b)| = 1$ and $L(a) \neq L(b)$ as we can restrict the lists if they are larger. Let b, a, x, y be a part of a facial walk along the outer face of G, see Figure 3.8. Note that it may happen that $b = y$. Let $N = N(x) \setminus \{a, y\}$ and let $C = L(x) \setminus L(a)$. Assume without loss of generality that $|C| = 2$ as we can just throw away some colors in case $|C| > 2$. Let G' be an induced subgraph of G on $V(G) \setminus \{x\}$. We define L' to be:

$$L'(u) := \begin{cases} L(u) \setminus C & \text{if } u \in N\,, \\ L(u) & \text{otherwise.} \end{cases}$$

Note that vertices of N have lists L of size five as they are not in the outer face of G. They lost at most two colors from their lists L' but they are in the outer face of G'. Hence G' and L' satisfy the assumptions of the theorem and by induction we obtain an L'-coloring φ of G'.

We extend φ to an L-coloring of G by assigning $\varphi(x) \in C \setminus \{\varphi(y)\}$. Note that the assignment gives a proper coloring as vertices in $N(x) \setminus \{y\}$ are not colored by colors from C. The L-coloring φ contradicts that G is not L-colorable. $\square$

Chapter 4

6-critical graphs

This chapter is based on papers *6-critical graphs on the Klein bottle* by Kawarabayashi, Kráľ, Kynčl and Lidický [31] and *5-colouring graphs with 4 crossings* by Erman, Havet, Lidický and Pangrác [17].

4.1 Introduction

This chapter is devoted to critical graph. Let us recall that a graph G is *k-critical* if $\chi(G) = k$ but for every proper subgraph H of G holds that $\chi(H) < k$. Critical graphs are in some sense the smallest obstacles for coloring a graph with less colors than k. Let G be a graph with $\chi(G) \geq k$. Then it contains a subgraph H, which is k-critical. It can be easily obtained from G by removing edges and vertices as long as the chromatic number stays at least k. If further removal of any vertex or edge causes chromatic number to drop below k, we have a k-critical subgraph.

The previous observation has the following algorithmic consequence. Let $\mathcal{G}$ be class of graphs containing only finitely many k-critical graphs. Then it can be decided in polynomial time if a graph $G \in \mathcal{G}$ can be colored by less than k colors simply by checking if G contains any of the finitely many k-critical graphs.

The question of k-colorability is $\mathcal{NP}$-complete in general [30]. So it is one of the motivations for deciding if a class of graphs has finitely many k-critical graphs and enumerating them. Also k-critical graphs come handy when one is proving that some graph may or may not be colored by $k-1$ colors as the k-critical graphs guarantee some structure.

Let us first review critical graphs on surfaces. A well-known result of Heawood [27] asserts that the chromatic number of a graph embedded on a surface of Euler genus g is bounded by the Heawood number $H(g) = \left\lfloor \frac{7+\sqrt{24g+1}}{2} \right\rfloor$. Dirac Map Color Theorem [9, 10] asserts that a graph G embedded on a surface of Euler genus $g \neq 0, 2$ is $(H(g) - 1)$-colorable unless G contains a complete graph of order $H(g)$ as a subgraph. Dirac's theorem in the language of critical graphs says the following: the only $H(g)$-critical graph that can be embedded on a surface of Euler genus $g \neq 0, 2$ is the complete graph of order $H(g)$.

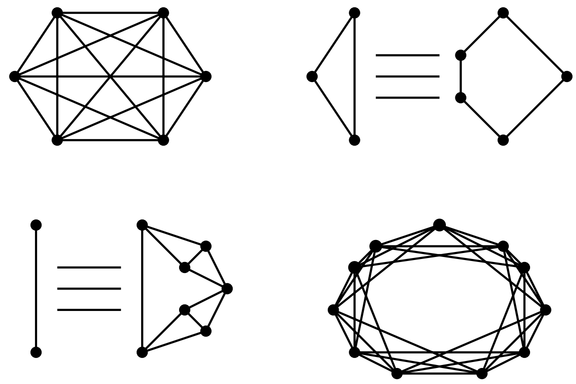


Figure 4.1: The list of all four 6-critical graphs that can be embedded on the torus. Some of the edges are only indicated in the figure: the straight edges between two parts represent that the graph is obtained as the join of the two parts and the vertices with stars of edges leaving them are adjacent to all vertices in the graph.

In fact, Dirac [9] showed that there are only finitely many k-critical graphs, $k \geq 8$, that can be embedded on a fixed surface. The number of 7-critical graphs that can be embedded on a fixed surface is also finite by classical results of Gallai [23, 24] as pointed out by Thomassen in [43]. Later, Thomassen [45] established that the number of 6-critical graphs that can be embedded on any fixed (orientable or non-orientable) surface is also finite (see also [39] for related results on 7-critical graphs). This result is best possible as there are infinitely many k-critical graphs, $3 \leq k \leq 5$, that can be embedded on any fixed surface different from the plane [21].

In this chapter, we focus on 6-critical graphs. First, we discuss 6-critical graphs on the Klein bottle and then we move to 6-cirtical graphs with small crossing number. As every plane graph is 4-colorable [2, 3, 38], there are no 6-critical graphs in the plane. Dirac Map Color Theorem implies that the complete graph of order six is the only 6-critical projective planar graph. Thomassen [43] gave a complete list of 6-critical toroidal graphs: the only 6-critical graphs that can be embedded on the torus are the complete graph K_6, the join of the cycles C_3 and C_5 (recall that the join of two graphs G_1 and G_2 is the graph obtained by adding all edges between G_1 and G_2), the graph obtained by applying Hajos' construction to two copies of K_4 and then by adding K_2 joined to all other vertices, and the third distance power of the cycle C_{11} (which is further denoted by T_{11}), see Figure 4.1.

Thomassen then posed a question whether the toroidal 6-critical graphs distinct from T_{11} and the graph obtained by applying Hajos's to two copies of K_6 are the only 6-critical graphs that can be embedded on the Klein bottle. It turned out that conjecture is false and that there are nine 6-critical graphs on the Klein bottle. We present the proof of this fact in Subsection 4.2.

Every graph with crossing number g can be embedded on a surface of Euler genus g simply by using cross caps instead of crossings. Hence some results for graphs on surfaces directly transfers to graphs with bounded crossing number. In particular, we know that there are finitely many k-critical graphs with bounded crossing number for $k \geq 6$. Note that the result cannot be improved to $k \geq 5$ as there are infinitely many 5-critical graphs with just one crossing. Consider a wheel W with an apex vertex. It is straightforward to verify that it is indeed a 5-critical graph with crossing number one.

There are no 6-critical graphs with crossing number at most two as graphs with crossing number two are embeddable to the Klein bottle and the 6-critical with the smallest crossing number embeddable in the Klein bottle, equal to 3, is K_6. Oporowski and Zhao [37] proved that K_6 is the unique 6-critical graph with crossing number 3. They posed a question if K_6 is unique also for crossing numbers 4 and 5. It is not unique for 6 crossings as the join of C_3 and C_5 has crossing number 6. The answer is positive only for 4 crossings. In Section 4.3 we give a proof that K_6 is the unique 6-critical graph with four crossings and exhibit a 6-critical graph with crossing number 5 different from K_6.

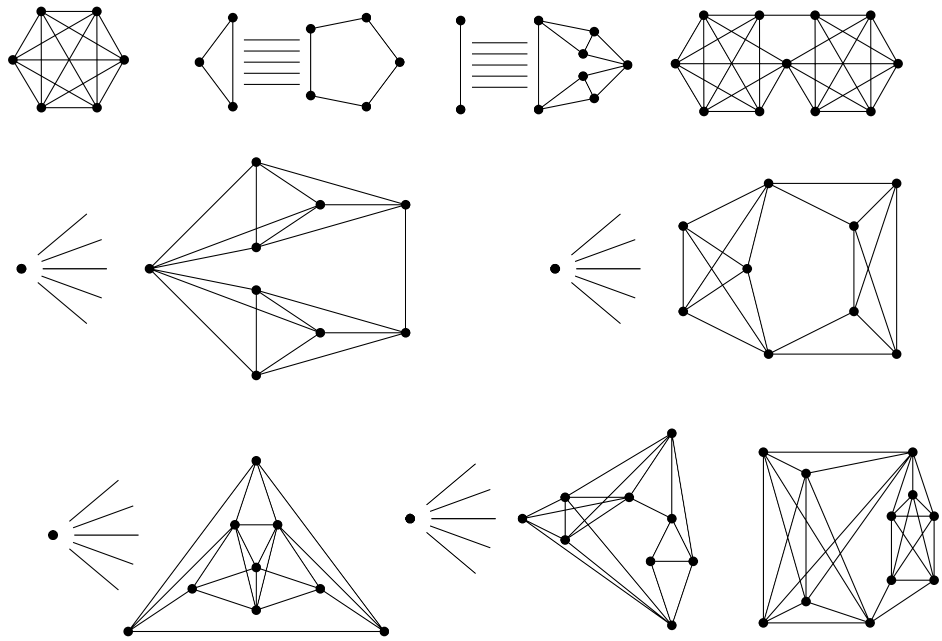


Figure 4.2: The list of all nine 6-critical graphs that can be embedded in the Klein bottle. Some of the edges are only indicated in the figure: the straight edges between two parts represent that the graph is obtained as the join of the two parts and the vertices with stars of edges leaving them are adjacent to all vertices in the graph.

4.2 6-critical graphs on the Klein bottle

This section is based on paper [31].

4.2.1 Introduction

This section is devoted to enumerating all 6-critical graphs that can be embedded on the Klein bottle (the graphs are depicted in Figure 4.2).

Theorem 4.1. *There are nine non-isomorphic* 6*-critical graphs that can be embedded on the Klein bottle which are depicted in Figure 4.2. The graphs have altogether a single non-2-cell embedding and* 18 *non-isomorphic* 2*-cell embeddings on the Klein bottle, which are depicted in Figures 4.5 and 4.6.*

It disproves a conjecture of Thomassen's who conjectured that there are only four them. The same result was independently obtained by Chenette, Postle, Streib, Thomas and Yerger [7]. Chenette et al. investigated the structure of 6-critical graphs on the Klein bottle, i.e., found all 6-critical graphs that could possibly be embedded on the Klein bottle. Our approach is based on a systematic generating of all embeddings of 6-critical graphs on the Klein bottle from the complete graph K_6. Our proof is computer-assisted (unlike the proof of Chenette et al.) but it additionally yields the list of all non-isomorphic embeddings of 6-critical graphs on the Klein bottle.

As we have mentioned, our proof is computer-assisted. We outline the main concepts we use and explain the procedure used to generate all embeddings of 6-critical graphs on the Klein bottle. In order to verify the correctness of our programs, we have separately prepared two different programs implementing our procedures and compared their outputs. Further details of the implementation and the source code of our programs can be found at `http://kam.mff.cuni.cz/~bernard/klein`. Here, we establish the correctness of used algorithms and refer the reader for details on implementation to the web page.

4.2.2 6-critical graphs

In this subsection, we observe basic properties of 6-critical graphs on the Klein bottle. Euler's formula implies that the average degree of a graph embedded on the Klein bottle is at most six. As Sasanuma [40] established that every 6-regular graph that can be embedded on the Klein bottle is 5-colorable, we have the following proposition (observe that no 6-critical graph contains a vertex of degree four or less):

Proposition 4.2. *The minimum degree of every* 6*-critical graph on the Klein bottle is five.*

Let G be a 6-critical graph on the Klein bottle and v a vertex of degree five in G. Further let v_i, $1 \leq i \leq 5$, be the neighbors of v in G. If all vertices v_i and v_j, $1 \leq i < j \leq 5$, are adjacent, the vertices v and v_i, $1 \leq i \leq 5$, form a clique of order six in G. As G is 6-critical, G must then be a complete graph of order six. Hence, we can conclude the following:

Proposition 4.3. *Let G be a* 6*-critical graph embedded on the Klein bottle. If G is not a complete graph of order six, then G contains a vertex v of degree five that has two non-adjacent neighbors v' and v''.*

We now introduce the following *reduction*: let G be a 6-critical graph embedded on the Klein bottle that is not isomorphic to K_6 and let v, v' and v'' be three vertices as in Proposition 4.3. $G|vv'v''$ is the graph obtained from G by removing all the edges incident with v except for vv' and vv'' and contracting the edges vv' and vv'' to a new vertex w. The obtained graph can have parallel edges but it does not have loops as the vertices v' and v'' are not adjacent. Observe that the graph $G|vv'v''$ is not 5-colorable: otherwise, consider a 5-coloring of $G|vv'v''$ and color the vertices v' and v'' with the color assigned to the vertex w. Next, extend the 5-coloring to v—this is possible since v has five neighbors and at least two of them (v' and v'') have the same color. Hence, we obtain a 5-coloring of G contradicting our assumption that G is 6-critical. Since $G|vv'v''$ has no 5-coloring, it contains a 6-critical subgraph—this subgraph will be denoted by $|G|vv'v''|$ and we say that G can be *reduced* to $|G|vv'v''|$.

Observe that the reduction operation can again be applied to $|G|vv'v''|$ until a graph that is isomorphic to K_6 is obtained (the process eventually terminates since the order of the graph is decreased in each step).

We continue with a simple observation on the graph $|G|vv'v''|$.

Proposition 4.4. *Let G be a 6-critical graph embedded on the Klein bottle, v a vertex of degree five in G and v' and v'' two non-adjacent neighbors of v. The graph $|G|vv'v''|$ contains the vertex w obtained by contracting the path $v'vv''$. Moreover, the vertex w has a neighbor w' in $|G|vv'v''|$ that is a neighbor of v' in G but not of v'' and it also has a neighbor w'' that is neighbor of v'' but not of v' in G.*

Proof. If $|G|vv'v''|$ does not contain the vertex w, then $|G|vv'v''|$ is a subgraph of $G \setminus \{v, v', v''\}$. Since both $|G|vv'v''|$ and G are 6-critical graphs, this is impossible. Hence, $|G|vv'v''|$ contains the vertex w.

Assume now that $|G|vv'v''|$ contains no vertex w' as described in the statement of the proposition, i.e., all neighbors of w in $|G|vv'v''|$ are neighbors of v'' in G. This implies that $|G|vv'v''|$ is isomorphic to a subgraph of $G \setminus \{v, v'\}$ (view the vertex v'' as w) which is impossible since both G and $|G|vv'v''|$ are 6-critical. A symmetric argument yields the existence of a vertex w''. □

The strategy of our proof is to generate all 6-critical graphs by reversing the reduction operation. More precisely, we choose a vertex w of a 6-critical graph G and partition the neighbors of w into two non-empty sets W_1 and W_2. We next replace the vertex w with a path w_1ww_2 and join the vertex w_i, $i = 1, 2$, to all vertices in the set W_i. Let $G[w, W_1, W_2]$ be the resulting graph. We say that $G[w, W_1, W_2]$ was obtained by *expanding* the graph G. By Proposition 4.4, the following holds (choose w as in the statement of the proposition):

Proposition 4.5. *Let G be a 6-critical graph embedded on the Klein bottle and let v be a vertex of degree five of G with two non-adjacent neighbors v' and v''. The graph $G' = |G|vv'v''|$ contains a vertex w such that $G'[w, W_1, W_2] \subseteq G$ for some partition W_1 and W_2 of the neighbors of the vertex w.*

4.2.3 Minimal graphs

Our plan is to generate all 6-critical graphs from the complete graph K_6 by expansions and insertions of new graphs into faces. In this subsection, we describe the graphs we have to insert into the faces to be sure that we have generated all 6-critical graphs.

A plane graph G with the outer face bounded by a cycle C of length k is said to be *k-minimal* if for every edge $e \in E(G) \setminus C$, there exists a proper precoloring φ_e of C with five colors that cannot be extended to G and that can be extended to a proper 5-coloring of $G \setminus e$ (the graph G with the edge e removed). Note that the precolorings φ_e can differ for various choices of e.

The cycle C_k of length k is k-minimal (the definition vacuously holds); we say that C_k is a *trivial* k-minimal graph. For $k = 3$, it is easy to observe that C_3 is the only 3-minimal graph since the colors of the vertices of C_3 must differ and every planar graph is 5-colorable. Similarly, C_4 and the graph obtained from C_4 by adding a chord are the only 4-minimal graphs. As for $k = 5$, Thomassen [43] showed that if G is a plane graph with the outer face bounded by a cycle C of length five and C is chordless, then a precoloring of C with five colors can be

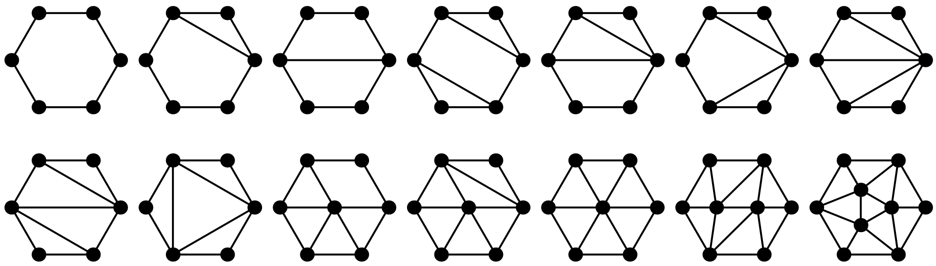


Figure 4.3: The list of all 6-minimal graphs.

extended to G unless G is the 5-wheel and the vertices of C are precolored with all five colors. Hence, C_5, C_5 with one chord, C_5 with two chords and the 5-wheel are the only 5-minimal graphs. The analogous classification result of Thomassen [43] implies that the only 6-minimal graphs (up to an isomorphism) are those depicted in Figure 4.3.

The following lemma justifies the use of k-minimal graphs in our considerations:

Lemma 4.6. *Let G be a 6-critical graph embedded on the Klein bottle. If C is a contractible cycle of G of length k, then the subgraph G' of G inside the cycle C (G' includes the cycle C itself) is k-minimal.*

Proof. We verify that G' is k-minimal. Let e be an edge of G' that is not contained in C. Since G is 6-critical, $G \setminus e$ has a 5-coloring c. Let φ_e be the coloring c restricted to the cycle C. If φ_e could be extended to G', then the extension of φ_e to the subgraph G' combined with the coloring c outside C would yield a 5-coloring G. This establishes the existence of a precoloring φ_e as in the definition of k-minimal graphs and the proof of the lemma is now finished. □

In the light of Lemma 4.6, our next goal is to find all k-minimal graphs for small values of k. The following proposition enables us to systematically generate all k-minimal graphs for any fixed k from the lists of k'-minimal graphs for $3 \leq k' < k$.

Proposition 4.7. *If G is a non-trivial k-minimal graph, $k \geq 3$, with the outer cycle C, then either the cycle C contains a chord or G contains a vertex v adjacent to at least three vertices of the cycle C. In addition, if C' is a cycle of G of length k' and G' is the subgraph of G bounded by the cycle C' (inclusively), then G' is a k'-minimal graph.*

Proof. First assume that C is chordless and each vertex v of G is adjacent to at most two vertices of C. Let G' be the subgraph of G induced by the vertices not lying on C. We consider the following list coloring problem: each vertex of G' not incident with the outer face receives a list of all five available colors and each vertex incident with the outer face is given a list of the colors distinct from the colors assigned to its neighbors on C in G. By our assumption, each such vertex has a list of at least three colors. A classical list coloring result of Thomassen [42] on list 5-colorings of planar graphs yields that G' has a coloring from the constructed lists. Hence, every precoloring of the boundary of G can be extended to the whole graph G and thus G cannot be k-minimal. This establishes the first part of the proposition. The proof of the fact that every cycle of length k' bounds a k'-minimal subgraph is very analogous to that of Lemma 4.6 and omitted. □

Proposition 4.7 suggests the following algorithm for generating k-minimal graphs. Assume that we have already generated all ℓ-minimal graphs for $\ell < k$ and let M_ℓ be the list of all ℓ-minimal graphs. Note that we have explicitly described the lists M_3, M_4, M_5 and M_6. The list M_k is then generated by the following procedure (the vertices of outer boundary are denoted by $v_1, \ldots, v_k$):

```
M_k := { the cycle C_k on v_1,...,v_k }
repeat
  M' := M_k
  forall 1 <= a < b <= k with b-a >= 2 do
    G := the cycle C_k on v_1,...,v_k with the chord v_av_b
    forall G_1 in M_{b-a+1} and G_2 in M_{k+a-b+1} do
      H := G with G_1 and G_2 pasted into its faces
      if H is k-minimal and H is not in M_k then
        add H to M_k
    endfor
  endfor
  forall 1 <= a < b < c <= k do
    G := the cycle C_k on v_1,...,v_k with the vertex v
         adjacent to v_a, v_b and v_c
    forall G_1 in M_{b-a+2}, G_2 in M_{c-b+2} and
           G_3 in M_{k+a-c+2} do
      H := G with G_1, G_2 and G_3 pasted into its faces
      if H is k-minimal and H is not in M_k then
        add H to M_k
    endfor
  endfor
until M_k = M'
```

Proposition 4.7 implies that the list M_k contains all k-minimal graphs after the termination of the procedure: if G is a k-minimal graph, it contains either a chord or a vertex v adjacent to three vertices on the outer cycle and the graphs inside the faces of the skeleton formed by the outer cycle and the chord / the edges adjacent to v are also minimal. The verifications whether the graph G is isomorphic to one of the graphs in M_k and whether G is k-minimal are straightforward and the reader can find the details in the program available at `http://kam.mff.cuni.cz/~bernard/klein`.

The numbers of non-isomorphic k-minimal graphs for $3 \leq k \leq 10$ can be found in Table 4.1. We finish this subsection by justifying our approach with showing that the number of k-minimal graphs is finite for every k; in particular, the procedure always terminates for each value of k.

Proposition 4.8. *The number of k-minimal graphs is finite for every $k \geq 3$.*

k	3	4	5	6	7	8	9	10
$\lvert M_k\rvert$	1	2	4	14	46	291	2124	19876
n_k	0	0	1	3	4	6	7	9

Table 4.1: The numbers of non-isomorphic k-minimal graphs for $3 \le k \le 10$ and the largest number n_k of internal vertices of a k-critical graph.

Proof. Let A_k be the number of k-minimal graphs and $A_{k,\ell}$ the number of k-minimal graphs G such that exactly ℓ precolorings of the boundary of G with five colors can be extended to G. Clearly, $A_{k,\ell} = 0$ for $\ell > 5 \cdot 4^{k-1}$ since there are at most $5 \cdot 4^{k-1}$ proper precolorings of the boundary of G. We prove that the numbers $A_{k,\ell}$ are finite by the induction on $5^k + \ell$. More precisely, we establish the following formula:

$$\begin{aligned} A_{k,\ell} \quad \le \quad & k \cdot \sum_{i=3}^{k-1} 4i(k+2-i)A_iA_{k+2-i} + && (4.1)\\ & k \cdot \sum_{i=4}^{k-1} \sum_{i'=4}^{k+3-i} 8ii'(k+6-i-i')A_iA_{i'}A_{k+6-i-i'} + && (4.2)\\ & k \sum_{i=1}^{\ell-1} 2kA_{k,i} && (4.3) \end{aligned}$$

Fix k and ℓ. By Proposition 4.7, every k-minimal graph G with ℓ extendable precolorings of its boundary cycle C either contains a chord or a vertex v adjacent to three vertices on C. In the former case, the cycle C and the chord forms cycles of length i and $k+2-i$. Since these cycles bound i-minimal and $(k+2-i)$-minimal graphs by Proposition 4.7, the number of such k-minimal graphs is at most A_iA_{k+2-i}. After considering at most k possible choices of the chord (for fixed i) and $2i$ and $2(k+2-i)$ possible rotations and/or reflections, we obtain the term (4.1).

Let us analyze the case that G contains a vertex v adjacent to three vertices on C. If the neighbors of v are not three consecutive vertices of C, then the edges between v and its neighbors delimit cycles of lengths $i \ge 4$, $i' \ge 4$ and $k+6-i-i'$. These cycles bound i-minimal, i'-minimal and $(k+6-i-i')$-minimal graphs and their number (including different rotations and reflections) is estimated by the term (4.2).

Assume that the neighbors of v on C are consecutive. Let v', v'' and v''' be the neighbors of v and G' the subgraph of G inside the cycle C' where C' is the cycle C with the path $v'v''v'''$ replaced with the path $v'vv'''$ (see Figure 4.4). Fix a precoloring φ_0 of the vertices of C except for v''. Let α be the number of ways in which φ_0 can be extended to v that also extends to G'. Similarly, α' the number of ways in which φ_0 can be extended to v'' that also extends to G.

We show that $\alpha \le \alpha'$. If $\alpha = 0$, then $\alpha' = 0$. If $\alpha = 1$, then $\alpha' > 1$. Finally, if $\alpha > 1$, then $\alpha \le \alpha'$ as any extension of φ_0 to C also extends to G (note that α' is 3 or 4 depending on $\varphi_0(v')$ and $\varphi_0(v''')$). We conclude that the number of precolorings of C' that can be extended to G' does not exceed the number of

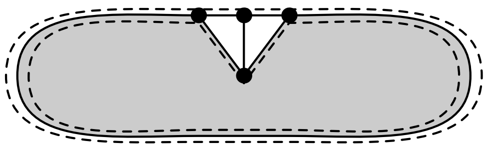


Figure 4.4: The notation used in the proof of Proposition 4.4.

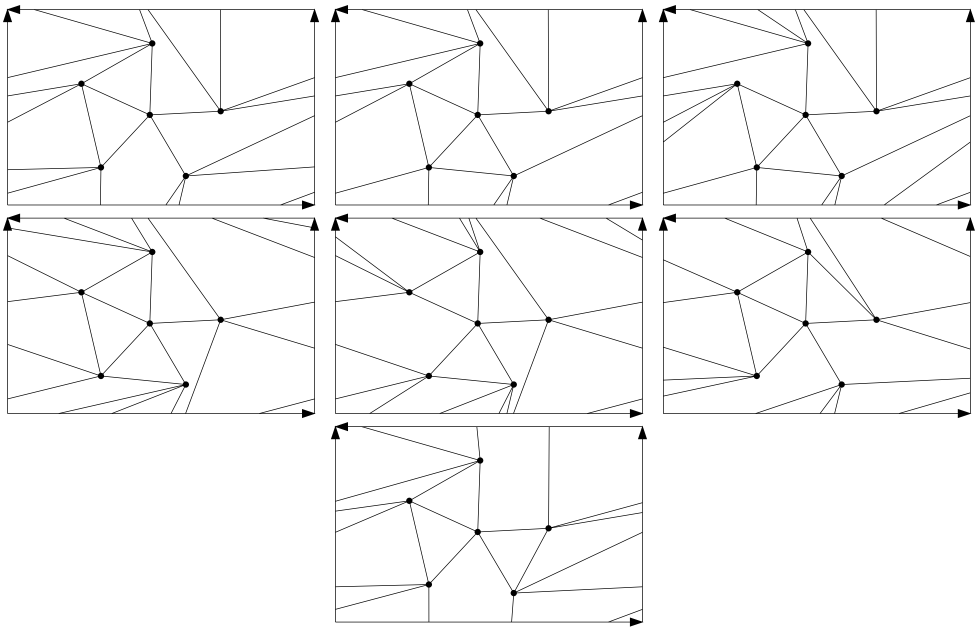


Figure 4.5: The list of all seven non-isomorphic 2-cell embeddings of K_6 on the Klein bottle.

precolorings of C extendable to G.

Let φ be the precoloring of C that cannot be extended to G but that can be extended to $G \setminus vv''$ and let φ_0 be the restriction of φ to $C \setminus v''$. It is easy to infer that the value of α for this particular precoloring φ_0 must be equal to one and consequently $\alpha' > 1$ for φ_0. Hence, the number of precolorings of C' that can be extended to G' is strictly smaller than the number of precolorings of C that can be extended to G. Since G' is a k-minimal graph with fewer precolorings of the boundary that can be extended to G' than the number of precolorings of C extendable to G, the number of k-minimal graphs G with a vertex v with three consecutive neighbors on C including their possible rotations and reflections is estimated by (4.3). This finishes the proof of the inequality and thus the proof of the whole proposition. $\square$

4.2.4 Embeddings of K_6 on the Klein bottle

Subsequent applications of our reduction procedure to a 6-critical graph on the Klein bottle eventually lead to an embedding of the complete graph K_6. The resulting embedding of K_6 is either a 2-cell embedding or not. Recall that an embedding is said to be 2-*cell* if every face is homeomorphic to a disc.

If the resulting embedding of K_6 is not 2-cell, the embedding must be isomorphic to the embedding obtained from the unique embedding of K_6 in the projective plane by inserting a cross-cap into one of its faces. Otherwise, the embedding is isomorphic to one of the seven embeddings of K_6 depicted in Figure 4.5. All 2-cell embeddings of K_6 on the Klein bottle can be easily generated by a simple program that ranges through all 2-cell embeddings of K_6 on surfaces: for each vertex v of K_6, the program generates all cyclic permutations of the other vertices (corresponding to the order in which the vertices appear around v) and chooses which edges alter the orientation. Each such pair of cyclic permutations and alterations of orientations determines uniquely both the embedding and the surface. It is straightforward to compute the genus of the surface and test whether the constructed embedding is not isomorphic to one of the previously found embeddings. The source code of the program can be found at `http://kam.mff.cuni.cz/~bernard/klein`.

4.2.5 Expansions of 2-cell embeddings of K_6

In this subsection, we focus on embeddings of 6-critical graphs that can be reduced to a 2-cell embedding of K_6. All such 6-critical graphs can easily be generated, using the expansion operation and Lemma 4.6, by the following procedure:

```
G_1, G_2, G_3, G_4, G_5, G_6, G_7 :=
  non-isomorphic embeddings of K_6 on the Klein bottle
k := 7
i := 1
while i <= k do
  for all vertices w of G_i do
    for all partitions of N(w) into W_1 and W_2 do
       H_0 := G[w,W_1,W_2]
       for all H obtained from H_0 by pasting
                 minimal graphs into its faces do
         if H is not isomorphic to any of G_1, ..., G_k then
           k := k + 1; G_k := H
         endfor
      endfor
    endfor
  i := i + 1
  done { while }
output G_1, ..., G_k
```

The source code of the program implementing the above procedure can be found at `http://kam.mff.cuni.cz/~bernard/klein`. The program eventually terminates outputting 11 embeddings of 6-critical graphs on the Klein bottle, which are depicted in Figure 4.6, in addition to the seven 2-cell embeddings of K_6. Hence, Proposition 4.5 and Lemma 4.6 now yield:

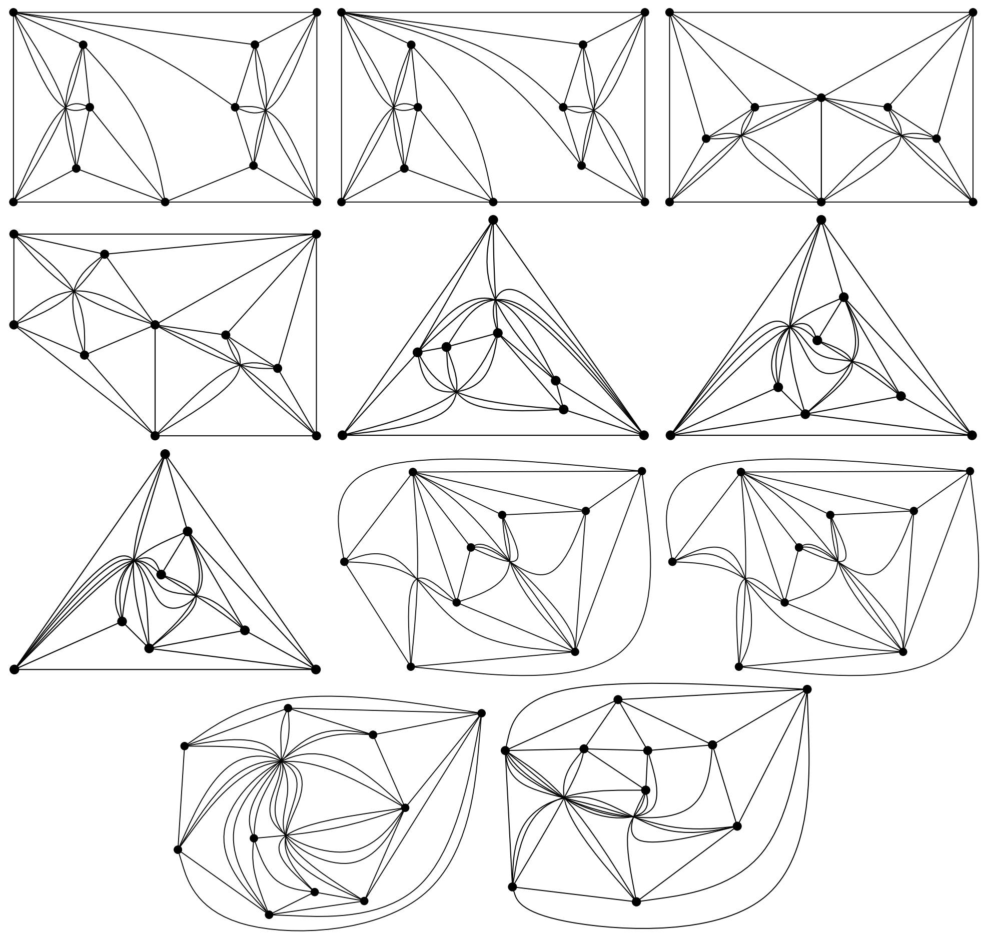


Figure 4.6: The list of 11 non-isomorphic embeddings of 6-critical graphs on the Klein bottle that are distinct from K_6. The graphs are drawn in the plane with two cross-caps.

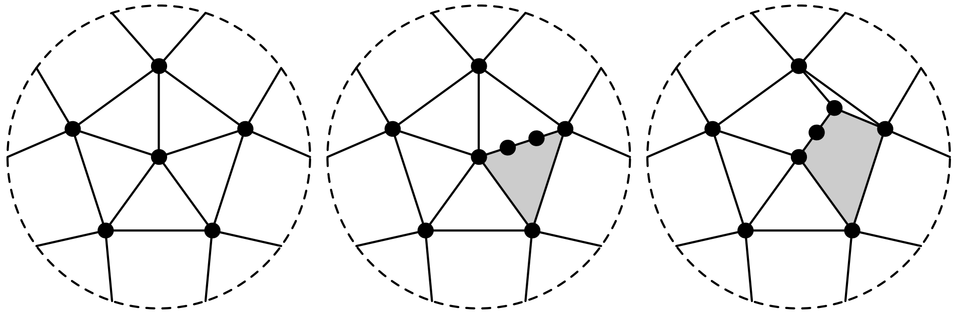


Figure 4.7: The unique embedding of K_6 in the projective plane and its two possible expansions.

Lemma 4.9. *Let G be an embedding of a 6-critical graph on the Klein bottle that is distinct from K_6. If G can sequentially be reduced to a 2-cell embedding of K_6 on the Klein bottle, then G is isomorphic to one of the eleven embeddings depicted in Figure 4.6.*

4.2.6 Expansions of non-2-cell embedding of K_6

As we have already analyzed embeddings of 6-critical graphs that can be reduced to a 2-cell embedding of K_6 on the Klein bottle, it remains to analyze 6-critical graphs that can be reduced to a non-2-cell embedding of K_6. We eventually show that all such embeddings are isomorphic to one of those depicted in Figure 4.6.

Lemma 4.10. *Let G be a 6-critical graph embedded on the Klein bottle. If G can be reduced to a non-2-cell embedding of K_6, then G is isomorphic to one of the embeddings depicted in Figure 4.6.*

Proof. Let G be a 6-critical graph on the Klein bottle with the smallest order that can be reduced to a non-2-cell embedding of K_6 and that is not isomorphic to any of the embeddings in Figure 4.6. Observe that the choice of G implies that any possible reduction of G yields a non-2-cell embedding of K_6 on the Klein bottle (otherwise, Lemma 4.9 yields that the reduced graph is a smaller graph missing in Figure 4.6 which contradicts our choice of G).

Let H be the unique embedding of K_6 in the projective plane and w a vertex of H. By Proposition 4.5, G contains $H[w, W_1, W_2]$ for some partition of the neighborhood of w into non-empty sets W_1 and W_2. By symmetry, $|W_1| = 1$ or $|W_1| = 2$. We first analyze the case that $|W_1| = 1$, i.e., G contains the embedding drawn in the middle of Figure 4.7 as a subgraph. The face which is not 2-cell is drawn using the gray color.

Let G_{15} be the subgraph of G contained inside the cycle $C_{15} = ww'w''w_1w_5$ and G_{12} the subgraph contained inside the cycle $C_{12} = ww'w''w_1w_2$. By Lemma 4.6, G_{12} is either the cycle C_{12} with zero, one or two chords or a 5-wheel bounded the cycle C_{12}. The interiors of the remaining 2-cell faces of $H[w, W_1, W_2]$ must be empty (since they are triangles).

Assume that $G_{12} \neq C_{12}$. The graph G without the interior of the cycle C_{12} is 5-colorable since G is 6-critical. Observe that the vertices w and w_1 must get the same color in any such 5-coloring (since adding an edge ww_1 to G would form a clique of order six). However, it is always possible to permute the colors of the vertices of G_{15} preserving the colors of w, w_1 and w_5 in such a way that the 5-coloring can be extended to G_{12}. This contradicts our assumption that G is 6-chromatic. Hence, $G_{12} = C_{12}$.

Since G is 6-critical, the graph G_{15} is 5-colorable. Moreover, the vertices w and w_1 receive distinct colors in every 5-coloring of G_{15}: if the vertices w and w_1 have the same color, the 5-coloring of G_0 can be extended to the whole graph G.

Let G' be the graph obtained from G_{15} by identifying the vertices w and w_1. Since G_{15} can be drawn in the projective plane with the cycle C_{15} bounding a face, G' can also be drawn in the projective plane. As no 5-coloring assigns the vertices w and w_1 the same color, G' contains K_6 as a subgraph. On the other hand, since G is 6-critical, G does not contain K_6 as a subgraph and thus the subgraph of G' isomorphic to K_6 contains the vertex obtained by the identification of w and w_1. In addition, G' does not contain any edges except for the edges of the complete graph and the path ww_5w_1 (removing any additional edge from G would yield a graph that is also not 5-colorable contrary to our assumption that G is 6-critical). We conclude that G_{15} is comprised of

1. the path ww_5w_1, a complete graph on a 5-vertex set X such that $\{w', w''\} \subset X$ and $w_5 \notin X$, and such that $N(w)$ and $N(w_1)$ partition X, or

2. the path ww_5w_1, a complete graph on a 5-vertex set X, $\{w', w'', w_5\} \subset X$, such that $N(w) \setminus \{w_5\}$ and $N(w_1) \setminus \{w_5\}$ partition $X \setminus \{w_5\}$.

In the former case, the graph G is isomorphic to the first or the second embedding on the first line in Figure 4.6; in the latter case, G is isomorphic to the third or the fourth embedding on the first line in the figure. This finishes the analysis of the case that $|W_1| = 1$.

We now assume that $|W_1| = 2$, i.e., $G[w, W_1, W_2]$ is the graph depicted in the right part of Figure 4.7. We can also assume that w is not adjacent to w_2 in G since otherwise we could choose $W_1 = \{w_1\}$ which would bring us to the previous case. Similarly, the vertices w, w_1 and w_5 do not form a triangular face of G. Let C_{15} be the cycle $ww'w''w_1w_5$, C_{23} the cycle $ww'w''w_2w_3$, G_{15} the subgraph of G inside the cycle C_{15}, and G_{23} the subgraph inside the cycle C_{23}. As in the previous case, G_{23} is either the cycle C_{23} with zero, one or two chords or a 5-wheel bounded by C_{23}.

It is straightforward (but tedious) to check that any coloring c of G_{15} with five colors extends to a coloring of G unless:

- the vertices w and w'' are assigned the same color in c, or

- all the five vertices w, w', w'', w_1 and w_5 are assigned mutually distinct colors and G contains edges w_3w' and w_3w'' (see the embedding in the left part of Figure 4.8).

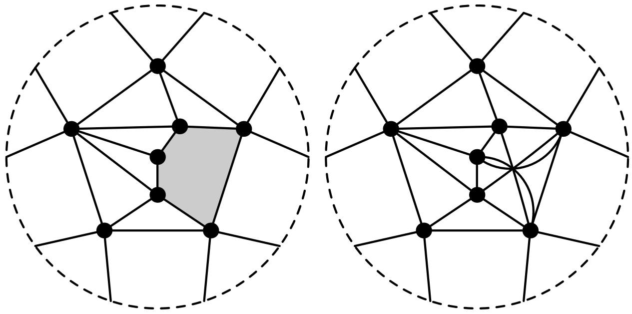


Figure 4.8: The embeddings obtained in the analysis in the proof of Lemma 4.10.

The reader is asked to verify the details him-/her-/itself.

We first show that there is a coloring of G_{15} of the latter type. Let G' be the graph obtained from G_{15} by adding the edge ww''. Assume that G' contains a complete graph of order six as a subgraph. If G_{23} contains an inner edge e, consider a 5-coloring of $G \setminus e$ which exists since G is 6-critical. The coloring must assign the vertices w and w'' the same color (since otherwise, c restricted to G_{15} would also be a proper coloring of G'). Consequently, none of the vertices w_i, $1 \leq i \leq 5$, can be assigned the common color of w and w'' which is impossible since the vertices w_i, $1 \leq i \leq 5$, form a clique. We conclude (under the assumption that G' contains K_6 as a subgraph) that G_{23} is formed by the cycle C_{23} only. As in the previous case, we can now establish that G' is formed by a subgraph isomorphic to K_6 and the path ww_5w_1w'' (which need not to be disjoint); in particular, the vertex w' is contained in the subgraph isomorphic to K_6. It is now easy to verify that the embedding of G must be isomorphic to the first or the last embedding in the first line in Figure 4.6.

Since G' does not contain K_6 as a subgraph, there is a coloring of G_{15} with five colors which assigns the vertices w and w'' distinct colors. Since G is not 5-colorable, G_{15} has a coloring assigning all the vertices w, w', w'', w_1 and w_5 distinct colors and G must be of the type depicted in the left part of Figure 4.8. Since the vertices w and w_2 are not adjacent in G and the degree of w_4 is five, we can consider the graph $|G|w_4ww_2|$; let G_0 be this graph. By the choice of G, G_0 is a non-2-cell embedding of K_6 in the projective plane and Proposition 4.4 implies that G_0 contains the vertex w_0 obtained by contracting the path ww_4w_2 in G.

If G_0 does not contain the vertex w_3, consider a coloring of G_{15} assigning the vertices w, w', w'', w_1 and w_5 five distinct colors. This coloring restricted to G_0 is a proper coloring of $G_0 = K_6$ with five colors since G_0 can contain only the edges w_0w'' and w_0w_1 in addition to those contained in G_{15} (viewing the vertices w and w_0 to be the same vertex). Hence, G_0 contains the vertex w_3. Since the only neighbors of w_3 in $G|w_4ww_2$ are the vertices w_0, w', w'', w_1 and w_5, the vertex set of G_0 must be $\{w_0, w', w'', w_1, w_5, w_3\}$. In particular, the vertex w_5 is adjacent to w' and w'' in G. A symmetric argument applied to $|G|w_2w''w_4|$ implies that the vertex w_1 is adjacent to w' and w'' in G. This brings us to the embedding depicted in the right part of Figure 4.8, which is isomorphic to the third embedding on the second line in Figure 4.6. $\square$

4.2.7 Wrapping together

We now wrap the results obtained in the previous subsections. The discussion in Subsection 4.2.4 and Lemmas 4.9 and 4.10 yield Theorem 4.1.

Immediate corollaries of Theorem 4.1 are:

Corollary 4.11. *Let G be a graph that can be embedded on the Klein bottle. G is* 5*-colorable unless it contains one of the nine graphs depicted in Figure 4.2 as a subgraph.*

Corollary 4.12. *Let G be a graph embedded on the Klein bottle. G is* 5*-colorable unless it contains a subgraph with embedding isomorphic to one of the* 19 *embeddings depicted in Figures 4.5, 4.6 and 4.7.*

Eppstein [14, 15] showed that testing the existence of a subgraph isomorphic to a fixed graph H of a graph embedded on a fixed surface can be solved in linear time. As we have found the explicit list of 6-critical graphs on the Klein bottle, we also obtain the following corollary:

Corollary 4.13. *There is an explicit linear-time algorithm for testing whether a graph embedded on the Klein bottle is* 5*-colorable.*

4.3 6-critical graphs with few crossings

This section is based on paper [17].

4.3.1 Introduction

This section is devoted to showing that K_6 is the only 6-critical graph with crossing number at most four.

Theorem 4.14. *The only 6-critical graph with crossing number at most four is* K_6*.*

It is strengthening a result of Oporowski and Zaho [37] which is claiming that K_6 is the only 6-critical graph with at most three crossings. They also conjectured that K_6 is the unique 6-critical graph even for at most five crossings. We disprove the conjecture by exhibiting a 6-critical graph different from K_6 with crossing number five.

As a first step towards proving Theorem 4.14 we show that K_6 is the only 6-critical graph which may become planar after removing three of its edges.

Theorem 4.15. *Let G be a 6-critical graph and let F be a set of its three edges. If $G - F$ is planar then G is K_6.*

The proof of Theorem 4.15 is interesting as it is not complicated yet it uses a variety of basic graph coloring techniques and it uses Theorem 4.1.

A graph contains K_6 as a subgraph if and only if it has a clique of size 6. Hence it is possible to rephrase Theorem 4.14 as a claim that a graph G with crossing number at most two is 5-colorable if and only if $\omega(G) \leq 5$. Analogously, it is also possible to rephrase Theorem 4.15.

4.3.2 Drawings with crossings

Recall that a drawing of G is *optimal* if it minimizes the number of crossings. Note that two edges may intersect several times, in either endvertices or crossings. However, thanks to the two following lemmas, we will only consider *nice* drawings, i.e. drawings such that two edges intersect at most once.

Lemma 4.16. *Every graph with crossing number k has a nice drawing with at most k crossings.*

Proof. Let G be a graph with crossing number k. Consider an optimal drawing of G that minimizes the number of crossings between edges with a common vertex. Suppose, by contradiction, that two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ intersect at least twice. Let a and b be two points in the intersection of e_1 and e_2. Without loss of generality we may assume that u_1, u_2, v_1, and v_2 are in the exterior of the closed curve C which is the union of the (a, b)-portion P_1 on u_1v_1 and the

(a, b)-portion P_2 on u_2v_2. We may also assume that P_1 contains at least as many crossings as P_2.

Then one can redraw u_1v_1 along the (u_1, a)-portion of e_1, P_2, and the (b, v_1)-portion of e_1 slightly in the exterior of C so that e_1 and e_2 do not cross anymore. Doing so, all the crossings of P_1 including a and b (if they were crossings) disappear while a crossing is created per crossings of P_2 distinct from a and b. Since one of $\{a, b\}$ must be a crossing (there are no parallel arcs), we obtain a drawing with one crossing less, a contradiction. $\square$

Similarly, one can show the following lemma.

Lemma 4.17. *Every graph with a set F of edges whose deletion results in a planar graph has a nice drawing in which each crossing contains at least one edge from F.*

In this section, we consider only nice drawings. Thus a crossing is uniquely defined by the pair of edges it belongs to. Henceforth, we will often confound a crossing with this set of two edges.

A *face* of a drawing $\tilde{G}$ is a connected component of the space obtained by deleting $\tilde{V} \cup \tilde{E}$ from the plane. We let $F(\tilde{G})$ (or simply F) be the set of faces of $\tilde{G}$. We say that a vertex v or a portion of an edge is incident to $f \in \tilde{F}$ if v is contained in the closure of f. The boundary of f, denoted by $bd(f)$ consists of the vertices and maximum (with respect to inclusion) portions of edges incident to it. An *embedding* of a graph is the set of boundaries of the faces of some drawing of G in the plane.

Lemma 4.18. *Up to a permutation of the vertices, there is only one embedding of K_6 using exactly three crossings. (See Figure 4.9.)*

Proof. Let A be an embedding of K_6 using three crossings. Let us show that it is unique. First we observe that every edge is crossed at most once. Otherwise, there will be two edges whose removal leaves the graph planar which is a contradiction to Proposition 2.1. As every cluster of a crossing contains four vertices, there must be a vertex v contained in two of them. Note that v cannot be in all three clusters since $K_6 - v$ (which is isomorphic to K_5) is not planar. Let $e_1 = vx$ and $e_2 = vy$ be the two crossed edges adjacent to v and e_3 one of the edges of the crossing whose cluster does not contain v. The graph $K_6 \setminus \{e_1, e_2, e_3\}$ is a planar triangulation T where $\deg(v) = 3$.

We denote a, b, c the neighbours of v in T. They must induce a triangle. Without loss of generality, ab and bc are the edges crossed by e_1 and e_2, respectively.

As T is a triangulation abx and bcy form triangles. Moreover, xby is also a triangle as x and y are consecutive neighbours around b. The last two edges, which are not discussed yet, are xc and ya. They must cross inside $bxyc$ (one of them is e_3). Hence A is unique. $\square$

Lemma 4.19. *A drawing of K_5 with all vertices incident to the same face requires 5 crossings.*

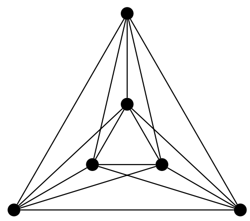


Figure 4.9: Drawing of K_6 with three crossings.

Proof. Let us number the vertices of K_5 v_1, v_2, v_3, v_4, v_5 in the clockwise order around the boundary of the face f incident to them. Then free to redraw the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5 and v_5v_1, we may assume that the boundary is the cycle $v_1v_2v_3v_4v_5$ and that f is its interior. Now both v_1v_3 and v_2v_4 are in the exterior of C and thus must cross. Similarly, $\{v_2v_4, v_3v_5\}$, $\{v_3v_5, v_4v_1\}$, $\{v_4v_1, v_5v_2\}$ and $\{v_5v_2, v_1v_3\}$ are crossings. $\square$

Lemma 4.20. *A drawing of $K_{2,3}$ such that vertices of each part are in a common face requires at least one crossing.*

Proof. Let $(\{u_1, u_2\}, \{v_1, v_2, v_3\})$ be the bipartition of $K_{2,3}$. Suppose by contradiction that $K_{2,3}$ has a drawing such that each part of the bipartition is in a common face. Then adding a vertex u_3 is the face incident to the vertices v_1, v_2 and v_3 and connecting u_3 to those vertices by new edges yields a drawing of $K_{3,3}$ with no crossing which contradicts the fact that $K_{3,3}$ is not planar. $\square$

4.3.3 Properties of 6-critical graphs

A *stable crossing cover* of a drawing of a graph is a set of vertices that is both stable and a crossing cover.

Lemma 4.21. *Every graph with a stable crossing cover is 5-colourable.*

Proof. Let G be a graph having a stable crossing cover W. Use the Four Colour Theorem on $G - W$ and extend the colouring of $G - W$ to G by using a fifth colour on W. $\square$

Let G be a graph and u, v be vertices of G. The operation of *identification* of u and v in G results in a graph denoted by $G/\{u, v\}$, which is obtained from $G - \{u, v\}$ by adding a new vertex w and the set of edges $\{wz \mid uz \text{ or } vz \text{ is an edge of } G\}$.

Lemma 4.22. *Let G be a graph and v be a degree five vertex of G. Let u and w be two non-adjacent neighbours of v. If $(G-v)/\{u, w\}$ is 5-colourable, then so is G.*

Proof. A proper 5-colouring of $(G-v)/\{u, w\}$ corresponds to a proper 5-colouring of $G-v$ such that u and w are coloured by the same colour. So it can be extended to a proper 5-colouring of G by assigning a colour to v. $\square$

Let G be a graph embedded in the plane. A cycle is *separating* if it has a vertex in its interior and a vertex in its exterior. A cycle C is *non-crossed* if all its edges are non-crossed. It is *regular* if any cluster of a crossing containing an edge of C contains at least three vertices of C.

Lemma 4.23. *In every drawing in the plane of a* 6*-critical graph there is no separating regular triangle.*

Proof. Let G be a 6-critical graph drawn in the plane. Suppose, by way of contradiction, that there is a regular triangle C. Let G_1 be the graph induced by the vertices in C and inside C and let G_2 be a graph induced by the vertices in C and outside C. Since C is separating, both G_1 and G_2 have fewer vertices than G. Hence, by 6-criticality of G, they are 5-colourings of those graphs. In addition, in these colourings of G_1 and G_2, the colours of the vertices of C are distinct. So, free to permute the colours, one can assume that the two 5-colourings of G_1 and G_2 agree on C. Hence their union yields a 5-colouring of G. □

Lemma 4.24. *Let G be a* 6*-critical graph distinct from K_6. In every nice drawing of G, there is no separating triangle such that*

- *at most one of its edges is crossed, and*
- *there is at most one crossing in its interior.*

Proof. Suppose, by way of contradiction, that such a cycle $C = x_1x_2x_3$ exists. Then by Lemma 4.23, one of its edges, say x_2x_3, is crossed. Let uv be the edge crossing it with u inside C and v outside. By Lemma 4.23, C is not regular, so $u \neq x_1$. Moreover, $u \notin \{x_2, x_3\}$ since the drawing is nice.

Let G_1 be the graph induced by C and the vertices outside C. Then G_1 admits a 5-colouring c_1 since G is 6-critical.

Let G_2 be the graph obtained from the graph induced by C and the vertices inside C by adding the edges ux_1, ux_2 and ux_3 if they do not exist. Observe that G_2 has a planar drawing with at most 2 crossings. Indeed the edge ux_1 may be drawn along uv and then a path in the outside of C and the edges ux_2 and ux_3 may be drawn along the edges of the crossing $\{x_2x_3, uv\}$. Thus G_2 admits a 5-colouring c_2.

In both colourings, the colours of the vertices of C are distinct. So, free to permute the colours, we may assume that c_1 and c_2 agree on C. One can also choose for u a colour of $\{1, \dots, 5\} \setminus \{c_2(x_1), c_2(x_2), c_2(x_3)\}$ so that $c_2(u) \neq c_1(v)$. Then the union of c_1 and c_2 is a 5-colouring of G. □

Lemma 4.25. *Let G be a* 6*-critical graph. In every drawing of G in the plane, there is no non-crossed* 4*-cycle C such that*

- *C has a chord in its exterior,*
- *C and its interior is a plane graph, and*
- *the interior of C contains at least one vertex.*

Proof. Suppose, by way of contradiction, that there is a 4-cycle $C = tuvw$ satisfying the properties above with vt a chord in its exterior. Consider the graph G_1, which is obtained from G by removing the vertices inside C. Since G is 6-critical, G_1 admits a 5-colouring c_1 in $\{1, 2, 3, 4, 5\}$. Without loss of generality, we may assume that $c_1(v) = 5$. Hence $\{c_1(t), c_1(u), c_1(w)\} \subset \{1, 2, 3, 4\}$.

Now consider the graph G_2 which is obtained from G by removing the vertices outside C. If $c_1(u) = c_1(w)$, let H be the graph obtained from $G_2 - v$ by identifying u and w. If $c_1(u) \neq c_1(w)$, let H be the graph obtained from $G_2 - v$ by adding the edge uw if it does not already exist. In both cases H is a planar graph. Hence H admits a 4-colouring c_2 in $\{1, 2, 3, 4\}$. Moreover, by construction of H, $c_2(u) = c_2(w)$ if and only if $c_1(u) = c_1(w)$. Hence free to permute the colours, we may assume that c_1 and c_2 agree on $\{t, u, w\}$.

Hence the union of c_1 and c_2 is a 5-colouring of G. $\square$

4.3.4 6-critical graph of crossing number 5

We prove Theorem 4.26 by exhibiting a drawing of a 6-critical graph G using 5 crossings which is not K_6.

Theorem 4.26. *The graph G depicted in Figure 4.10 is 6-critical.*

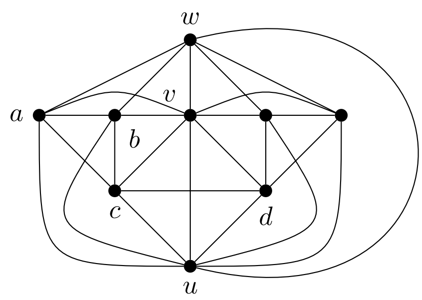


Figure 4.10: A 6-critical graph of crossing number 5.

Proof. We show by contradiction that G is not 5-colourable. We refer the reader to Figure 4.10 for names of vertices. Assume that ϱ is a 5-colouring of G. As vertices u, v and w form a triangle, they must get distinct colours. Without loss of generality, assume that $\varrho(u) = 1, \varrho(v) = 2$ and $\varrho(w) = 3$. The vertices a and b are adjacent to each other and to all the vertices of the triangle uvw, hence $\{\varrho(a), \varrho(b)\} = \{4, 5\}$. Thus $\varrho(c) = 3$ as c is adjacent to a, b, u and v. By symmetry we obtain that $\varrho(d)$ is also 3, which is a contradiction since cd is an edge.

It can be easily checked that every proper subgraph of G is 5-colourable. So G is 6-critical. $\square$

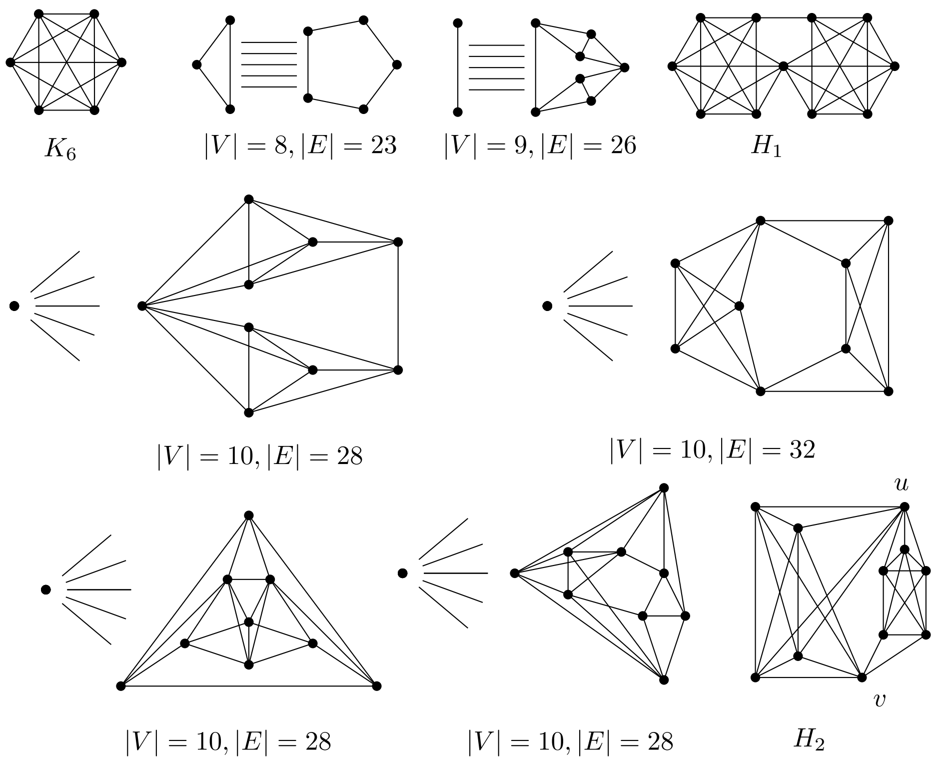


Figure 4.11: The list of 6-critical graphs embeddable on the Klein bottle. The first three of them are embeddable on torus as well.

4.3.5 Colouring graphs whose crossings are covered by few edges

The main topic of this subsection is Theorem 4.15. In the proof of Theorem 4.15, we use the list of all 6-critical graphs embeddable on the torus, which was obtained by Thomassen [43], and the list of all 6-critical graphs embeddable on the Klein bottle, which was the topic of previous section.

Lemma 4.27. *If three edges are deleted from a* 6*-critical graph embeddable on the torus other than* K_6*, then the resulting graph is nonplanar.*

Proof. Recall thet all 6-critical graphs embeddable on the torus are depicted in Figure 4.1 For all of them except K_6, we have $|E| > 3|V| - 3$. Thus the graphs are not planar after removing three edges according to Proposition 2.1 □

Lemma 4.28. *If three edges are deleted from a* 6*-critical graph embeddable on the Klein bottle other than* K_6*, then the resulting graph is nonplanar.*

Proof. We know the complete list of graphs which must be checked due to Theorem 4.1, see Figure 4.11. For all of those graphs except K_6, H_1 and H_2, we have $|E| > 3|V| - 3$. Thus those graphs are not planar after removing three edges according to Proposition 2.1.

Now we need to deal with the last two graphs H_1 and H_2. Let us first examine H_1. It contains two edge disjoint copies of K_6 without one edge. Each of these copies needs at least two edges to be removed by Proposition 2.1, so H_1 needs at least four edges to be removed.

Let us now examine H_2. Let F be a set of edges such that $H_2 \setminus F$ is planar. Let us denote by u and v the two vertices of the only 2-cut of H_2, see Figure 4.11. Observe that $H_2 - \{u, v\}$ is a disjoint union of K_5 and K_4. Since K_5 is not planar, one edge e of this K_5 is in F. But there is still a (u, v)-path P in $K_5 \setminus e$. Then the union of the graph induced by u, v, the vertices of the K_4 and the path P is a subdivision of K_6. Thus, by Proposition 2.1 for K_6, at least three of its edges must be in F. Thus $|F| \geq 4$. □

To make the basic part of the proof of Theorem 4.15 simpler, we first prove the following Theorem, which is a generalization of claim that there is no 6-critical graph which is planar after removing at most two of its edges.

Theorem 4.29. *Let G be a graph. If there is a set F of at most $2k$ edges such that $G \setminus F$ is planar then G is $(4 + k)$-colourable.*

Proof. We procced by induction on k. The result holds when $k = 0$ by the Four Colour Theorem.

Suppose that the result is true for k. Let $G = (V, E)$ be a graph with a set F of at most $2k + 2$ edges such that $G \setminus F$ is planar. Without loss of generality, we may assume that F is minimal, i.e. for any proper subset $F' \subset F$, $G \setminus F'$ is not planar.

Consider a planar drawing of $G \setminus F$. It yields a drawing of G such that each crossing contains an edge of F.

Suppose that $|F| \leq 2k + 1$. Let $e = uv$ be an edge of F. By the induction hypothesis, $G - v$ is $(4 + k)$-colourable because $F \setminus e$ is a set of $2k$ edges whose removal leaves $G - v$ planar. Hence $\chi(G) \leq \chi(G - v) + 1 \leq 4 + k + 1$.

So we may assume that $|F| = 2k + 2$.

If two edges e and f of F have a common vertex v, then $G - v$ is $(4+k)$-colourable because $F \setminus \{e, f\}$ is a set of $2k$ edges whose removal leaves $G - v$ planar. So $\chi(G) \leq \chi(G - v) + 1 \leq 4 + k + 1$. So we may assume that the edges of F are pairwise non-adjacent.

Let $e = \{u_1, u_2\}$ and $f = \{v_1, v_2\}$ be two edges in F. Then the endvertices of these two edges induce a K_4. Suppose for contradiction that u_1 and v_1 are not adjacent. Then $G - \{u_1, v_1\}$ is $(4+k)$-colourable because $F \setminus \{e, f\}$ is a set of $2k$ edges whose removal leaves $G - \{u_1, v_1\}$ planar and u_1 and v_1 can get the same colour. So $\chi(G) \leq \chi(G - \{u_1, v_1\}) + 1 \leq 4 + k + 1$. Hence $X = \{u_1, u_2, v_1, v_2\}$ induces a K_4.

We further distinguish two possible cases:

$k = 0$: Let the edges of F be $e = \{u_1, u_2\}$ and $f = \{v_1, v_2\}$ and let $X = \{u_1, u_2, v_1, v_2\}$. Let C be the 4-cycle induced by X in the plane graph $G \setminus \{e, f\}$. Note that C is a separating cycle, otherwise $G \setminus e$ would be planar. We cut G along C and obtain two smaller graphs G_1 and G_2, where both of them contain

X. We 5-colour them by induction. A colouring of G can be then obtained from the 5-colourings of G_1 and G_2 by permuting colours on X so that the these two colourings agree on $V(C)$.

$k \geq 1$: Note that union of all endvertices of edges from F induce a complete graph $K_{2|F|}$. A $K_{2|F|}$ must be planar after removing at most $|F|$ edges. Hence the following Euler's formula holds:

$$\begin{aligned} |E| &\leq 3|V| - 6 + 2k + 2 \\ \binom{4k+4}{2} &\leq 3(4k+4) + 2k - 4 \\ 8k^2 - 2 &\leq 0 \end{aligned}$$

Hence this case is not possible.

□

And now comes the proof of Theorem 4.15.

Proof. Let G be a 6-critical graph distinct from K_6 and let F be a set of at most three edges. Assume for a contradiction that $G \setminus F$ is planar.

Let us consider a nice drawing of G. By Lemma 4.21, G has no stable crossing cover.

If $|F| \leq 2$, then Theorem 4.29 contradicts the fact that G is not 5-colourable. Hence we assume that $F = \{e_1, e_2, e_3\}$. Set $e_i = u_i v_i$ for $i \in \{1, 2, 3\}$.

Claim 1. *The three edges of F are pairwise vertex-disjoint.*

Proof. If there is a vertex v shared by all three edges, then $\{v\}$ is a stable crossing cover, a contradiction. Hence a vertex u is shared by at most two edges of F. Let s be the number of degree two vertices in the graph induced by F.

We now derive a contradiction for each value of $s > 0$. So $s = 0$, which proves the claim.

s = 1: Without loss of generality, $u = u_1 = u_2$. None of $\{u, u_3\}$ and $\{u, v_3\}$ is a stable crossing cover so uu_3 and uv_3 are edges. We redraw the edge e_3 along the path $u_3 u v_3$ such that it crosses only edges incident to u. See Figure 4.12(A). Then u is a stable crossing cover, a contradiction.

s = 2: Without loss of generality, $u = u_1 = u_2$ and $v = v_2 = v_3$. Then F induces a path. None of $\{v_1, v\}$ and $\{u, u_3\}$ is a stable crossing cover, so $v_1 v$ and uu_3 are edges. We add a handle between vertices u and v. Then we draw edges of F using the handle, see Figure 4.12(B). Hence G can be embedded on the torus, which is a contradiction to Lemma 4.27.

s = 3: Without loss of generality, $u = u_1 = u_2$ is one of the shared vertices. Let v and w be the other two. Note that F induces a triangle. By Proposition 2.1, we have $|E(G)| \leq 3|V(G)| - 3$. Hence there must be at least 6 vertices of degree five as the minimum degree of G is five.

Let x be a degree five vertex different from u, v and w. By minimality of G, there exists a 5-colouring ϱ of $G - x$. Free to permute the colours, we may assume that $\varrho(u) = 1$, $\varrho(v) = 2$ and $\varrho(w) = 3$. Moreover, the neighbours of x are coloured all differently. We denote by y and z the neighbours of x, which are coloured 4 and 5 respectively. We assume that G is embedded in the plane such that all crossings are covered by F. There are two consecutive neighbours of x in the clockwise order such that they have colours in $\{1, 2, 3\}$. We denote these vertices by a and b. Without loss of generality let the clockwise order around x be z, y, a, b and $\varrho(a) = 1$ and $\varrho(b) = 2$. See Figure 4.12(C).

Let A be the connected component of a in the graph induced by the vertices coloured 1 and 5. If A does not contain z, we can switch colours on it. Then x can be coloured by 1 and we have a contradiction. Note that the colour switch is correct even if u is in A because the new colour of u will be 5 which different from 2 and 3. Thus there must be a path between a and z of vertices coloured 1 and 5. A similar argument shows that there is a path between b and y of vertices coloured 2 and 4. These paths must be disjoint and they are not using edges of F. But they cannot be drawn in the plane without crossings, a contradiction.

□

Claim 2. *For any distinct integers $i, j \in \{1, 2, 3\}$, an endvertex of e_i is adjacent to at most one endvertex of e_j.*

Proof. Suppose not. Then without loss of generality, we may assume that u_2 is adjacent to u_1 and v_1. First we redraw the edge e_1 along the path $u_1u_2v_1$. Then every edge crossed by e_1, which is not e_3, is incident to e_2. Since $\{u_2, u_3\}$ and $\{u_2, v_3\}$ are not stable crossing covers, u_2u_3 and u_2v_3 are edges. We redraw e_3 along the path $u_3u_2v_3$. Then, again, every edge crossed by e_3, which is not e_1, is incident to e_2. Moreover, the edges e_1 and e_3 cross otherwise $\{u_2\}$ would be a stable crossing cover. See Figure 4.12(D).

We distinguish several cases according to the number p of neighbours of v_2 among u_1, v_1, u_3 and v_3.

$p = 0$: The vertex v_2 and a pair of two non-adjacent vertices among u_1, v_1, u_3 and v_3 would form a stable crossing cover. Hence $\{u_1, v_1, u_3, v_3\}$ induces a K_4. See Figure 4.12(E). By Lemma 4.23, there is no vertex inside each of the triangles $u_2u_1u_3$, $u_2u_3v_1$, $u_2v_1v_3$ and $u_2u_1v_3$. Hence all the vertices are inside the 4-cycle $u_1u_3v_1v_3$. It includes the vertex v_2. We redraw e_1 such that it is crossing only e_3 and u_2v_3. Then $\{v_3, v_2\}$ is a stable crossing cover, a contradiction. See Figure 4.12(F).

$p = 1$: Without loss of generality we may assume that the neighbour of v_2 is u_1. None of $\{v_2, v_1, u_3\}$ and $\{v_2, v_1, v_3\}$ is a stable crossing cover so u_3v_1 and v_1v_3 are edges. By Lemma 4.23, there is no vertex inside each of the triangles

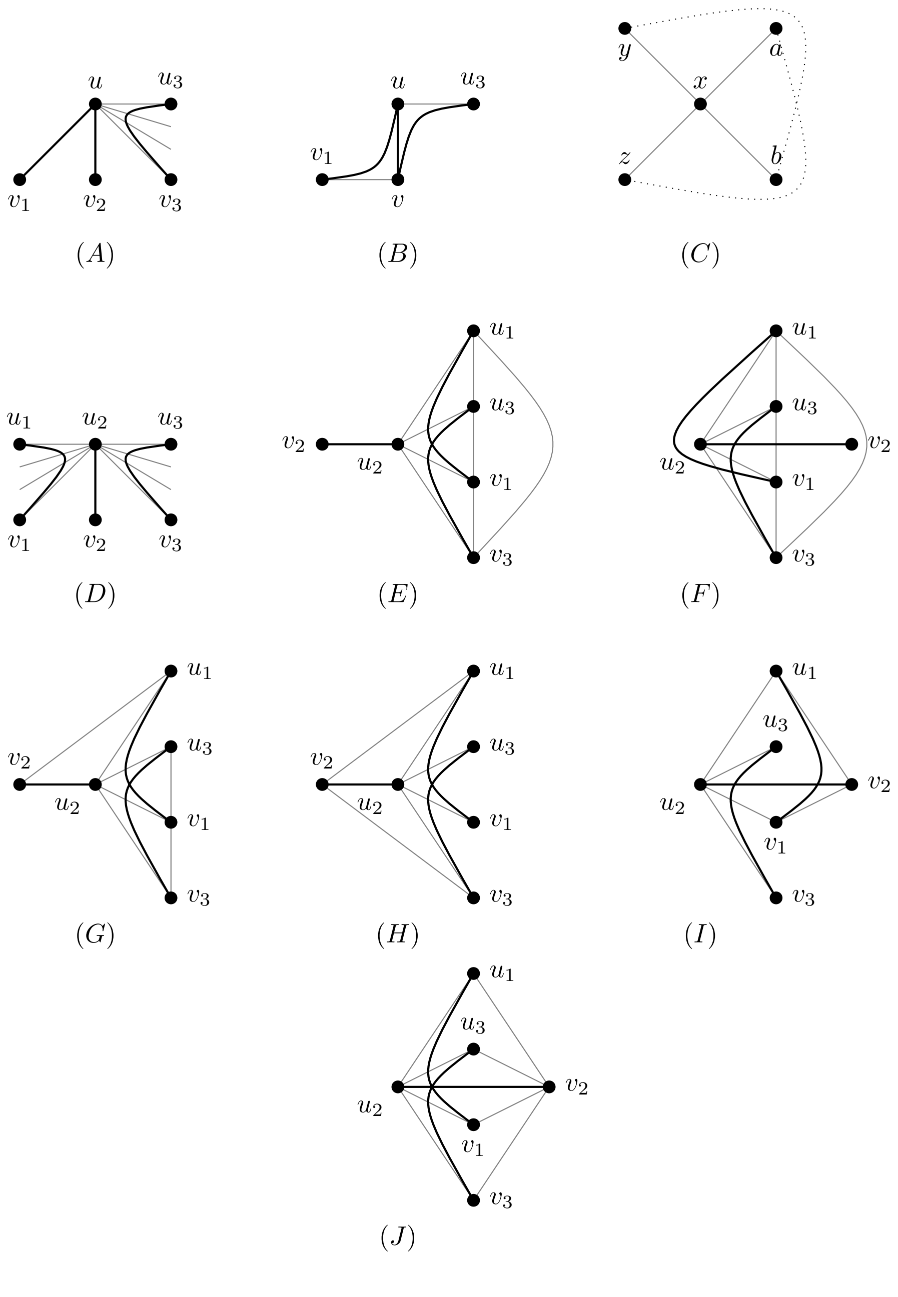


Figure 4.12: The three black edges are covering all the crossings.

$u_2u_3v_1$ and $u_2v_1v_3$. See Figure 4.12(G). Thus the edge e_3 could be drawn inside these triangles and the set F can be changed to $F' = \{e_1, e_2, u_2v_1\}$. Two edges of F' share an endvertex which is a contradiction to Claim 1.

$p \in \{2, 3\}$: We further distinguish two sub-cases. Either two neighbours of v_2 in $\{u_1, v_1, u_3, v_3\}$ are joined by an edge of F or not.

In the second case, without loss of generality, we may assume that the vertices adjacent to v_2 are u_1 and v_3. Now by Lemma 4.25 there is no vertex inside the 4-cycle $v_2u_1u_2v_3$. Hence e_2 can be drawn inside this cycle. See Figure 4.12(H). Since the removal of $\{e_1, e_3\}$ does not make G planar, v_1v_3 is inside $v_2u_1u_2v_3$. Hence the set $F' = \{e_1, e_3, u_1v_3\}$ contradicts Claim 1.

In the first case, we may assume Without loss of generality, that v_2 is adjacent to u_1 and v_1. We first redraw e_1 along the path $u_1v_2v_1$. Now all the edges crossing e_1 are incident to v_2. Thus $\{v_2, u_3\}$ or $\{v_2, v_3\}$ form a stable crossing cover. See Figure 4.12(I).

$p = 4$: See Figure 4.12(J). We repeatedly use Lemma 4.25 which implies that the 4-cycles $u_2u_3v_2u_1$, $u_2u_3v_2v_1$, $u_2v_1v_2v_3$ and $u_2v_3v_2u_1$ are not separating. This means that the graph contains only six vertices. This is a contradiction because the unique 6-critical graph on six vertices is K_6.

□

Since $\{u_1, u_2, u_3\}$ is not a stable crossing cover, it must induce at least one edge, say u_1u_2. Then Claim 2 implies that u_1v_2 and v_1u_2 are not edges. Now $\{v_1, u_2, u_3\}$ and $\{v_1, u_2, v_3\}$ are not stable crossing covers. Thus, by symmetry, we may assume that u_2u_3 and v_1v_3 are edges. $\{u_1, v_2, u_3\}$ is not a stable crossing cover so u_1u_3 is an edge; $\{v_1, v_2, u_3\}$ is not a stable crossing cover so v_1v_2 is an edge; $\{u_1, v_2, v_3\}$ is not a stable crossing cover so v_2v_3 is an edge. Hence there are two triangles $u_1u_2u_3$ and $v_1v_2v_3$, which are not separating by Lemma 4.23.

Without loss of generality, two possibilities occur. Either the edges of F do not cross each other or one pair of them is crossing. If they do not cross (Figure 4.13(A)), G can be embedded on the torus by adding a handle into the triangles and drawing the edges of F on the handle, which contradicts Lemma 4.27.

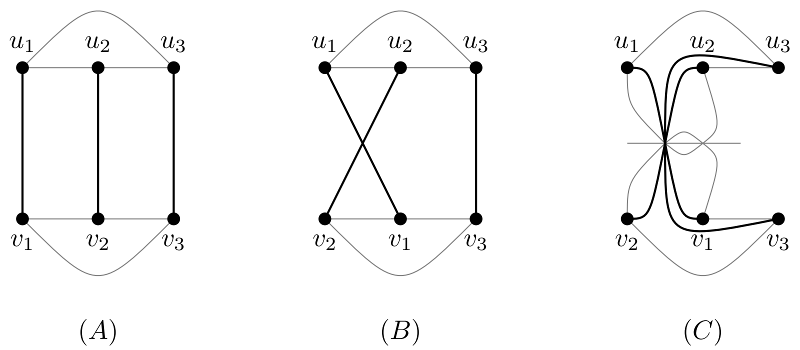


Figure 4.13: The last case of Theorem 4.15.

If they cross (Figure 4.13(B)), it is possible to draw G on the Klein bottle, see Figure 4.13(C), which contradicts Lemma 4.28. □

4.3.6 5-colouring graphs with 4 crossings

In this subsection we prove the Theorem 4.14. We repeat it for reades conveinence.

Theorem. *The unique* 6*-critical graph with crossing number at most four is* K_6*.*

Proof. Suppose, by way of contradiction, that $G = (V, E)$ is a 6-critical graph with crossing number at most 4 distinct from K_6. Moreover, one may assume that G is such a critical graph with the minimum number of vertices and with the maximum number of edges on $|V(G)|$ vertices.

Moreover, assume that we have a nice optimal drawing of G. By Theorem 4.15, there are four crossings and every edge is crossed at most once.

Since G is 6-critical, every vertex has degree at least 5. By Proposition 2.1, we have $|E| \leq 3|V| - 6 + \mathrm{cr}(G) \leq 3|V| - 2$. Hence there are at least four vertices of degree 5.

Let v be an arbitrary degree five vertex and v_i, $1 \leq i \leq 5$ be the neighbours of v in the counterclockwise order around v. By criticality of G, the graph $G - v$ admits a 5-colouring ϕ. Necessarily, all the v_i are coloured differently, otherwise ϕ could be extended to v.

For any $i \leq j$, there is a path, denoted by $v_i - v_j$, from v_i to v_j such that all its vertices are coloured in $\phi(v_i)$ or $\phi(v_j)$. Otherwise, v_j is not in the connected component A of v_i in the graph induced by the vertices coloured $\phi(v_i)$ and $\phi(v_j)$. Hence by exchanging the colours $\phi(v_i)$ and $\phi(v_j)$ on A, we obtain a 5-colouring ϕ' of $G - v$ such that no neighbour of v is coloured $\phi(v_i)$. Hence by assigning $\phi(v_i)$ to v we obtain a 5-colouring of G, a contradiction.

Let q be the number of crossed edges incident to v.

Claim 3. $q \neq 0$.

Proof. The union of the $v_i - v_j$, for $i \neq j$, is a subdivision of K_5 in $G - v$. If $q = 0$, then the v_i, $1 \leq i \leq 5$, are in one face after the removal of v. By Lemma 4.19, such a subdivision requires 5 crossings which contradicts the assumption of at most four crossings. □

Claim 4. $q \neq 1$.

Proof. Suppose to the opposite that $q = 1$. Without loss of generality, we may assume that the crossed edge is vv_1.

The path $v_2 - v_4$ must cross the two paths $v_1 - v_3$ and $v_3 - v_5$. Since every edge is crossed at most once, then v_2v_4 is not an edge.

Let G' be the graph obtained from $G - v$ by identifying v_2 and v_4 into a new vertex v'. By Lemma 4.22, G' is not 5-colourable. Now G' has at most three crossings because we removed the crossed edge vv_1 together with v. So, by minimality of G, the graph G' contains a subgraph H isomorphic to K_6. Moreover, H must contain v' since G contains no K_6. Since G' has only three crossings we can use Lemma 4.18. Let u_1 and u_2 be vertices of H which form a triangular face together with v' and let u_3, u_4 and u_5 be the vertices forming the other triangular face. Without loss of generality, we may assume that $u_3u_4u_5$ is inside $v'u_1u_2$ as in Figure 4.14(A).

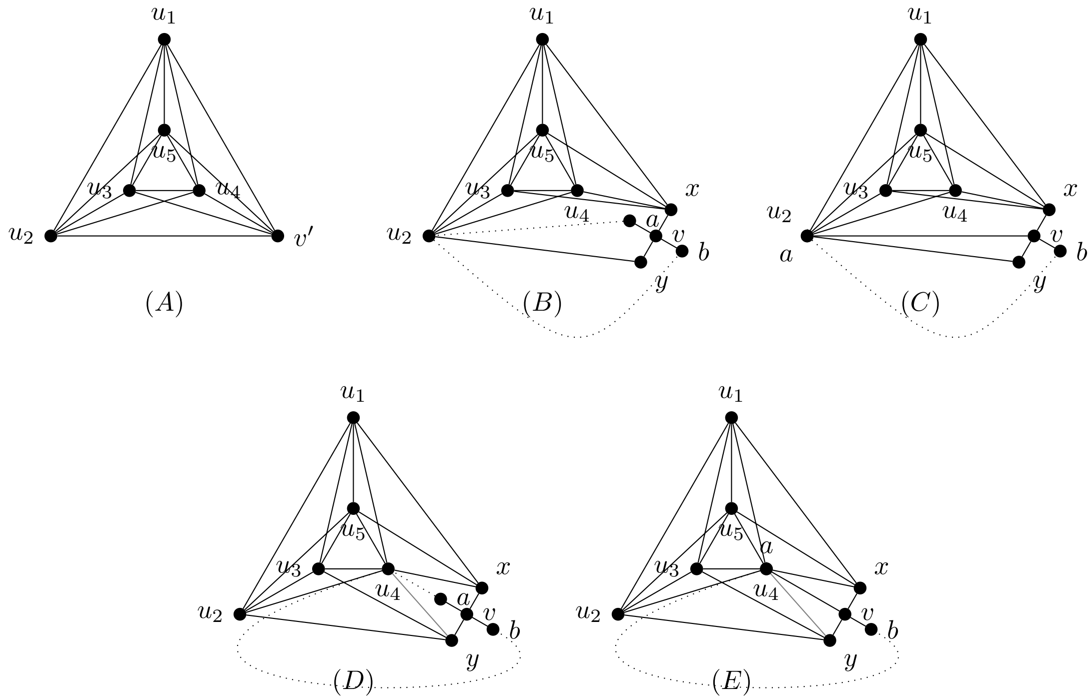


Figure 4.14: K_6 when identifying two neighbours of v.

Let us now consider the situation in G. Instead of discussing many rotations of K_6 we rather fix K_6 and try to investigate possible placings of v and its neighbours. We denote the neighbours of v which were identified by x and y (i.e. $\{v_2, v_4\} = \{x, y\}$). Let a and b be the two other neighbours of v such that va and vb are not crossed ($\{a, b\} = \{v_3, v_5\}$). Moreover, we assume that in the counterclockwise order around v, the sequence is x, a, y, b. Note that the vertex v_1 may be inserted anywhere in the sequence.

One of the identified vertices, say x, is adjacent to at least two vertices of $\{u_3, u_4, u_5\}$.

1) Assume first that x is adjacent to u_3,u_4 and u_5. Then since G has no K_6, it is not adjacent to some vertex in $\{u_1, u_2\}$, say u_2. Thus $yu_2 \in E$.

   The vertex a is either inside u_2yvx or is u_2. See Figure 4.14(B) and (C), respectively. The path $a - b$ (represented by dotted line in the figure) necessarily uses u_2. Since colours $\phi(a)$ and $\phi(b)$ alternate on $a - b$, this path cannot contain x nor u_3, u_4 and u_5. The paths $a - b$ and avb separate x and y and there must be paths $v_1 - x$ and $v_1 - y$. Thus at least one of them must cross the path $a - b$. But none of the four crossings is available for that, a contradiction.

2) Let us now assume that x is adjacent to only two vertices of $\{u_3, u_4, u_5\}$, say u_4 and u_5. Then u_3 is adjacent to y. (Possibly u_4 and y are adjacent too.) The path $a - b$ must go through u_4 and then continue to u_1 or u_2. It cannot go through u_3 or u_5 since the colours on the path alternate. See Figure 4.14(D) and (E).

   The path $x - y$ must cross $a - b$. Hence either $x - y$ goes through u_3y and $a - b$ through u_4u_2 or $x - y$ goes through xu_5 and $a - b$ through u_4u_1. In both cases, one of the paths $v_1 - x$ and $v_1 - y$ must cross $a - b$. But there are no more crossings available.

This completes the proof of Claim 4. □

Claim 5. *$q \neq 2$.*

Proof. Suppose to the opposite that $q = 2$.

We first prove the following assertion that will be used several times.

Assertion *Let x and y be two neighbours of v. Then x and y are adjacent if one of the following holds:*

- *vx and vy are not crossed;*
- *$\{x, y\}$ is included in the cluster of some crossing.*

Observe that $G - v$ has at most two crossings. Suppose that x and y are not adjacent. If vx and vy are not crossed, we can identify x and y along xvy without adding any new crossing. If $\{x, y\}$ is included in the cluster of some crossing, we can identify x and y along the edges of this crossing without adding any new crossing. Hence in both cases $(G - v)/\{x, y\}$ has a planar drawing with at most 2 crossings. Then Lemma 4.22 and Theorem 4.29 yield a contradiction. This proves the Assertion.

Assume that the crossed edges are consecutive, say vv_1 and vv_2. By the Assertion, v_3v_5 is an edge. See Figure 4.15(A). If v_3v_5 is not crossed or crosses either vv_1 and vv_2, then the cycle vv_3v_5 is regular, which contradicts Lemma 4.23. If v_3v_5 is crossed by another edge, then the cycle vv_3v_5 contradicts Lemma 4.24. Henceforth, we may assume that the two crossed edges are not consecutive, say vv_2 and vv_5.

By the Assertion, v_1v_3, v_1v_4 and v_3v_4 are edges. If v_1v_3 is not crossed, then the triangle vv_1v_3 is separating because v_2 and v_4 are on the opposite sides. This contradicts Lemma 4.23. If v_1v_3 is crossed it can be redrawn along the path v_1vv_3 with one crossing with vv_2. Symmetrically, we assume that v_1v_4 is crossing vv_5. See Figure 4.15(B).

By the Assertion, $\{v_1v_2, v_2v_3, v_4v_5, v_5v_1\} \subset E(G)$. See Figure 4.15(C).

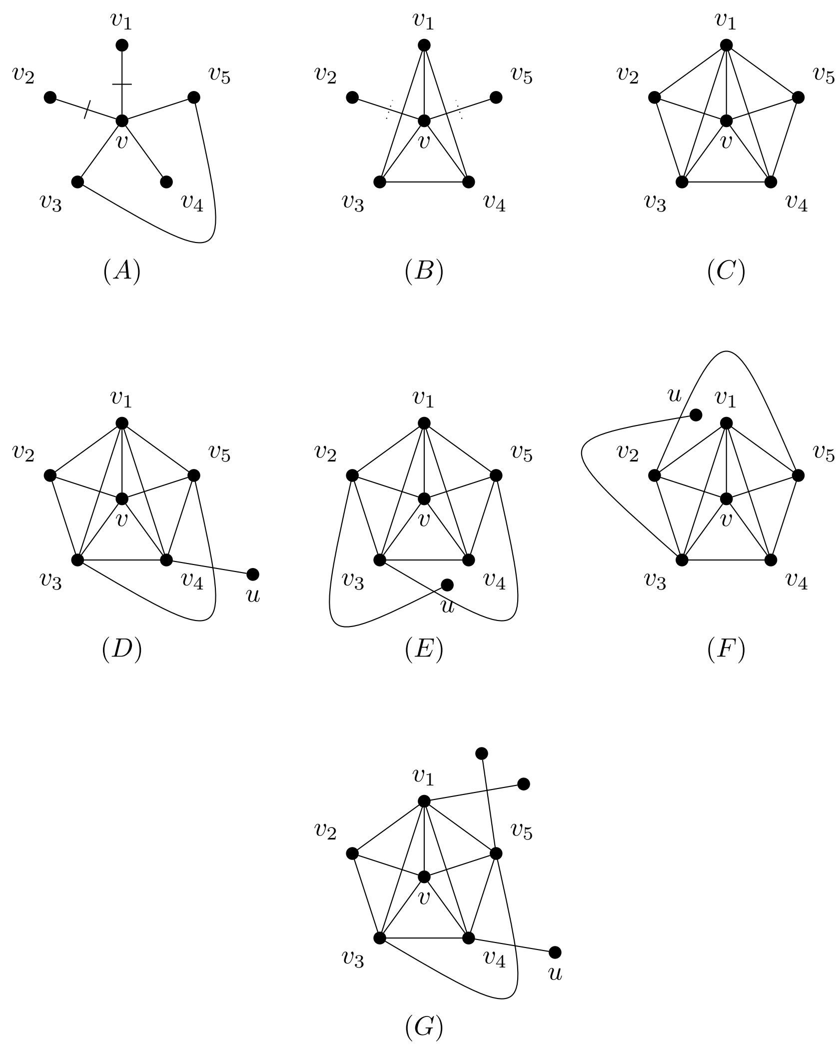


Figure 4.15: Two crossed edges.

Let $C = \{c_1c_2, c_3c_4\}$ and $D = \{d_1d_2, d_3d_4\}$ be the two crossings not having v in their cluster. For convenience and with a slight abuse of notation, we denote by C (resp. D) both the crossing C (resp. D) and its cluster. For $X \in \{C, D\}$, let $a(X) := |X \cap N(v)|$. Without loss of generality, we may assume that $a(C) \leq a(D)$.

A vertex u is a *candidate* if it is not adjacent to v. There is no candidate u common to both C and D otherwise $\{u, v\}$ would be a stable crossing cover. There are no non-adjacent candidate vertices $c \in C$ and $d \in D$ otherwise $\{v, c, d\}$ would be a stable crossing cover.

Assume that $a(D) = 4$. The vertex v_1 cannot be in D because it is already adjacent to all the other neighbours of v by edges not in D. Thus $D = \{v_2, v_3, v_4, v_5\}$. But then, by the Assertion, v_2v_5 is an edge. So $N(v) \cup \{v\}$ induces a K_6, a contradiction.
Hence $a(C) \leq a(D) \leq 3$.

Suppose now that $X \in \{C, D\}$ does not induce a K_4. Then two vertices x_1 and x_2 of X are not adjacent. One can add the edge x_1x_2 and draw it along the edges of the crossing such that no new crossing is created. Hence by the choice of G, the obtained graph $G \cup x_1x_2$ contains a K_6. Since K_6 has crossing number 3, one of the crossings containing v in its cluster must be used. So v belongs to the K_6 and hence the K_6 is induced by $\{v\} \cup N(v)$. In such case edges v_2v_4 and v_3v_5 cross and hence form C or D, which is not possible since $a(C) \leq a(D) \leq 3$.

Hence both C and D induce a K_4. Thus the candidates in $C \cup D$ induce a complete graph. So there are at most five of them. Since $C \cap D$ contains no candidate, we have $a(C) + a(D) \geq 3$ and so $2 \leq a(D) \leq 3$

Assume that $a(D) = 2$ and thus $1 \leq a(C) \leq 2$. Then C (resp. D) contains a set C' (resp. D') of two candidates. All the vertices of C' are adjacent to all the vertices of D'. But since both C and D contain a vertex in $N(v)$, drawing all the edges between these two sets requires one more crossing, a contradiction.
Hence $a(D) = 3$.

Thus, an edge of D has its two endvertices in $N(v)$ and so it is v_2v_5, v_2v_4 or v_3v_5. Let u be the unique candidate of D.

Assume first that $v_1 \in D$. Then v_1u is an edge of D. Moreover, C must be on the paths $v_2 - v_4$ and $v_3 - v_5$. Since edges are crossed at most once $D = \{v_1u, v_2v_5\}$. Let w be a candidate vertex in C. Then w is outside the cycle vv_2v_5. But the only neigbour of v_1 outside this cycle is u which is distinct from w because the crossings C and D have no candidate vertex in common. Thus $\{w, v_1\}$ is a stable crossing cover, a contradiction to Lemma 4.21.
So $v_1 \notin D$.

By symmetry, we may assume that D is either $\{v_3v_5, v_4u\}$ (Figure 4.15(D)) or $\{v_3v_5, v_2u\}$ (Figure 4.15(E)) or $\{v_2v_5, v_3u\}$ (Figure 4.15(F)). In the second and third cases, Lemma 4.24 is contradicted by the cycle $v_3v_4v_5$ and $v_1v_2v_5$ respectively.
Hence $D = \{v_3v_5, v_4u\}$.

The set $\{v_2, v_4\}$ is stable and covers the three crossings distinct from C. Hence $\{v_2, v_4\}$ does not intersect C, otherwise it would be a stable crossing cover. So

$C \cap N(v) \subset \{v_1, v_3, v_5\}$. The edge v_1v_5 is not crossed, otherwise it could be redrawn along the edges of the crossing $\{vv_5, v_1v_4\}$ to obtain a drawing of G with fewer crossings. Furthermore, v_1v_3 and v_3v_5 are not in C because they are in some other crossing. Hence $a(C) \leq 2$.

Let B be the set of candidates of C. Recall that all vertices of B are adjacent to u. Moreover, every vertex $b \in B$ is adjacent to a vertex of $\{v_2, v_4\}$ otherwise $\{v_2, v_4, b\}$ is a stable crossing cover. But v_4 and u are separated by $v_3v_4v_5$, so all vertices of B are adjacent to v_2. Now the graph induced by the edges between B and $\{u, v_2\}$ is a complete bipartite graph. Moreover, its induced drawing has no crossing and the vertices of each part are in a common face. Thus, by Lemma 4.20, $|B| \leq 2$.
So $a(C) = 2$.

Recall that $C \cap N(v) \subset \{v_1, v_3, v_5\}$. Suppose that $C \cap N(v) = \{v_1, v_3\}$. The closed curve formed by the path v_3vv_1 and the two"half-edges" connecting v_1 to v_3 in C separates v_2 and u. Then the vertices of B cannot be adjacent to both u and v_2, a contradiction. Similarly, we obtain a contradiction if $C \cap N(v) = \{v_3, v_5\}$. Hence we may assume that $C \cap N(v) = \{v_1, v_5\}$. But then connecting the vertices of B to those of $\{v_2, v_4\}$ would require one more crossing. See Figure 4.15(G).

This completes the proof of Claim 5. □

Claim 6. $q \neq 3$.

Proof. Suppose that $q = 3$.

Let C be the crossing whose cluster does not contain v. It contains no candidate u otherwise $\{u, v\}$ would be a stable crossing cover. Hence $C \subset N(v)$.

Assume first that the three crossed edges incident to v are consecutive, say the crossed edges are vv_1, vv_2 and vv_5. By the Assertion, v_3v_4 is an edge. See Figure 4.16(A). Up to symmetry, the cluster of C is one of the following three sets $\{v_1, v_2, v_3, v_4\}$ or $\{v_2, v_3, v_4, v_5\}$ or $\{v_1, v_2, v_4, v_5\}$.

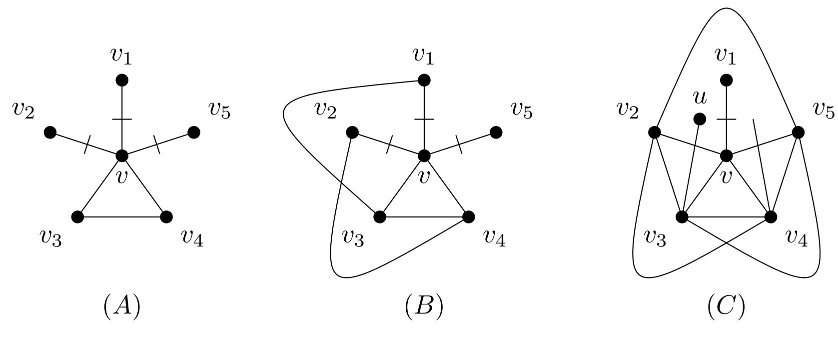


Figure 4.16: Three consecutive crossed edges.

- $C = \{v_1, v_2, v_3, v_4\}$. Then the edges of C are not v_1v_4 and v_2v_3 because it is impossible to draw them such that each is crossed exactly once. Hence $C = \{v_1v_3, v_2v_4\}$. The Jordan curve formed by the path v_1vv_4 and the two "half-edges" connecting v_1 to v_4 in C separates $\{v_2, v_3\}$ and v_5. See Figure 4.16(B). Moreover, it is crossed only once (on edge v_1v), while two crossings are needed, one for each of the disjoint paths $v_2 - v_5$ and $v_3 - v_5$, a contradiction.

- $C = \{v_2, v_3, v_4, v_5\}$. Then the edges of C are not v_2v_3 and v_4v_5 because it is impossible to draw them such that each is crossed exactly once. Hence $C = \{v_2v_4, v_3v_5\}$. Hence by the Assertion, v_2v_3, v_4v_5 and v_2v_5 are edges. The triangle vv_2v_3 has only one crossed edge. So, by Lemma 4.24, it is not separating. Thus its interior is empty and the edge crossing vv_2 is incident to v_3. Let u be the second endvertex of this edge. By symmetry, the interior of vv_4v_5 is empty and the edge crossing vv_5 is v_4t for some vertex t.

  If $u = t = v_1$, then by the Assertion v_1v_2 and v_1v_5 are edges. So $N(v) \cup \{v\}$ induces a K_6, a contradiction. Hence without loss of generality we may assume that $u \neq v_1$. See Figure 4.16(C).

  The interiors of the cycles vv_2v_3, vv_3v_4 and $v_2v_3v_4$ contain no vertices by Lemma 4.23. Hence v_3 is a degree five vertex. Moreover, its two neighbours u and v are not adjacent and $(G - v_3)/\{u, v\}$ has at most two crossings. Then Theorem 4.29 and Lemma 4.22 yield a contradiction.

- $C = \{v_1, v_2, v_4, v_5\}$. The crossing C is neither $\{v_1v_2, v_4v_5\}$ nor $\{v_1v_5, v_2v_4\}$ since it is impossible to draw so that every edge is crossed exactly once. Hence $C = \{v_1v_4, v_2v_5\}$. By the Assertion, $v_2v_4 \in E(G)$. Then the triangle vv_2v_4 contradicts Lemma 4.24.

Suppose now that the three crossed edges incident to v are not consecutive. Without loss of generality, we assume that these edges are vv_1, vv_3 and vv_4.

By the Assertion, v_2v_5 is an edge. If v_2v_5 is not crossed, then vv_2v_5 is a separating triangle, contradicting Lemma 4.23. So v_2v_5 is crossed. It could not cross vv_3 nor vv_4 otherwise vv_2v_5 would be a regular cycle contradicting Lemma 4.23. Moreover, v_2v_5 cannot be in C otherwise vv_2v_5 would contradict Lemma 4.24. Hence v_2v_5 crosses vv_1.

By the Assertion, v_1v_2 and v_1v_5 are edges. Moreover they are not crossed, otherwise they could be redrawn along the edges of the crossing $\{vv_1, v_2v_5\}$ to obtain a drawing of G with fewer crossings. See Figure 4.17(A).

Consider the paths $v_2 - v_4$ and $v_3 - v_5$. If they cross, it is through C. Since $C \subset N(v)$, the paths $v_2 - v_4$ and $v_3 - v_5$ are actually edges. See Figure 4.17(B). But one can redraw v_2v_5 along the edges of C to obtain a drawing of G with fewer crossings, a contradiction.

Suppose now that $v_2 - v_4$ and $v_3 - v_5$ do not cross. By symmetry, we may assume that $v_2 - v_4$ cross vv_3. The paths $v_1 - v_4$ and $v_3 - v_5$ cross. It must be through C so v_1v_4 and v_3v_5 are both edges. See Figure 4.17(C). By the Assertion, v_1v_3, v_3v_4 and v_4v_5 are edges.

If v_2v_4 is also an edge, the Assertion implies that v_2v_3 is also an edge. Then $N(v) \cup \{v\}$ induces a K_6, a contradiction. Hence $v_2v_4 \notin E(G)$.

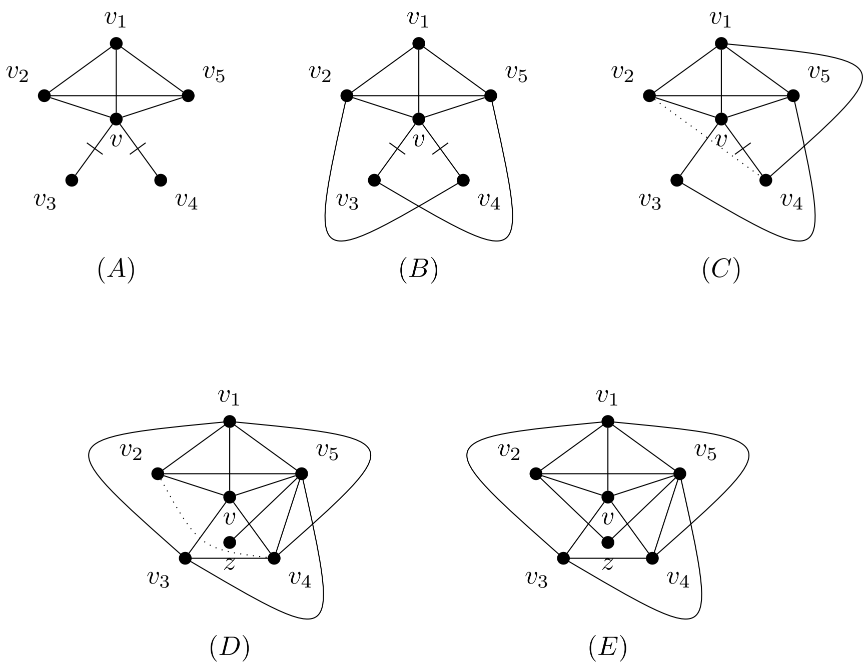


Figure 4.17: Three non-consecutive crossed edges.

By Lemma 4.24, the cycle vv_4v_5 is not separating, so its interior contains no vertex and vv_4 is crossed by an edge with v_5 as an endvertex. Let z be the other endvertex of this edge. As an edge is crossed at most once, z is inside vv_3v_4. See Figure 4.17(D).

Let ab be the edge which is crossing vv_3. The sets $\{v_5, a\}$ and $\{v_5, b\}$ are not stable otherwise they would be a stable crossing cover. Hence v_5a and v_5b are both edges. Thus $ab = v_2z$. See Figure 4.17(E). Now v_1z is not an edge and hence $\{v_1, z\}$ is a stable crossing cover, contradicting Lemma 4.21.

This completes the proof of Claim 6. □

Claim 7. $q \neq 4$

Proof. By way of contradiction, suppose that $q = 4$. Then $\{v\}$ is a stable crossing cover, a contradiction. □

Combining Claims 3, 4, 5, 6 and 7 yields a contradiction. This finishes the proof of Theorem 4.14. □

Chapter 5

3-choosability of planar triangle-free graphs

This chapter is based on papers *On 3-choosability of plane graphs without 3-, 7- and 8-cycles* [11] and *3-choosability of triangle-free planar graphs with constraints on 4-cycles* [12] by Dvořák, Lidický and Škrekovski.

5.1 Introduction

Recall that a graph G is k-choosable if every vertex of G can be properly colored whenever every vertex has a list of at least k available colors.

A celebrated Four color theorem states that every planar graph is 4-colorable. Grötzsch's theorem states that every planar triangle-free graph is 3-colorable. One might then ask - what other sets of forbidden cycles guarante 3-colorability? Or more generally, what are the sufficient conditions imposed on short cycles to imply 3-colorability? Montassier [36] is collecting all related results and presents them in a catalogue style manner.

In this chapter we consider analogue of 3-color problem for choosability.
Thomassen [42] proved that every planar graph is 5-choosable. Voigt [48] showed that not all planar graphs are 4-choosable. We would like to have an analogue of Grötzsch's theorem for 3-choosability. By 3-degenericity, every planar triangle-free graph is 4-choosable. Voigt [49] exhibited an example of a non-3-choosable triangle-free planar graph. Hence the analogue of Grötzsch's theorem for 3-choosability needs to have some additional conditions.

For example, forbidding all odd cycles is a sufficient condition as every planar bipartite graph is 3-choosable [1].

There is a bunch of other results on this topic. We summarize them in Table 5.1. Current state is also available at Montassier's web page [36].

There are many other possible combinations of cycles one may try to forbid. We would like to explicitly mention one, which was our initial motivation to study 3-choosability of planar graphs:

3	4	5	6	7	8	9	authors	year
$\times$	$\times$						Thomassen [44]	1995
$\times$			$\times$	$\times$		$\times$	Zhang and Xu [52]	2004
$\times$		$\times$			$\times$	$\times$	Zhang [51]	2005
$\times$		$\times$	$\times$				Lam, Shiu and Song [32]	2005
$\times$					$\times$	$\times$	Zhang, Xu and Sun [53]	2006
$\times$					$\times$	$\times$	Zhu, Lianying and Wang [54]	2007
$\times$			$\times$	$\times$	$\times$		Lidický [34]	2009
$\times$				$\times$	$\times$		Dvořák, Lidický and Škrekovski [11]	2009
$\times$			$\times$	$\times$			Dvořák, Lidický and Škrekovski [12]	2010

Table 5.1: Forbidden cycles in planar graphs implying 3-choosability.

Problem 5.1. *Is there k such that forbidding all odd cycles of length $\leq k$ is a sufficient condition for 3-choosability of planar graphs?*

Such a condition makes a graph locally bipartite and would strengthen the result of Alon and Tarsi [1] that every bipartite planar graph is 3-choosable.

In the following section we give the proof that every triangle-free planar graph without 7-cycles and 8-cycles in 3-choosable. The proof is by Discharging method. The basic idea of the method is to start with a counterexample and show that is has some structure. Next, Euler's formula is used to show that the counterexample is not planar. Discharging is a very powerful technique for attacking planar graphs. For example The Four color theorem was proven by a brutal discharging.

In Section 5.3 we show that every triangle-free planar graph without 6-cycles and 7-cycles in 3-choosable. It is actually a corollary of a theorem which is forbidding triangles and some configurations of 4-cycles. Hence it is also strengthening a result of Thomassen [44] which is forbidding 4-cycles completely. Moreover, is also strengthens the results of Lidický [34], Zhang and Xu [52], Lam et al. [32] and Li [33]. Recently, Guoa and Wang [26] published the same result as ours claiming that they are strenghtening and fixing the result of Li [33]. Although the result looks similar the result presented in the next section, the proof method is different. We use a method developed by Thomassen and used for showing 5-choosability of planar graphs [42]. The basic idea is to identify a vertex (or vertices) in the outer face which can be removed and use induction on the smaller graph. The method is not used as much as Discharging. On the other hand, proof by Thomassen's method tends to be more elegant.

5.2 Graphs without 3-,7-,8-cycles

This section is based on paper [11].

Our goal is to prove the following theorem.

Theorem 5.2. *Every plane graph G without 3-, 7- and 8-cycles is 3-choosable. Moreover, any precoloring of a 4- or 5-face h can be extended to a list coloring of G provided that each vertex not in $V(h)$ has at least three available colors.*

Proof. Suppose that Theorem 5.2 is false, and let G be a minimal counterexample. In case that h is precolored, we assume that h is the outer face of G. We shall get a contradiction by using the Discharging Method. Here is an overview of the proof: First we study some reducible configurations which cannot occur in the smallest counterexample because of the minimality. Next, we identify some additional configurations which are forbidden by the assumptions of the theorem. Finally, we show that there is no planar graph satisfying all the constraints. To prove it, we assign each vertex and face an initial charge such that the total charge is negative. Afterwards, the charge of faces and vertices is redistributed according to prescribed rules in such a way that the total charge stays unchanged, and thus negative. Under the assumption that the identified configurations are not present in G, we show that the final charge of each vertex and each face is non-negative, which is a contradiction.

Lemma 5.3. *No 4- or 5-cycle is separating.*

Proof. Let C be a separating 4- or 5-cycle. By the minimality of G, color first the part of G outside of C, and then extend the coloring of C to the part of G inside C. □

5.2.1 Reducible configurations

We use the term *configuration* for a graph H, possibly with degree constraints on its vertices when considering H as a potential subgraph of G. We say that a configuration H is *reducible* if it cannot appear in the minimal counterexample G.

Lemma 5.4. *The following configurations of non-precolored vertices are reducible:*

(1) *a* (≤ 2)*-vertex* v*;*

(2) *an even cycle* C_{2k} *whose vertices have degree* 3*;*

(3) *two 4-cycles* $v_1v_2v_3v_4$ *and* $v_1v_5v_6v_7$ *consisting of mutually distinct vertices* $v_1, \ldots, v_7$*, such that* v_1 *is a* 4*-vertex and* v_i *has degree* 3 *for* $2 \leq i \leq 7$*, see Figure 5.1.*

Proof. Let L be an arbitrary list assignment of G such that each vertex is assigned precisely 3 colors. We show that G is L-colorable provided that it contains one of the three configurations.

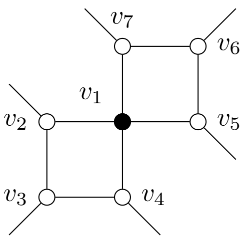


Figure 5.1: A reducible configuration.

If G has a non-precolored 2-vertex v, then by the minimality of G, the graph $G-v$ is L-colorable. This coloring can be extended to v, since it has three available colors and at most two neighbors.

Suppose now that G contains an even cycle C of non-precolored 3-vertices. Let φ be an L-coloring of $G - C$. For each $v \in V(C)$, if v has a neighbor w in $G - C$, then let $L'(v) = L(v) \setminus \{\varphi(w)\}$. Otherwise (if all three neighbors of v belong to C), let $L'(v) = L(v)$. The graph induced by the vertices of C, say $G[C]$, is a 2-connected graph different from a clique and an odd cycle, such that $\deg_{G[C]}(v) = |L'(v)|$ for each $v \in V(G[C])$. Hence, $G[C]$ is L'-colorable by [16]. This completes the proof of Lemma 5.4.(2).

Finally suppose that G contains the third configuration K. Note that v_i for $2 \leq i \leq 7$ has two neighbors in K and the third neighbor, denoted by w_i, must be in $G-K$. Otherwise, G contains a triangle, which is forbidden by the assumptions of the theorem, or a separating 4- or 5-cycle which contradicts Lemma 5.3.

Let φ be an L-coloring of $G - K$. Let $L'(v_1) = L(v_1)$ and let $L'(v_i) = L(v_i) \setminus \{\varphi(w_i)\}$ for $2 \leq i \leq 7$. We show that there exists a proper L'-coloring φ' of v_2, v_3 and v_4 such that $|L'(v_1) \setminus \{\varphi'(v_2), \varphi'(v_4)\}| \geq 2$. Consider the following cases:

- $L'(v_2) \cap L'(v_4) \neq \emptyset$: Let a be a common color of v_2 and v_4. We color v_2 and v_4 by a, and extend this coloring to v_3.
- $L'(v_2) \cap L'(v_4) = \emptyset$: Then $|L'(v_2) \cup L'(v_4)| \geq 4$. Hence, there exists a color $a \in (L'(v_2) \cup L'(v_4)) \setminus L'(v_1)$. Without loss of generality assume that $a \in L'(v_2)$. We assign a to v_2, and afterwards L'-color v_3 and v_4.

Since the 4-cycle $v_1v_5v_6v_7$ is 2-choosable, we can extend φ' to an L'-coloring of K, giving an L-coloring of G. □

We can assume that *the outer face h of G is a precolored 4- or 5-cycle*: if G has no precolored 4- or 5-face, then every vertex has degree ≥ 3 according Lemma 5.4(1). Euler's formula implies that G has a 4- or 5-face f. So we can fix some coloring of the vertices of f and redraw G such that f becomes the outer face.

Lemma 5.5. *A 4-face $f \neq h$ cannot be adjacent to 5- or 6-face. Moreover, f can share at most two edges with other 4-faces. If a 4-face shares edges with two other 4-faces, then they surround a vertex of degree three.*

Proof. Let $f = v_1v_2v_3v_4$ be a 4-face sharing at least one edge with a face $f' = v_1v_2u_3 \dots u_t$, where $t \in \{4, 5, 6\}$. As G has no triangles, $u_3 \neq v_4$ and $u_t \neq v_3$. If $u_3 = v_3$, then $\deg(v_2) = 2$ and thus $v_1v_2v_3$ is a part of the outer face h. Observe that $f' = h$ since 2-vertex v_2 can be shared by at most two faces and $h \neq f$. In this case, we remove v_2 and color v_4 instead. Therefore, $u_3 \neq v_3$, and by symmetry, $u_t \neq v_4$.

Suppose that $t = 5$. If $u_4 \notin \{v_3, v_4\}$, then $v_1u_5u_4u_3v_2v_3v_4$ would be a 7-cycle, and if $u_4 \in \{v_3, v_4\}$, then G contains a triangle, which is a contradiction. Therefore, G does not contain a 4-face adjacent to a 5-face.

Consider the case that $t = 6$. If $\{u_4, u_5\} \cap \{v_3, v_4\} = \emptyset$, then $v_1u_6u_5u_4u_3v_2v_3v_4$ would be an 8-cycle, thus assume that say $u_4 \in \{v_3, v_4\}$. As G does not contain triangles, $u_4 \neq v_3$, and hence $u_4 = v_4$. But, the 4-cycle $v_4v_1v_2u_3$ separates v_3 from u_5, which is a contradiction. It follows that G does not contain a 4-face adjacent to a 6-face.

Suppose now that $t = 4$ and that f shares an edge with one more 4-face f''. Assume first that $f'' = v_3v_4u_5u_6$. Observe that $\{u_5, u_6\} \cap \{v_1, v_2\} = \emptyset$. If $\{u_5, u_6\} \cap \{u_3, u_4\} = \emptyset$, then $v_1u_4u_3v_2v_3u_6u_5v_4$ is an 8-cycle, thus assume that say $u_5 \in \{u_3, u_4\}$. As G does not contain triangles, $u_5 \neq u_4$, thus $u_5 = u_3$. However, G then contains a separating 4-cycle $u_3v_2v_1v_4$.

It follows that $f'' = v_1v_4u_5u_6$. By symmetry, f does not share the edge v_2v_3 with a 4-face, thus f does not share edges with three 4-faces. Also, as G does not contain 8-cycles, $\{u_5, u_6\} \cap \{u_3, u_4\} \neq \emptyset$. Note that $u_5 \neq u_3$ because of the separating 4-cycle $u_3v_2v_1v_4$, and $u_5 \neq u_4$ and $u_6 \neq u_3$, as G does not contain triangles. It follows that $u_4 = u_6$, thus v_1 has degree three and it is surrounded by 4-faces f, f' and f''. □

Lemma 5.6. *No two 5-faces f and f' distinct from h are adjacent.*

Proof. Let $f = v_1v_2v_3v_4v_5$ and $f' = v_1v_2u_3u_4u_5$. As $f \neq h$ and $f' \neq h$, v_1 and v_2 have degree at least three, thus $v_3 \neq u_3$ and $v_5 \neq u_5$. As G does not contain triangles, $v_3 \neq u_5$ and $v_5 \neq u_3$. As $v_2v_3v_4v_5v_1u_5u_4u_3$ is not an 8-cycle, $\{v_3, v_4, v_5\} \cap \{u_3, u_4, u_5\} \neq \emptyset$. By symmetry, we may assume that $v_4 \in \{u_3, u_4\}$. As G does not contain triangles, $v_4 \neq u_3$, thus $v_4 = u_4$. However, at least one of 4-cycles $u_4u_3v_2v_3$ or $u_4u_5v_1v_5$ is distinct from h, contradicting Lemma 5.3 or Lemma 5.5. □

5.2.2 Initial charges

We assign the initial charge to each non-precolored vertex v and the initial charge to each face $f \neq h$, respectively, by

$$\mathrm{ch}(v) := 2\deg(v) - 6 \quad \text{and} \quad \mathrm{ch}(f) := \ell(f) - 6.$$

A precolored vertex v of h has initial charge $\mathrm{ch}(v) := 2\deg(v) - 4$ and the outer face h has initial charge $\mathrm{ch}(h) := 0$.

It is easy to see that every vertex has non-negative initial charge, and that only the (≤ 5)-faces $\neq h$ have negative charge. We are interested in the total amount of charge of G. By Euler's formula, the total amount of charge is

$$\begin{aligned}
\sum_{v\in V(G)} \mathrm{ch}(v) + \sum_{f\in F(G)} \mathrm{ch}(f) &= \sum_{v\in V(G)} (2\deg(v) - 6) + \\
&\quad 2\ell(h) + \sum_{f\in F(G)} (\ell(f) - 6) + 6 - \ell(h) \\
&= (4|E(G)| - 6|V(G)|) + \\
&\quad (2|E(G)| - 6|F(G)|) + 6 + \ell(h) \\
&= 6(|E(G)| - |V(G)| - |F(G)|) + 6 + \ell(h) \\
&= -6 + \ell(h).
\end{aligned}$$

As $\ell(h) \le 5$, the total charge is negative.

5.2.3 Discharging rules

We use the following discharging rules to redistribute the initial charge, see Figure 5.2. A vertex v is *big* if $\deg(v) \ge 4$ or it is precolored and $\deg(v) = 3$.

Rule 1. *Let a (≥ 9)-face f share an edge e with a 4-face $g \ne h$. If g contains only one big vertex, then f sends charge $1/3$ to g through the edge e.*

Rule 2. *Let two (≥ 9)-faces f_1 and f_2 share a 3-vertex v with a 4-face $g \ne h$ which contains only one big vertex. Let e be the common edge of f_1 and f_2 that is incident with v. Then each of f_1 and f_2 sends charge $1/6$ to g through the edge e.*

Rule 3. *Let a (≥ 9)-face f share a common edge uv with a 4-face g, which has no precolored vertex, and $\deg(v) = 4$. Let uvw be a part of the facial walk of f. If v is the only big vertex of g, then f sends charge $1/6$ to g through the edge vw.*

Rule 4. *A (≥ 9)-face sends charge $1/3$ to an adjacent 5-face $g \ne h$ through their common edge $e = uv$, if u and v are of degree three.*

Rule 5. *A 6-face sends charge $1/4$ to an adjacent 5-face $g \ne h$ through their common edge $e = uv$, if u and v are of degree three.*

Rule 6. *A big vertex v sends charge to an incident 4-face $g \ne h$. If $\deg(v) = 4$ and v is not precolored, or $\deg(v) = 3$ (and v is precolored), then v sends charge 1. Otherwise, v sends charge $4/3$ to g.*

Rule 7. *A big vertex sends charge $1/2$ to every adjacent 5- or 6-face $g \ne h$.*

Note that rules apply simultaneously. Hence, for example Rule 1 and Rule 2 can both send charge from one face to some other. Also multiplicity is considered, for example, a face can send charge to another face through several edges.

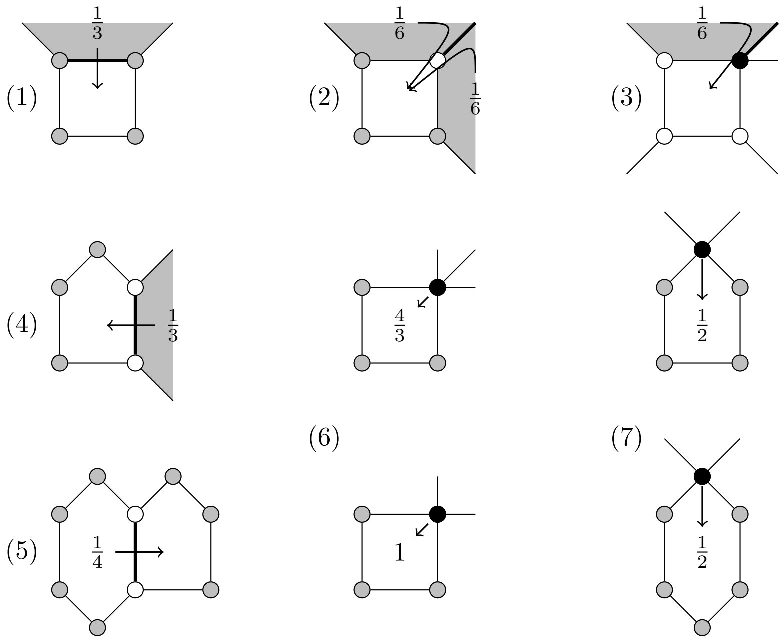


Figure 5.2: The discharging Rules 1–7. A black vertex denotes a big vertex, a white vertex denotes a non-precolored 3-vertex, and a gray vertex can be of any degree in G. A thick edge is used for transferring charge and a gray face is a (≥ 9)-face.

5.2.4 Final charges

We use $\mathrm{ch}^*(x)$ to denote the final charge of a vertex or face x. Next we show that the final charge of every vertex and face is non-negative, thus establishing the theorem.

Let v be a vertex of degree d of G. If v is not big, then its initial charge is zero, and no charge is sent or received by it, hence its final charge is zero as well. Therefore, assume that v is big. If $d = 3$, then v is incident with h, hence its initial charge is 2, As v sends charge of at most 1 to each of the two incident faces distinct from h, its final charge is nonnegative. Therefore, assume that $d \geq 4$.

The vertex v sends charge by Rules 6 and 7 to 4-, 5-, and 6-faces. Let a be the number of 4-faces distinct from h incident with v. Let b be the number of 5-faces and 6-faces (other than h) incident with v. The final charge of v is

$$\mathrm{ch}^*(v) \;\geq\; 2d - 6 - \frac{4}{3}a - \frac{1}{2}b.$$

If $a = 0$, then the final charge of v is at least $2d - 6 - \frac{1}{2}b \geq \frac{3d}{2} - 6 \geq 0$. Suppose now that $a > 0$. A 4-face distinct from h cannot be adjacent to a 5- or 6-face by Lemma 5.5. Hence if v is not incident with h, there must be at least two (≥ 7)-faces incident with v, and if v is incident with h, then there must be at least one (≥ 7)-face incident with v. In both cases, $a + b \leq d - 2$. The final charge of v is at least $2d - 6 - \frac{4}{3}(a + b) \geq \frac{2d-10}{3}$, which is nonnegative if $d \geq 5$.

Finally, consider the case that $d = 4$. Since $a > 0$, we have $a + b \leq 2$. If v is incident with h, then its initial charge is 4, and the final charge is at least

$4 - \frac{4}{3}(a+b) \geq \frac{4}{3}$. If v is not incident with h, then its initial charge is 2, and it sends at most one to each incident face of length at most 6, thus its final charge is at least $2 - (a+b) \geq 0$. We conclude that the final charge of each vertex is nonnegative.

Let f be an arbitrary face of G. If f is the outer face h, then $\mathrm{ch}^*(h) = \mathrm{ch}(h) = 0$. Therefore, we assume that $f \neq h$.

We consider the following cases regarding $\ell(f)$:

$\ell(f) \geq 9$: We show that f *sends charge of at most* $1/3$ *through each of its edges.* Then,

$$\mathrm{ch}^*(f) \geq \ell(f) - 6 - \frac{\ell(f)}{3} \geq \frac{2\ell(f)}{3} - 6 \geq 0.$$

Let $e = uv$ be an edge of f and let g be the face incident with e distinct from f. If $g = h$, then no charge is sent through e, hence assume that $g \neq h$. Note that if f sends charge through e only once, then this charge is at most $1/3$. We consider the following subcases regarding the size of g:

- $\ell(g) = 4$ *and g is incident with only one big vertex:* f sends charge $1/3$ to g through e by Rule 1. The face f can send further charge through e only by Rule 3. Then, we may assume that v is a 4-vertex, vw is an edge of f and it is incident with some 4-face g' for which v is also the only big incident vertex, and no vertex of g' is precolored. As v is the only big vertex of g, no vertex of g is precolored as well. But then g and g' form a reducible configuration, by Lemma 5.4(3).
- $\ell(g) = 4$ *and g is incident with more than one big vertex:* then the charge is sent through e only by Rule 3, for the total of at most $1/6 + 1/6 = 1/3$.
- $\ell(g) = 5$: In this case, f sends either at most $1/3$ through e by Rule 4 (if both u and v have degree three) or at most twice $1/6$ by Rule 3 (if u or v have degree four).
- $\ell(g) = 6$: The face f sends at most twice $1/6$ through e by Rule 3.
- $\ell(g) \geq 9$: The charge of $1/6$ is sent at most twice through e by Rule 2 or Rule 3.

This case analysis establishes the claim.

If $\ell(f) \leq 6$, then the boundary of f is a cycle, thus if f contains a precolored vertex of degree two, then it contains at least two precolored vertices of degree at least three, and these two vertices are big. Similarly, if $\ell(f) \leq 6$ and f is incident with a precolored vertex of degree three, then f contains at least two big vertices.

$\ell(f) = 6$: By Lemma 5.4(2), f cannot consist of only non-precolored 3-vertices, thus f contains a big vertex v. The face f receives $1/2$ from v by Rule 7, and at most twice sends $1/4$ by Rule 5 (as two 5-faces distinct from h cannot share an edge by Lemma 5.6 and f contains a big vertex). Therefore, $\mathrm{ch}^*(f) \geq 0 + 1/2 - 2/4 = 0$.

$\ell(f) = 5$**:** The face f has initial charge -1 and it sends no charge. By Lemmas 5.5 and 5.6, f is not adjacent to any face of length at most 5 distinct from h. We consider several possibilities regarding the number of big vertices incident with f.

If f contains at least two big vertices, then Rule 7 applies twice, and thus $\mathrm{ch}^*(f) \geq -1 + 2/2 = 0$.

If f contains one big vertex v, then no vertex of f except possibly for v is precolored. Note that Rule 7 applies once. Moreover, f contains three edges whose endvertices are non-precolored vertices of degree 3. The charge is received by f through these three edges by Rules 4 and 5. Thus, $\mathrm{ch}^*(f) \geq -1 + 1/2 + 3/4 > 0$.

If f is incident with no big vertex, then all its vertices are of degree 3 and are not precolored. Then, f receives charge by Rules 4 and 5 through each incident edge, and $\mathrm{ch}^*(f) \geq -1 + 5/4 > 0$.

$\ell(f) = 4$**:** By Lemma 5.4(2), the face f must contain a big vertex. If f contains at least two big vertices, then Rule 6 applies twice, and $\mathrm{ch}^*(f) \geq -2 + 2 = 0$. Therefore, we may assume that f is incident with exactly one big vertex v. In particular, no vertex of f other than v is precolored, and if v is precolored, then $\deg(v) \geq 4$.

If at most one edge of f is shared with another 4-face, then at least three edges of f are incident with faces of size at least 9 by Lemma 5.5. After applying Rule 6 and three times Rule 1, we obtain $\mathrm{ch}^*(f) \geq -2+1+3/3 = 0$. By Lemma 5.5, the 4-face f cannot share three edges with other 4-faces. Therefore, we may assume that f shares exactly two edges with other 4-faces f_1 and f_2, and the three 4-faces surround a 3-vertex y. Note that $v \neq y$, otherwise, v is precolored and hence f contains at least two big vertices.

If v is incident with f_1 or f_2, then Rule 6, twice Rule 1 and twice Rule 2 apply and $\mathrm{ch}^*(f) \geq -2 + 1 + 2/3 + 2/6 = 0$. Now assume that v is not adjacent to any of the other two 4-faces. If v is precolored or $\deg(v) \geq 5$, then Rule 6 and twice Rule 1 apply and $\mathrm{ch}^*(f) \geq -2 + 4/3 + 2/3 = 0$. Finally, if v is a non-precolored 4-vertex, then Rule 6, twice Rule 1, and twice Rule 3 apply, and we infer that $\mathrm{ch}^*(f) \geq -2 + 1 + 2/3 + 2/6 = 0$.

□

5.3 Graphs with constraints on 4-cycles

This section is based on paper [12].

5.3.1 Introduction

In this section we prove the following theorem:

Theorem 5.7. *Any planar triangle-free graph without* 4*-cycles adjacent to* 4*- and* 5*-cycles is* 3*-choosable.*

Let us point out the result of Li [33], strengthening the result of Thomassen [44]: every planar triangle-free graph such that no 4-cycle shares a **vertex** with another 4- or 5-cycle is 3-choosable. Theorem 5.7 is forbidding only 4-cycles sharing an **edge** with other 4- or 5-cycles: Hence it is also strengthening the result of Li [33]. Recently, Guoa and Wang [26] published an alternative proof of Theorem 5.7.

Moreover, we obtain the following corollary is a direct consequence.

Corollary 5.8. *Any planar graph without* 3*-,* 6*- and* 7*-cycles is* 3*-choosable.*

Using the proof technique of precoloring extension developed by Thomassen [44], we show the following extension of Theorem 5.7:

Theorem 5.9. *Let* G *be a triangle-free planar graph without* 4*-cycles adjacent to* 4*- and* 5*-cycles, with outer face* C*, and* P *a path of length at most three such that* $V(P) \subseteq V(C)$*. The graph* G *can be* L*-colored for any list assignment* L *such that*

- $|L(v)| = 3$ *for all* $v \in V(G) \setminus V(C)$*;*
- $2 \leq |L(v)| \leq 3$ *for all* $v \in V(C) \setminus V(P)$*;*
- $|L(v)| = 1$ *for all* $v \in V(P)$*, and the colors in the lists give a proper coloring of the subgraph of* G *induced by* $V(P)$*;*
- *the vertices with lists of size two form an independent set; and*
- *each vertex with lists of size two has at most one neighbor in* P*.*

Note that we view the single-element lists as a precoloring of the vertices of P. Also, P does not have to be a part of the facial walk of C, as we only require $V(P) \subseteq V(C)$.

5.3.2 Short outer face corollary

Theorem 5.9 has the following easy consequence about short outer face. We give it before the proof of Theorem 5.9 as it comes handy during induction.

Corollary 5.10. *Let* G *be a triangle-free planar graph without* 4*-cycles adjacent to* 4*- and* 5*-cycles, with the outer face bounded by an induced cycle* C *of length at most* 9*. Furthermore, assume that*

- *if $\ell(C) = 8$, then at least one edge of C does not belong to a 4-cycle; and*
- *if $\ell(C) = 9$, then some two consecutive edges of C do not belong to 4- and 5-cycles.*

Let L be an assignment of lists of size 1 to the vertices of C and lists of size 3 to the other vertices of G. If L prescribes a proper coloring of C, then G can be L-colored.

Proof. The claim follows from Theorem 5.9 for $\ell(C) = 4$. If $\ell(C) \in \{5, 6, 7\}$, then let $u_1w_1vw_2u_2$ be an arbitrary subpath of C. Let L' be the list assignment obtained from L by removing the color $L(v)$ from the lists of vertices adjacent to v. We also set the lists of w_1 and w_2 to 2-lists such that the precoloring of the other vertices of C forces the prescribed color $L(w_1)$ on w_1 and $L(w_2)$ on w_2, i.e., $L'(w_1) = L(w_1) \cup L(u_1)$ and $L'(w_2) = L(w_2) \cup L(u_2)$. As all the vertices x with $|L'(x)| = 2$ are neighbors of a single vertex v, the graph $G - v$ together with the list assignment L' satisfies the assumptions of Theorem 5.9. It follows that we can L'-color $G - v$, giving an L-coloring of G.

Let us now consider the case that $\ell(C) = 8$, and let $C = w_1uvw_2r_1r_2r_3r_4$, where the edge uv does not belong to a 4-cycle. Let us delete vertices u and v from G, remove the color in $L(u)$ from the lists of neighbors of u and the color in $L(v)$ from the lists of neighbors of v, and change the list of w_1 to $L(w_1) \cup L(r_4)$ and the list of w_2 to $L(w_2) \cup L(r_1)$, so that the precoloring of the path $P = r_1r_2r_3r_4$ forces the right colors on w_1 and w_2. As uv does not belong to a 4-cycle, the vertices with lists of size two form an independent set. As C is induced, both w_1 and w_2 have only one neighbor in the 3-path P. Let x be a neighbor of u other than v and w_1. The vertex x cannot be adjacent to both r_1 and r_4, as the 4-cycle uxr_4w_1 would be adjacent to a 5-cycle $xr_1r_2r_3r_4$. Similarly, x cannot be adjacent to both r_1 and r_3 or both r_2 and r_4. As G does not contain triangles, x has at most one neighbor in P. By symmetry, this is also true for the neighbors if v. Therefore, the graph satisfies assumptions of Theorem 5.9, and can be colored from the prescribed lists.

Finally, suppose that $\ell(C) = 9$, and let $C = w_1uvww_2r_1r_2r_3r_4$, where the edges uv and vw are not incident with 4- and 5-cycles. We argue similarly as in the previous case. We delete vertices u, v and w from G and remove their colors from the lists of their neighbors. We also set the list of w_1 to $L(w_1) \cup L(r_4)$ and the list of w_2 to $L(w_2) \cup L(r_1)$, so that the precoloring of the path $r_1r_2r_3r_4$ forces the right colors on w_1 and w_2. Observe that the resulting graph satisfies assumptions of Theorem 5.9, hence it can be colored. □

5.3.3 Proof of Theorem 5.9

Before we proceed with the proof of Theorem 5.9, let us describe the notation that we use in figures. We mark the precolored vertices of P by full circles, the vertices with list of size three by empty circles, and the vertices with list of size two by empty squares. The vertices for that the size of the list is not uniquely determined in the situation demonstrated by the particular figure are marked by crosses.

Proof of Theorem 5.9. Suppose G together with lists L is the smallest counterexample, i.e., such that $|V(G)| + |E(G)|$ is minimal among all graphs that satisfy the assumptions of Theorem 5.9, but cannot be L-colored, and $\sum_{v \in V(G)} |L(v)|$ is minimal among all such graphs. Let C be the outer face of G and P a path with $V(P) \subseteq V(C)$ as in the statement of the theorem. We first derive several properties of this counterexample. Note that each vertex v of G has degree at least $|L(v)|$.

Lemma 5.11. *The graph G does not contain separating cycles of length at most seven. Every edge of each separating* 8*-cycle K belongs to a* 4*-cycle lying inside K. And, at least one of every two consecutive edges of each separating* 9*-cycle K belongs to a* 4*- or* 5*-cycle lying inside K.*

Proof. Let K be the separating cycle. We may assume that K is induced, as otherwise we could consider a shorter separating cycle of length at most 7. Let G_1 be the subgraph of G induced by the exterior of K (including K) and G_2 the subgraph of G induced by the interior of K (including K). By the minimality of G, Theorem 5.9 holds for G_1 and G_2 and their subgraphs. Therefore, there exists a coloring of G_1 from the prescribed lists, and this coloring can be extended to G_2 by Corollary 5.10. This is a contradiction, as G cannot be colored from the lists. $\square$

A *chord* of a cycle K is an edge in G joining two distinct vertices of K that are not adjacent in K. As G does not have triangles and 4-cycles adjacent to 4- and 5-cycles, a cycle of length at most 7 does not have a chord. Therefore, Lemma 5.11 implies that every cycle of length at most 7 is a face. Similarly, a cycle K of length 8 with an edge that does not belong to a 4-cycle in the interior of K is either an 8-face, or it has a chord splitting it to a 4-face and a 6-face, or two 5-faces.

Lemma 5.12. *The graph G is* 2*-connected.*

Proof. Obviously, G is connected. Suppose now that v is a cut vertex of G and G_1 and G_2 are nontrivial induced subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$. Both G_1 and G_2 satisfy the assumptions of Theorem 5.9. If v is precolored, then by the minimality of G there exist L-colorings of G_1 and G_2, and they combine to a proper L-coloring of G. If v is not precolored, then we may assume that $P \subseteq G_1$. An L-coloring of G_1 assigns a color c to v. We change the list of v to $\{c\}$, color G_2 and combine the colorings to an L-coloring of G. $\square$

By Lemma 5.12, C is a cycle. A *k-chord* of C is a path $Q = q_0 q_1 \ldots q_k$ of length k joining two distinct vertices of C, such that $V(C) \cap V(Q) = \{q_0, q_k\}$ (e.g., 1-chord is just a chord).

Lemma 5.13. *The cycle C has no chords.*

Proof. Suppose $e = uv$ is a chord of C, separating G to two subgraphs G_1 and G_2 intersecting in e. If both u and v are precolored, then we L-color G_1 and G_2 by the minimality of G and combine their colorings. Otherwise, by symmetry

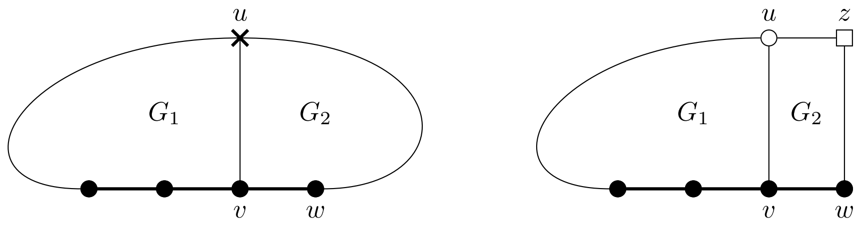


Figure 5.3: A chord of C

assume that $u \notin V(P)$, and that $|V(P) \cap V(G_1)| \geq |V(P) \cap V(G_2)|$. In particular, $|(V(P) \cap V(G_2)) \setminus \{u, v\}| \leq 1$. Furthermore, let us choose the chord in such a way that G_2 is as small as possible; in particular, the outer face of G_2 does not have a chord. Let us find an L-coloring of G_1 and change the lists of u and v to the colors assigned to them. If G_2 with these new lists satisfies assumptions of Theorem 5.9, then we find its coloring and combine the colorings to an L-coloring of G, hence assume that this is not the case.

Let $X = (V(P) \cap V(G_2)) \setminus \{u, v\}$. As G_2 does not satisfy assumptions of Theorem 5.9, there exists a vertex z with list of size two adjacent to two precolored vertices. As G is triangle-free, we conclude that X is not empty, say $X = \{w\}$ (see Figure 5.3), and z is adjacent to u and w. As G_2 does not contain chords and separating 4-cycles and $z \in V(C)$, G_2 is equal to the cycle $uvwz$. Since $|L(z)| = 2$, it holds that $|L(u)| = 3$. Let c_1 be the color of u in the coloring of G_1, and c_2 the single color in the list of w. If $L(z) \neq \{c_1, c_2\}$, then we can color z and finish the coloring of G, hence assume that $L(z) = \{c_1, c_2\}$. Let c be a color in $L(u) \setminus (\{c_1\} \cup L(v))$ (this set is nonempty, as $|L(v)| = 1$ and $|L(u)| = 3$).

Let us now color z by c_1 and set the list of u to $\{c\}$. If G_1 with this list at u satisfies assumptions of Theorem 5.9, we can color G_1, and thus obtain an L-coloring of G. Since G does not have such an L-coloring, the assumptions are violated, i.e., either u is adjacent to a vertex of P other than v, or G_1 contains a vertex (with list of size two) adjacent to both u and a vertex of P. This is a contradiction, as G would in both of these cases contain either a triangle, or a 4- or 5-cycle adjacent to the 4-cycle $uvwz$. □

By the previous lemma, P is a part of the facial walk of C, and C is an induced cycle.

Lemma 5.14. *$\ell(C) \geq 8$.*

Proof. Suppose that $\ell(C) \leq 7$. If $V(C) \neq V(P)$, then color the vertices of C properly from their lists. This can be done, as C is chordless and contains at least one vertex with list of size three. If $5 \leq \ell(C) \leq 7$, then the claim follows from the proof of Corollary 5.10, as by the minimality of G, all subgraphs of G satisfy Theorem 5.9. If $\ell(C) = 4$, then we delete one of the vertices of C and remove its color from the lists of its neighbors. It is easy to verify that the resulting graph satisfies the assumptions of Theorem 5.9, hence it has a proper coloring by the minimality of G. This coloring extends to an L-coloring of G, which is a contradiction. □

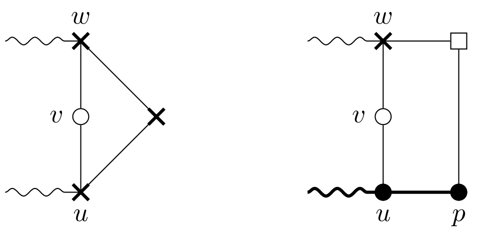


Figure 5.4: Possible 2-chords in G

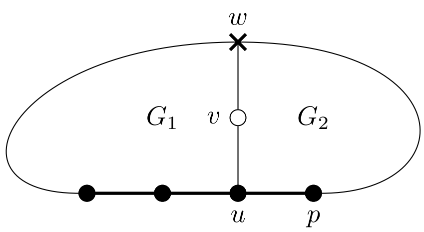


Figure 5.5: A 2-chord of C

Lemma 5.15. *No* 4*-cycle shares an edge with another* 4*- or* 5*-cycle.*

Proof. Suppose that $C_1 = v_1v_2v_3v_4$ and $C_2 = v_1v_2u_3 \dots u_t$ are cycles sharing the edge v_1v_2, $\ell(C_1) = 4$ and $t = \ell(C_2) \in \{4, 5\}$. Note that $C_1 \neq C$ and $C_2 \neq C$ by Lemma 5.14. By Lemma 5.11, both C_1 and C_2 bound a face. If $v_3 = u_3$, then v_2 would be a 2-vertex with list of size three. Thus, $v_3 \neq u_3$ and by symmetry, $v_4 \neq u_t$. As G does not contain triangles, $v_3 \neq u_t$ and $v_4 \neq u_3$, and in case that $t = 5$, $v_3 \neq u_4$ and $v_4 \neq u_4$. Therefore, C_1 and C_2 are adjacent, contradicting the assumptions of Theorem 5.9. □

Note that we can assume that $|V(P)| = 4$, as otherwise we can prescribe color for more of the vertices of C, without violating assumptions of Theorem 5.9. Let $P = p_1p_2p_3p_4$. We say that a k-chord Q of C *splits off* a face F from G if $F \neq C$ is a face of both G and $C \cup Q$. See Figure 5.4 for an illustration of 2-chords splitting off a face.

Lemma 5.16. *Every* 2*-chord uvw of C splits off a k-face F such that*

(a) $|V(F) \cap V(P)| \leq 2$ *and* $\{u, w\} \not\subseteq V(P)$*,*

(b) $k \leq 5$*, and*

(c) *if* $|V(F) \cap V(P)| \leq 1$*, then* $k = 4$*.*

In particular, the cycle C has no 2*-chord with* $|L(w)| = 2$ *and* $u \neq p_2, p_3$*.*

Proof. Suppose first that $u, w \in V(P)$. By Lemma 5.11, the 2-chord uvw together with a part of P bounds a face K. Color v by a color different from the colors of u and w, and remove $V(K) \setminus \{u, v, w\}$ from G, obtaining a graph G'. Note that a path of length at most three is precolored in G'. Since G cannot be L-colored, we may assume that G' does not satisfy assumptions of Theorem 5.9, i.e., there exists z with $|L(z)| = 2$ adjacent to both v and a vertex $y \in V(P) \cap V(G')$. As G is triangle-free, $y \notin \{u, w\}$. It follows that $yuvz$ or $ywvz$ is a 4-face. This is a contradiction, as K would be an adjacent 4-face. Therefore, $\{u, w\} \not\subseteq V(P)$, and by symmetry we assume that $w \notin V(P)$.

The 2-chord uvw splits G to two subgraphs G_1 and G_2 intersecting in uvw. Let us choose G_2 such that $|V(P) \cap V(G_2)| \leq |V(P) \cap V(G_1)|$, see Figure 5.5. Note that $|V(P) \cap V(G_2)| \leq 2$. Let us consider the 2-chord uvw such that $|V(P) \cap V(G_2)|$ is minimal, subject to the assumption that G_2 is not a face. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment for G_2 such that $L'(u) = \{\varphi(u)\}$, $L'(v) = \{\varphi(v)\}$, $L'(w) = \{\varphi(w)\}$ and $L'(x) = L(x)$ for $x \in V(G_2) \setminus \{u, v, w\}$. Let P' be the precolored path in G_2 (consisting of u, v, w, and possibly one other vertex p of P adjacent to u). As C has no chords and G_2 is not a face, P' is an induced subgraph. Since G cannot be L-colored, we conclude that G_2 cannot be L'-colored, and thus G_2 with the list assignment L' does not satisfy the assumptions of Theorem 5.9. Therefore, there exists a vertex z with $|L(z)| = 2$, adjacent to two vertices of P'.

Since G_2 is not a face, Lemmas 5.11 and 5.13 imply that z is not adjacent to both w and p. Similarly, z is not adjacent to both u and w. It follows that z is adjacent to v and p, and thus $|V(P) \cap V(G_2)| = 2$. Since we have chosen the 2-chord uvw so that $|V(P) \cap V(G_2)| = 2$ is minimal among the 2-chords for that G_2 is not a face, the 2-chord wvz splits off a face F' from G. Let x be the neighbor of z in F' other than v. Since $|L(z)| = 2$, it holds that $|L(x)| = 3$. As F' is a face, $\deg(x) = 2$, which is a contradiction. It follows that for every 2-chord, G_2 is a face. The choice of G_2 establishes (a).

Let $wvuv_4 \dots v_k$ be the boundary of the face G_2. Note that $V(P) \cap V(G_2) \subseteq \{u, v_4\}$, and $v_4, \dots, v_k$ have degree two. If $k > 5$, then at least one of v_5 and v_6 has list of size three, which is a contradiction, proving (b). Similarly, if $|V(F) \cap V(P)| \leq 1$ and $k = 5$, then at least one of v_4 and v_5 would be a 2-vertex with list of size three, proving (c).

Consider now a 2-chord uvw such that $|L(w)| = 2$ and $u \notin \{p_2, p_3\}$, and let x be the neighbor of w in G_2 distinct from v. As $u \notin \{p_2, p_3\}$, no vertex of $V(P) \setminus \{u\}$ lies in G_2. Therefore, $|L(x)| = 3$ and $\deg(x) = 2$, a contradiction. We conclude that no such 2-chord exists. □

Let us now consider the 3-chords of C:

Lemma 5.17. *Every* 3*-chord* $Q = uvwx$ *of* C *such that* $u, x \notin \{p_2, p_3\}$ *splits off a* 4*- or* 5*-face.*

Proof. Suppose that Q splits G into two subgraphs G_1 and G_2 intersecting in $uvwx$, such that $V(P) \cap V(G_2) \subseteq \{u, x\}$. Let us L-color G_1 and consider the vertices u, v, w and x of G_2 as precolored according to this coloring. If ux were an edge, then Q would split off a 4-face. It follows that Q is an induced path thus this precoloring of Q is proper. Similarly, as Q does not split off a 5-face, u and x do not have a common neighbor with list of size two. Neither v nor w is adjacent to a vertex with list of size 2 by Lemma 5.16. Therefore, G_2 satisfies

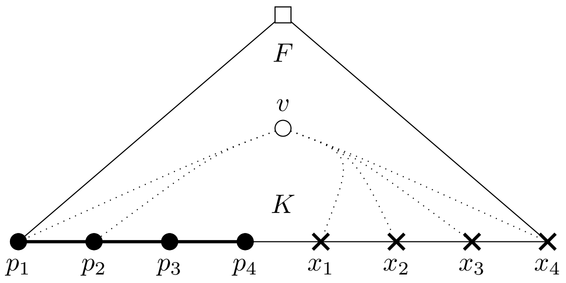


Figure 5.6: A 2-chord from p_1 or p_2 to $\{x_1, x_2, x_3, x_4\}$

assumptions of Theorem 5.9, and the coloring can be extended to G_2, giving an L-coloring of G. This is a contradiction. $\square$

5.3.4 Vertices in the outer face

Let $x_1x_2x_3x_4$ be the part of the facial walk of C such that x_1 is adjacent to p_4 and $x_2 \neq p_4$. By Lemma 5.14, $\{x_1, x_2, x_3, x_4\} \cap V(P) = \emptyset$. Let us now show a few properties of the vertices x_1, x_2, x_3, x_4 and their neighbors.

Lemma 5.18. *Let $Q = v_0v_1 \ldots v_k$ be a k-chord starting and ending in vertices of $x_1x_2x_3x_4$, or a cycle intersecting C in a single vertex $x \in \{x_1, x_2, x_3, x_4\}$. The following holds (for some $i \in \{1, 2, 3, 4\}$):*

- *If $\ell(Q) = 2$, then $Q = x_iv_1x_{i+2}$ splits off a 4-face.*
- *If $\ell(Q) = 3$, then Q splits off either a 4-face $x_ix_{i+1}v_1v_2$, or a 5-face $x_ix_{i+1}x_{i+2}v_1v_2$.*
- *If $\ell(Q) = 4$, then Q forms a boundary of a 4-face $x_iv_1v_2v_3$, or splits off a 5-face $x_ix_{i+1}v_1v_2v_3$, or splits off a 6-face $x_ix_{i+1}x_{i+2}v_1v_2v_3$.*

Proof. By a simple case analysis. The details are left to the reader. $\square$

Note also that if Q splits off a face of form $x_ix_{i+1}x_{i+2}v_1 \ldots v_{k-1}$, then $\deg(x_{i+1}) = |L(x_{i+1})| = 2$.

Lemma 5.19. *If Q is a k-chord with $k \leq 3$, starting in a vertex x_i (where $1 \leq i \leq 4$) and ending in a vertex with list of size two, then $k = 3$ and Q bounds a 4-face.*

Proof. Let $Q = q_0q_1 \ldots q_k$, where $q_0 \in \{x_1, x_2, x_3, x_4\}$ and $|L(q_k)| = 2$. By Lemmas 5.13 and 5.16, $k > 2$. If $k = 3$, then by Lemma 5.17, Q splits off a 4- or 5-face. However, the latter is impossible, as $|L(q_3)| = 2$, so the remaining vertex of the 5-face, whose degree is two, would have a list of size three. $\square$

Lemma 5.20. *There is no 2-chord from $\{p_1, p_2\}$ to $\{x_1, x_2, x_3, x_4\}$.*

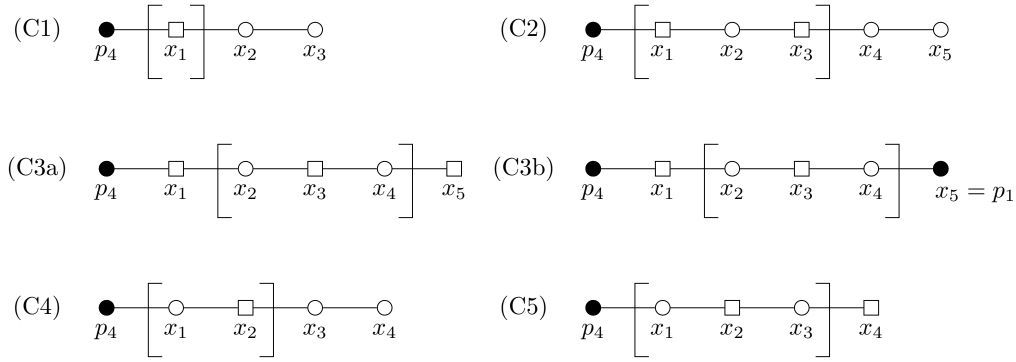


Figure 5.7: The construction of the set X_1

Proof. Suppose $Q = p_i v x_j$ is such a 2-chord, and let K be the cycle formed by Q and $p_i \dots p_4 x_1 \dots x_j$. Note that $\ell(K) \le 9$. Let us choose Q such that $\ell(K)$ is minimal. By Lemma 5.16, Q splits off a face F such that $\ell(F) \le 5$. Furthermore, if $\ell(K) = 9$, then $i = 1$, and hence $|V(P) \cap V(F)| = 1$. In that case, the claim (c) of Lemma 5.16 implies $\ell(F) = 4$. See Figure 5.6 for illustration. It follows that the edges $p_i v$ and $v x_j$ are not incident with a 4-face inside K, and if $\ell(K) = 9$, then they are not incident with a 5-face. By Lemma 5.11, K is not separating. If $\ell(K) \le 7$, then K is a face, and $\deg(v) = 2$, which is a contradiction. Similarly, if $\ell(K) > 7$, then K has a chord incident with v. By the minimality of $\ell(K)$, v is adjacent to p_3 or p_4. However, this contradicts Lemma 5.16(a). $\square$

If both x_1 and x_2 have lists of size three, then we remove one color from $L(x_1)$ and find a coloring by the minimality of L (note that x_1 is not adjacent to any vertex with list of size two, and has only one neighbor in P, as C does not have chords). Therefore, exactly one of x_1 and x_2 has a list of size two. Let x_5 be the neighbor of x_4 in C distinct from x_3. We now distinguish several cases depending on the lists of vertices in $\{x_1, x_2, x_3, x_4\}$, in order to choose a set $X_1 \subseteq \{x_1, x_2, x_3, x_4\}$ of vertices that we are going to color (and remove).

(C1) If $|L(x_1)| = 2$ and $|L(x_2)| = |L(x_3)| = 3$ (see Figure 5.7(1)), then we set $X_1 = \{x_1\}$.

(C2) If $|L(x_1)| = 2$, $|L(x_2)| = 3$, $|L(x_3)| = 2$, $|L(x_4)| = 3$ and $|L(x_5)| = 3$ (see Figure 5.7(2)), then we set $X_1 = \{x_1, x_2, x_3\}$.

(C3) If $|L(x_1)| = 2$, $|L(x_2)| = 3$, $|L(x_3)| = 2$, $|L(x_4)| = 3$ and $|L(x_5)| \le 2$ (see Figure 5.7(3)), then we set $X_1 = \{x_2, x_3, x_4\}$.

(C4) If $|L(x_1)| = 3$, $|L(x_2)| = 2$, $|L(x_3)| = 3$ and $|L(x_4)| = 3$ (see Figure 5.7(4)), then we set $X_1 = \{x_1, x_2\}$.

(C5) If $|L(x_1)| = 3$, $|L(x_2)| = 2$, $|L(x_3)| = 3$ and $|L(x_4)| = 2$ (see Figure 5.7(5)), then we set $X_1 = \{x_1, x_2, x_3\}$.

Let $m = \max\{i : x_i \in X_1\}$. Note the following properties of the set X_1:

- $|X_1| \le 3$.

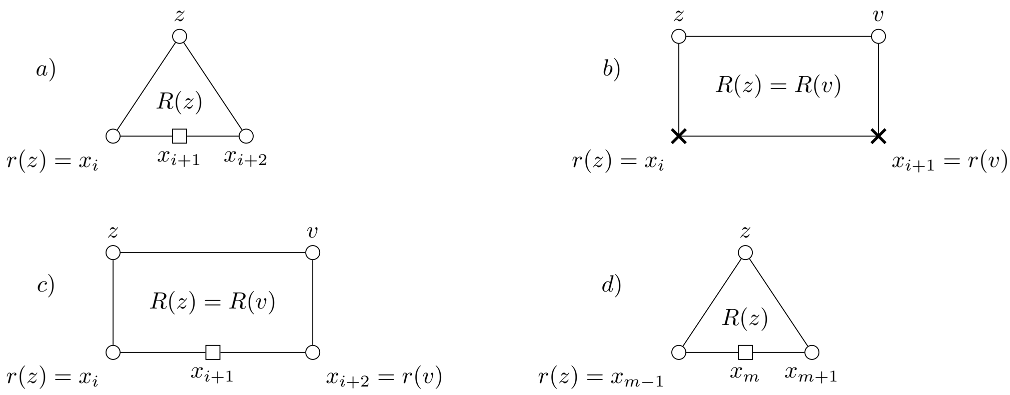


Figure 5.8: The construction of the set X_2

- If $|L(x_m)| = 2$, then $m \leq 3$ and $|L(x_{m+1})| = |L(x_{m+2})| = 3$.
- If $|L(x_m)| = 3$, then $|L(x_{m+1})| \leq 2$.

Let $\mathcal{F}$ be the set of faces of G incident with the edges of the path induced by X_1 ($\mathcal{F} = \emptyset$ in the case (C1)). We define a set $X_2 \subseteq V(G) \setminus V(C)$, together with functions $r : X_2 \to X_1$ and $R : X_2 \to \mathcal{F}$. A vertex $z \in V(G) \setminus V(C)$ belongs to X_2 if

- *z is adjacent to two vertices in X_1* (see Figure 5.8(a) for an example). By Lemma 5.18, z lies in a (uniquely determined) 4-face $F = x_i x_{i+1} x_{i+2} z$, where $x_i, x_{i+1}, x_{i+2} \in X_1$. We define $r(z) := x_i$ and $R(z) := F$. Or,
- *there exists a path $xzvy$ such that $x, y \in X_1$ and $v \notin \{p_1\} \cup X_1$* (see Figure 5.8(b), (c) and (d) for examples). If $v = x_{m+1}$, then by Lemma 5.16, the 2-chord xzv splits off a 4-face F. Otherwise the 3-chord $xzvy$ splits off a 4- or 5-face F by Lemma 5.18. We define $r(z) := x$ and $R(z) := F$. Note that $v \neq x_1$: otherwise, $x_1 \notin X_1$ and we are in case (C3), hence $|L(x_1)| = 2$ and the 2-chord $x_1 z x$ would contradict Lemma 5.16. It follows that v also belongs to X_2, unless $v = x_{m+1}$.

  Let us now show that $r(z)$ and $R(z)$ are well-defined. As a 4-face cannot be adjacent to a 4- or 5-face and G is triangle-free, z does not have another neighbor in X_1. Also, if there existed another path $xzv'y'$ with $y' \in X_1$ splitting off a face F', then both F and F' would be 5-faces; however, that would imply $|X_1| \geq 5$, which is a contradiction. Therefore, r and R are defined uniquely. Furthermore, v is the only neighbor of z in X_2, and $R(v) = R(z)$ (assuming that $v \neq x_{m+1}$).

We now find an L-coloring of $X_1 \cup X_2$ that we aim to extend to a coloring of G.

Lemma 5.21. *Let $H = G[V(P) \cup X_1 \cup X_2]$ be the subgraph of G induced by $V(P) \cup X_1 \cup X_2$. There exist an L-coloring φ_1 of X_1 and an L-coloring φ_2 of X_2 such that*

- *the coloring of H given by φ_1, φ_2 and the precoloring of P is proper,*

- *if $|L(x_{m+1})| \leq 2$, then $\varphi_1(x_m) \notin L(x_{m+1})$,*
- *if $x_1 \notin X_1$ (i.e., in the case (C3) of the definition of X_1), then $L(x_1) \neq L(p_4) \cup \{\varphi_1(x_2)\}$, and*
- *if $z \in X_2$ is adjacent to x_{m+1}, then $|L(x_{m+1}) \setminus \{\varphi_1(x_m), \varphi_2(z)\}| \geq 2$.*

Proof. Suppose first that there exists $z \in X_2$ adjacent to x_{m+1}. Note that z is unique, $m \geq 2$ and $R(z) = x_{m-1}x_mx_{m+1}z$ is a 4-face. As G does not contain a 2-vertex with list of size three, $|L(x_m)| = 2$ and $|L(x_{m-1})| = |L(x_{m+1})| = 3$. This happens only in the cases (C2) and (C4) of the definition of X_1, thus $x_1 \in X_1$ and $m \leq 3$. Furthermore, x_{m-1} is the only neighbor of z in X_1 and z is not adjacent to any other vertex of X_2. As $R(z)$ is a 4-face and G does not contain 4-cycles adjacent to 4- or 5-cycles, z is not adjacent to p_3 and p_4. By Lemma 5.20, z is not adjacent to p_1 and p_2, either, thus any choice of the color for z is consistent with the precoloring of P. Let us distinguish the following cases:

- If $L(z) \cap L(x_m) \neq \emptyset$, then choose $c \in L(z) \cap L(x_m)$ and let $\varphi_1(x_m) = \varphi_2(z) = c$.
- If $L(z) \neq L(x_{m+1})$, then choose $\varphi_2(z) \in L(z) \setminus L(x_{m+1})$ and $\varphi_1(x_m) \in L(x_m)$ arbitrarily.
- Finally, consider the case that $L(z) \cap L(x_m) = \emptyset$ and $L(z) = L(x_{m+1})$, i.e., the lists of x_m and x_{m+1} are disjoint. We choose $\varphi_1(x_m) \in L(x_m)$ and $\varphi_2(z) \in L(z)$ arbitrarily.

On the other hand, suppose that no vertex of X_2 is adjacent to x_{m+1}. If $|L(x_{m+1})| = 2$, then choose $\varphi_1(x_m) \in L(x_m) \setminus L(x_{m+1})$. Otherwise, choose $\varphi_1(x_m) \in L(x_m)$ arbitrarily (in case that $m = 1$, choose a color different from the one in $L(p_4)$)

In both of these cases, the precoloring of x_m (and possibly z) can be extended to a proper coloring ψ of the subgraph induced by $\{x_1, \ldots, x_m, z\}$ consistent with the precoloring of P. We fix φ_1 as the restriction of ψ to X_1.

Let us now construct (the rest of) the coloring φ_2. Consider a vertex $u \in X_2$ that is not adjacent to x_{m+1}. As $u \notin V(C)$, it holds that $|L(u)| = 3$. If u has no neighbor in X_2, then it has two neighbors $r(u), x \in X_1$ and $R(u)$ is a 4-face. We claim that u has no neighbor $p_i \in V(P)$. Otherwise, we obtain $i \geq 3$ by Lemma 5.20. By Lemma 5.16, the 2-chord $p_iur(u)$ splits off a 4- or 5-face. This face shares an edge with $R(u)$, which is a contradiction. Therefore, any choice of $\varphi_2(u) \in L(u) \setminus \{\varphi_1(x), \varphi_1(r(u))\}$ is consistent with the precoloring of P.

Finally, suppose that u has a neighbor $w \in X_2$. As we argued in the definition of X_2, each of u and w has exactly one neighbor in X_1, and u and w do not have any other neighbors in X_2. Also, w is not adjacent to x_{m+1}, as otherwise G would contain a triangle or two adjacent 4-cycles. By Lemma 5.16(a), each of u and w has at most one neighbor in P. If one of them does not have any such neighbor, then we can easily color u and w, hence assume that p_iu and p_jw are edges. By Lemma 5.20, $i, j \geq 3$. Without loss on generality, $j = 3$ and $i = 4$. This is a contradiction, as the 4-face p_3p_4uw shares an edge with $R(u)$. □

Consider the colorings φ_1 and φ_2 constructed in Lemma 5.21. Let $G' = G - (X_1 \cup X_2)$ and let L' be the list assignment such that $L'(v)$ is obtained from $L(v)$ by removing the colors of the neighbors of v in X_1 and X_2 for $v \neq x_1$, and $L'(x_1) = L(x_1)$ if $x_1 \notin X_1$. Suppose that G' with the list assignment L' satisfies assumptions of Theorem 5.9. Then there exists an L'-coloring φ of G', which together with φ_1 and φ_2 gives an L-coloring of G: this is obvious if $x_1 \in X_1$. If $x_1 \notin X_1$, then $|L(x_1)| = 2$, and $L(p_4) \subseteq L(x_1)$ by the minimality of G (otherwise, we could remove the edge p_4x_1). By the choice of φ_1, it holds that $\varphi_1(x_2) \neq \varphi(x_1)$. Since no other vertex of X may be adjacent to x_1 by Lemmas 5.13 and 5.16, φ together with φ_1 and φ_2 is a proper coloring of G. As G is a counterexample to Theorem 5.9, it follows that L' violates assumptions of Theorem 5.9, i.e.,

(a) a vertex $v \in V(G')$ with $|L'(v)| = 2$ is adjacent to two vertices of P; or

(b) $|L'(v)| \leq 1$ for some $v \in V(G') \setminus V(P)$; or

(c) two vertices $u, v \in V(G')$ with $|L'(u)| = |L'(v)| = 2$ are adjacent.

Let us now consider each of these possibilities separately.

(a) *A vertex $v \in V(G')$ with $|L'(v)| = 2$ is adjacent to two vertices of P.* By Lemmas 5.13 and 5.16(a), this is not possible.

(b) $|L'(v)| \leq 1$ *for some $v \in V(G') \setminus V(P)$.* If $|L(x_{m+1})| = 2$, then x_{m+1} does not have a neighbor in X_2 by Lemma 5.16 and hence $|L'(x_{m+1})| = 2$ by the choice of φ_1. If $|L(x_{m+1})| = 3$, then the choice of φ_1 and φ_2 according to Lemma 5.21 ensures $|L'(x_{m+1})| \geq 2$. Therefore, $v \neq x_{m+1}$.

Since G has neither chords nor 2-chords starting in X_1 and ending in a vertex with list of size two, it holds that $|L(v)| = 3$. Therefore, v has at least two neighbors $u_1, u_2 \in X_1 \cup X_2$. If at least one of u_1 and u_2 belonged to X_1, then v would be included in X_2, hence we may assume that $u_1, u_2 \in X_2$.

Consider the path $x_iu_1vu_2x_j$, where $x_i = r(u_1)$ and $x_j = r(u_2)$. We may assume that $i \leq j$. The cycle $x_i \dots x_ju_2vu_1$ has length at most six, thus it bounds a face F. Note that $i = j$, as each of $R(u_1)$ and $R(u_2)$ shares at least one edge with the path induced by X_1 and $F \neq R(u_1) \neq R(u_2) \neq F$. Therefore, F is a 4-face sharing an edge with 4-face $R(u_1)$ (and also with $R(u_2)$), which is a contradiction. Therefore, $|L'(v)| \geq 2$ for every $v \in V(G') \setminus V(P)$.

(c) *Two vertices $u, v \in V(G')$ with $|L'(u)| = |L'(v)| = 2$ are adjacent.* As the vertices with lists of size two form an independent set in G, we may assume that $|L(u)| = 3$. Let y_1 be a neighbor of u in $X_1 \cup X_2$.

Consider first the case that $|L(v)| = 2$. If $u \notin V(C)$, then by Lemma 5.16, $y_1 \notin V(C)$, and thus $y_1 \in X_2$ and $vuy_1r(y_1)$ is a 3-chord. By Lemma 5.19, this 3-chord splits off a 4-face F. Note that $F \neq R(y_1)$, as $u \notin X_2$. This is impossible, as the 4-face F would share an edge with $R(y_1)$. Therefore, $u \in V(C)$, and hence $v \neq x_1$. If $y_1 \in X_2$, then $uy_1r(y_1)$ is a 2-chord, and by Lemma 5.16, it splits off a 4-face adjacent to $R(y_1)$, which is again a contradiction. Assume now that $y_1 \in X_1$. As C does not have chords, it follows that $y_1 = x_m$ and $u = x_{m+1}$. However, in that case $v = x_{m+2}$ and $|L(x_{m+2})| = 2$, which contradicts the choice of X_1.

Consider now the case that $|L(v)| = 3$. Let y_2 be a neighbor of v in $X_1 \cup X_2$. As $u, v \notin X_2$, at least one of y_1 and y_2, say y_1, belongs to X_2. Let us consider the possibilities $y_2 \in X_1$ and $y_2 \in X_2$ separately:

- $y_2 \in X_1$: The cycle formed by $r(y_1)y_1uvy_2$ and a part of the path $x_1x_2x_3x_4$ between $r(y_1)$ and y_2 has length at most six, thus it bounds a face F. Note that $R(y_1)$ shares an edge with F. Let k_1 and k_2 be the number of edges that $R(y_1)$ and F, respectively, share with the path induced by X_1, $k_1 \geq \ell(R(y_1)) - 3 \geq 1$ and $k_2 = \ell(F) - 4 \geq 0$. Since $|X_1| \leq 3$, it holds that $k_1 + k_2 \leq 2$. If $k_1 = 1$, then $R(y_1)$ is a 4-face. Since 4- and 5-faces cannot be adjacent to $R(y_1)$, we obtain $\ell(F) \geq 6$. It follows that $k_2 \geq 2$, which is a contradiction. Similarly, if $k_1 = 2$, then F cannot be a 4-face, hence $\ell(F) \geq 5$ and thus $k_2 \geq 1$. This is again a contradiction.
- $y_2 \in X_2$: Let F be the cycle bounded by $r(y_1)y_1uvy_2r(y_2)$ and the part of the path $x_1x_2x_3x_4$ between $r(y_1)$ and $r(y_2)$. As $\ell(F) \leq 7$, F bounds a face. Note that $R(y_1) \neq R(y_2)$ and $\ell(R(y_1)) = \ell(R(y_2)) = 4$, as each of $R(y_1)$ and $R(y_2)$ shares an edge with the path induced by X_1. Since F shares edges with both $R(y_1)$ and $R(y_2)$, $\ell(F) \geq 6$. It follows that F shares at least one edge with the path induced by X_1 as well. However, this is impossible, since $|X_1| \leq 3$.

Therefore, the assumptions of Theorem 5.9 are satisfied by G' and L'. We conclude that we can find a proper coloring of G, which contradicts the choice of G as a counterexample to Theorem 5.9. $\square$

Chapter 6

Packing coloring

This chapter is based on a paper *The packing chromatic number of infinite product graphs* by Fiala, Klavžar and Lidický [19] and a manuscript *The packing chromatic number of the square lattice is at least 12* Ekstein, Fiala, Holub and Lidický [13].

6.1 Introduction

The concept of packing coloring comes from the area of frequency planning in wireless networks. This model emphasizes the fact that some frequencies are used more sparely than the others.

In graph terms, we ask for a partition of the vertex set of a graph G into disjoint classes $X_1, \ldots, X_k$ (representing frequency usage) according to the following constraints. Each color class X_i should be an *i-packing*, that is, a set of vertices with the property that any distinct pair $u, v \in X_i$ satisfies $\mathrm{dist}(u, v) > i$. Here $\mathrm{dist}(u, v)$ denotes the usual shortest path distance between u and v. Such partition is called a *packing k-coloring*, even though it is allowed that some sets X_i may be empty. The smallest integer k for which there exists a packing k-coloring of G is called the *packing chromatic number* of G and it is denoted by $\chi_\rho(G)$. This concept was introduced by Goddard et al. [25] under the name *broadcast chromatic number*. The term packing chromatic number was later (even if the corresponding paper was published earlier) proposed by Brešar et al. [5].

Sloper [41] followed with a closely related concept, the eccentric coloring. An *eccentric coloring* of a graph is a packing coloring in which a vertex v is colored with a color not larger than the eccentricity of v. His results among others imply that the infinite 3-regular tree has packing chromatic number 7.

The determination of the packing chromatic number is computationally difficult. In particular, it is NP-complete for general graphs [25]. In addition, in the same paper it was also proved that it is NP-complete to decide whether $\chi_\rho(G) \leq 4$. But things are even much worse: Fiala and Golovach showed that determining $\chi_\rho(G)$ is one of few problems that are NP-complete on trees [18].

The following interesting phenomena was the starting point for our investigations. The packing chromatic number of the infinite square lattice $\mathbb{Z}^2$ is finite, more precisely, Goddard et al. [25] showed that it lies between 9 and 23. On the other hand, Finbow and Rall [20] proved that the packing chromatic number of the infinite cubic lattice $\mathbb{Z}^3$ is unbounded. So where does a step from a finite number to the infinity occur? In Section 6.3 we prove that the packing chromatic number is unbounded already on two layers of the square lattice, that is, $\chi_\rho(P_2 \,\square\, \mathbb{Z}^2) = \infty$.

The upper bound for the square lattice was pushed to 17 by Holub and Soukal [28]. We improve the lower bound to 10 without using a computer and to 12 with use of a computer.

Just like square and cubic lattices, the hexagonal lattice $\mathcal{H}$ is important in different applications, for instance in the field of frequency assignment [22], which was the original motivation for packing chromatic cumber. Brešar et al. [5] showed that $6 \le \chi_\rho(\mathcal{H}) \le 8$ and asserted (without a proof) that the actual lower bound is 7. This was later indeed verified, using a computer, by Vesel [46]. In preprint version of [5], they asked if 8 is the correct bound. We prove an upper bound 7. As a consequence we now know that $\chi_\rho(\mathcal{H}) = 7$. We also investigate the situation of the hexagonal lattice with more hexagonal layers and we prove that $\chi_\rho(P_m \,\square\, \mathcal{H}) = \infty$ for every $m \ge 6$. Unlike for the square lattice we show that $\chi_\rho(P_2 \,\square\, \mathcal{H})$ is finite by exhibiting a coloring using 536 colors.

6.2 Density

Our approach on proving lower bounds or that a lattice L cannot be covered by a finite number of packings is based on arguments using the notion of the density of a packing. The idea is, roughly speaking, to assign first a unit area to every vertex of L. Then we redistribute the area to vertices covered by the packing such that areas at vertices from the packing are equal and as large as possible. In this way we can define a density for every vertex from the packing as the reciprocal of the area.

Formally we proceed as follows. Let X_k be a k-packing in L. For every x from L and a positive integer l we denote by $N_l(x)$ the set vertices at distance at most l from x, i.e. $N_l(x) := \{y : y \in L, \mathrm{dist}(x,y) \le l\}$. Observe that for arbitrary vertices u and v of X_k the sets $N_{\lfloor k/2 \rfloor}(u)$ and $N_{\lfloor k/2 \rfloor}(v)$ are disjoint, since the vertices u and v are at distance greater than k.

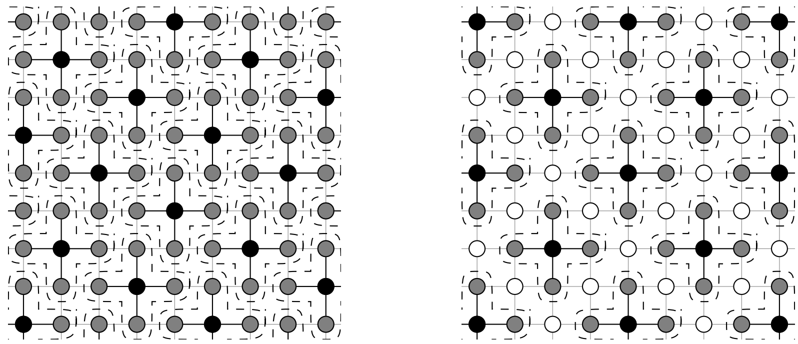
Let k be an odd number, x be a vertex from X_k, and y be a vertex at distance $\left\lceil \frac{k}{2} \right\rceil$ from x. Then there is no vertex from X_k in $N_{\lfloor k/2 \rfloor}(y)$. Hence y is not in $N_{\lfloor k/2 \rfloor}(z)$ of any vertex z from X_k. We redistribute the unit area assigned to y to vertices of X_k by sending the reciprocal of its degree to every of its neighboring sets $N_{\lfloor k/2 \rfloor}(x)$ as follows:

Definition 6.1. The k-*area* $A(x,k)$ assigned to a vertex $x \in V(L)$ is defined by

$$A(x,k) := \begin{cases} |N_{k/2}(x)| & \text{for } k \text{ even}, \\ |N_{\lfloor k/2 \rfloor}(x)| + \displaystyle\sum_{\substack{y \in V(G) \\ \mathrm{dist}(x,y) = \lceil k/2 \rceil}} \frac{|N_1(y) \cap N_{\lfloor k/2 \rfloor}(x)|}{\deg(y)} & \text{for } k \text{ odd}. \end{cases}$$

If the k-area is the same for all vertices of the lattice L we define $A(k) := A(x, k)$, where x is chosen arbitrarily.

By abuse of language we only speak of area instead of k-area if k is clear from the context. See Figure 6.1 for an example of distribution of the area in $\mathbb{Z}^2$. Note that the area $A(k)$ is in particular well-defined for lattices that are vertex transitive.



$A(2) = 5$ $\qquad$ $A(3) = 8$

Figure 6.1: Coverage of $\mathbb{Z}^2$ by X_2 on the left and by X_3 on the right. Vertices from the packings are black. The dotted cross shapes correspond to $N_1(x)$. The white vertices on the right are not covered by any set $N_1(x), x \in X_3$. For every white vertex, each adjoining set $N_1(x)$ receives $\frac{1}{4}$ or $\frac{2}{4}$ of its area, depending on the mutual position.

The definition of the area is justified in the following fundamental observation.

Proposition 6.2. *If a finite graph G has a packing k-coloring and all areas $A(i)$, $1 \le i \le k$, are well-defined, then*

$$\sum_{i=1}^{k} \frac{1}{A(i)} \ge 1 \,.$$

Proof. If G has n vertices then any color class X_i can contain at most $\frac{n}{A(i)}$ vertices. Therefore, $n = |V_G| = |X_1| + \cdots + |X_k| \le \frac{n}{A(1)} + \cdots + \frac{n}{A(k)}$, and the assertion follows. $\square$

Definition 6.3. Let G be a graph. Then the *density* of a set of vertices $X \subset V(G)$ is

$$d(X) := \limsup_{l \to \infty} \max_{x \in V} \left\{ \frac{|X \cap N_l(x)|}{|N_l(x)|} \right\} .$$

The following claim goes immediately:

Observation 6.4. *Let G be a graph and $X \subsetneq V(G)$. Then for every $\varepsilon > 0$ there exists l_0 such that for every vertex $x \in V(G)$ and $l > l_0$, it holds that*

$$\frac{|X \cap N_l(x)|}{|N_l(x)|} < d(X) + \varepsilon \,.$$

We now get an analogue of Proposition 6.2.

Lemma 6.5. *For every finite packing coloring with k classes $X_1, X_2, \dots, X_k$ of a graph G holds that*

$$\sum_{i=1}^{k} d(X_i) \geq d(X_1 \cup X_2) + \sum_{i=3}^{k} d(X_i) \geq d\Big(\bigcup_{i=1}^{k} X_i\Big) = 1\,.$$

Proof. We apply iteratively the following argument that for any vertex x and arbitrarily positive small ε, every sufficiently large l satisfies that

$$\frac{|N_l(x) \cap (X \cup Y)|}{|N_l(x)|} \leq \frac{|N_l(x) \cap X|}{|N_l(x)|} + \frac{|N_l(x) \cap Y|}{|N_l(x)|} \leq d(X) + d(Y) + \varepsilon.$$

□

Let x be a vertex of a graph G. We denote the boundary of $N_l(x)$ by $\Delta N_l(x) := \{y : \mathrm{dist}(y, x) = l\}$.

Lemma 6.6. *If for a graph G the area $A(k)$ is well-defined, and if*

$$\lim_{l \to \infty} \frac{|\Delta N_l(x)|}{|N_l(x)|} = 0\,,$$

then for any k-packing X_k it holds that $d(X_k) \leq \frac{1}{A(k)}$.

Proof. We choose a vertex x arbitrarily and use the following estimate: $|X_k \cap N_l(x)| \leq \frac{|N_l(x)|}{A_k} + |\{y : l - k \leq \mathrm{dist}(y, x) \leq l\}|$. Here the first summand estimates the maximum number of vertices z of X_k such that $N_{\lfloor k/2 \rfloor}(z) \subset N_l(x)$. The second summand is simply a rough estimate of all the remaining vertices of $N_l(x)$. According to our assumptions the right summand is negligible in comparison with $N_l(x)$ if l is large enough and the claim follows. □

6.3 Square lattices

In this section we focus on the case when two factors of the Cartesian product are 2-way infinite paths. In particular we prove that $\chi_\rho(P_m \,\Box\, \mathbb{Z}^2) = \infty$ for $m \geq 2$ and that $\chi_\rho(\mathbb{Z}^2) \geq 10$.

We now focus our attention on the lattice $P_2 \,\Box\, \mathbb{Z}^2$.

6.3.1 Two layers are not colorable

Lemma 6.7. *For every k and the lattice $P_2 \,\Box\, \mathbb{Z}^2$,*

$$A(k) = \begin{cases} k^2 + 2 & \text{for } k \text{ even}\,, \\ k^2 + 1 & \text{for } k \text{ odd}\,. \end{cases}$$

Proof. Observe that in a single layer of $\mathbb{Z}^2$ for any vertex $x \in \mathbb{Z}^2$ and integer i it holds that $|\{y : \mathrm{dist}(x, y) = i\}| = 4i$. Then the number of vertices at distance at most l in $\mathbb{Z}^2$ from any fixed vertex is $1 + \sum_{i=1}^{l} 4i$.

In the lattice $P_2 \,\square\, \mathbb{Z}^2$ we first consider the case of an even $k = 2l$. We count the size of N_l in both layers separately. By using the previous observation we get that:

$$A(k) = |N_l(x)| = 1 + \sum_{i=1}^{l} 4i + 1 + \sum_{i=1}^{l-1} 4i = 4l^2 + 2 = k^2 + 2\,.$$

If $k = 2l + 1$ is odd then we first discuss the case of $k = 1$. In this case $A(1) = 1 + \frac{5}{5} = 2$ since $N_0(x)$ is just a single vertex and it has 5 neighbors.

For the case of $l \geq 1$ we have to distinguish four kinds of vertices that are at distance $l + 1$ from some vertex x:

- four such vertices have one neighbor in $N_l(x)$ — those from the same $\mathbb{Z}^2$-layer as x that share a coordinate with x,
- $4l$ vertices have two neighbors in $N_l(x)$ — those remaining from the same layer,
- another four vertices have also two neighbors in $N_l(x)$ — those from the other layer but which share a coordinate with x,
- $4l - 4$ vertices have three neighbors in $N_l(x)$ — all the remaining vertices from the other layer.

In total we have:

$$A(k) = |N_l(x)| + 4\frac{1}{5} + 4l\frac{2}{5} + 4\frac{2}{5} + (4l - 4)\frac{3}{5} = 4l^2 + 2 + 4l = k^2 + 1\,.$$

□

We now are ready to prove the main result of this section, i.e. that the packing chromatic number of two layers of the square lattice is infinite.

Theorem 6.8. *For any $m \geq 2$, it holds that $\chi_\rho(P_m \,\square\, \mathbb{Z}^2) = \infty$.*

Proof. To get the result it is enough to prove the case $m = 2$. Let V be the vertex set of $P_2 \,\square\, \mathbb{Z}^2$.

We show that the sum of densities of all optimal k-packings is strictly less than one and get a contradiction with Lemma 6.5.

Since the lattice $P_2 \,\square\, \mathbb{Z}^2$ satisfies assumptions of Lemma 6.6 (cf. also Lemma 6.7), we can bound densities in terms of area, and for areas use an explicit expression given by Lemma 6.7.

However, this approach does not work such straightforwardly — the case of optimal 1- and 2-packings needs to be treated separately: Observe that the box

$P_2 \,\square\, P_2 \,\square\, P_2$ (the cube) cannot contain more than five vertices from $X_1 \cup X_2$. Hence we can bound the density of $d(X_1 \cup X_2)$ by $\frac{5}{8}$ since the whole lattice $P_2 \,\square\, \mathbb{Z}^2$ can be partitioned into such boxes.

We get a contradiction by the following estimate that holds for any packing coloring $X_1, \ldots, X_k$:

$$\begin{aligned} d\Big(\bigcup_{i=1}^{k} X_i\Big) \;&\leq\; d(X_1 \cup X_2) + \sum_{i=3}^{k} d(X_i) \;\leq\; \frac{5}{8} + \sum_{i=3}^{\infty} \frac{1}{A(i)} \;\leq \\ &\leq\; \frac{5}{8} + \sum_{i=3}^{15} \frac{1}{A(i)} + \int_{i=15}^{\infty} \frac{di}{i^2} \;\leq\; 0.9329 + \frac{1}{15} \;<\; 1\,. \end{aligned}$$

Here the exact value of the sum of the first 15 summands was obtained by a computer program. □

6.3.2 Lower bound 10

In this subsection we focus our attention on the square lattice $\mathbb{Z}^2$ and improve the lower bound of its packing chromatic number from 9 to 10. We base the argument on an observation that the best packing patterns for X_1 and for X_k with even k significantly overlap.

Lemma 6.9. *For the lattice $\mathbb{Z}^2$ and every k it holds that $A(k) = \left\lfloor \frac{k^2}{2} \right\rfloor + k + 1$.*

Proof. In the proof of Lemma 6.7 we have already observed that $|\{y : \mathrm{dist}(x,y) = i\}| = 4i$ for every vertex $x \in \mathbb{Z}^2$ and every i.

In the case of an even $k = 2l$ we have

$$A(k) = |N_l(x)| = 1 + \sum_{i=1}^{l} 4i = 2l^2 + 2l + 1 = \frac{k^2}{2} + k + 1\,.$$

In the case of an odd $k = 2l + 1$ we have four vertices at distance $l + 1$ from x that have a single neighbor in $N_l(x)$ and the remaining $4l$ vertices at distance $l + 1$ have two neighbors in $N_l(x)$. We get that

$$A(k) = |N_l(x)| + 4\frac{1}{4} + 4l\frac{2}{4} = 2l^2 + 4l + 2 = \left\lfloor \frac{k^2}{2} \right\rfloor + k + 1\,.$$

□

We now show that the best possible coverage of $\mathbb{Z}^2$ by $X_1 \cup X_2$ covers $\frac{5}{8}$ of the lattice which improves the bound $\frac{1}{2} + \frac{1}{6}$ corresponding to the case where X_1 and X_2 are treated separately.

Lemma 6.10. *The density $d(X_1 \cup X_2)$ on $\mathbb{Z}^2$ is at most $\frac{5}{8}$.*

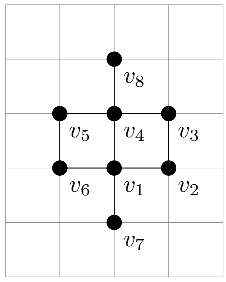


Figure 6.2: The graph O.

Proof. We first define a graph O on eight vertices consisting of a cycle $v_1, \ldots v_6, v_1$, a chord v_1v_4 and two vertices v_7 and v_8 of degree one adjacent to v_1 and v_4 respectively.

In Figure 6.2 is depicted an embedding of the graph O in $\mathbb{Z}^2$. We say that the position of O is $[x, y]$ if in such an embedding of O the vertex v_1 is placed at $[x, y]$.

The square lattice $\mathbb{Z}^2$ can be partitioned into copies of O, e.g. those copies of O placed at positions $[4i + 2j, 2j]$ where $i, j \in \mathbb{Z}$. This partition is depicted in Figure 6.3 and through the proof we assume that it is fixed.

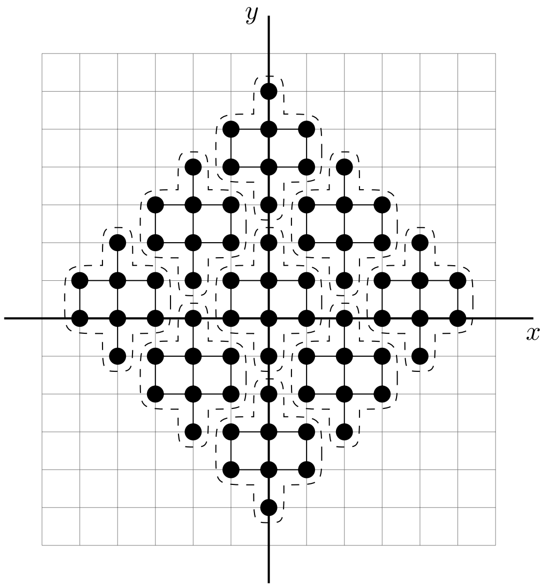


Figure 6.3: A partition of $\mathbb{Z}^2$ into isomorphic copies of O.

Assume that $X_1, \ldots, X_k$ is a packing k-coloring of $\mathbb{Z}^2$. Let X be the union of X_1 and X_2. We bound the density of X according to Definition 6.3, but first we present some properties of X and O. For this purpose, a copy of O is called a *z-copy* if it contains exactly z vertices of X.

The goal is to show that on average every copy of O contains at most 5 vertices of X.

We assume that the partition contains a 6-copy $O[x, y]$ and without loss of generality assume, that $v_3, v_6, v_7, v_8 \in X_1$ and $v_2, v_5 \in X_2$.

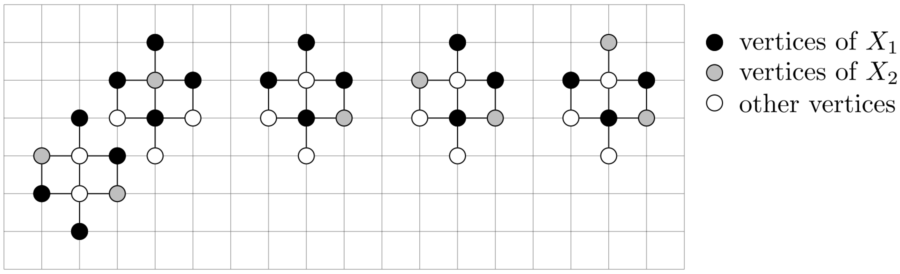


Figure 6.4: A 6-copy $O[x, y]$ is the bottom left copy of O. The others are possibilities for a 5-copy $O[x + 2, y + 2]$.

We claim that if the partition contains another 6-copy $O[x + 2i, y + 2i]$ for some $i > 0$ then there exists $j \in [0, i]$ such that $O[x + 2j, y + 2j]$ contains strictly less than 5 vertices of X.

Observe that v_6 and v_7 of $O[x + 2, y + 2]$ do not belong to X. There are four possibilities of extending X such that $O[x + 2, y + 2]$ contains five vertices of X. They are depicted in Figure 6.4. All four possibilities force that v_6 and v_7 from $O[x + 4, y + 4]$ do not belong to X. Hence it becomes an invariant which propagates through the diagonal up to $O[x+2i, y+2i]$. This contradiction proves the claim.

Note that in previous paragraph we went along the up-right diagonal. It was due to the configuration of the 6-copy $O[x, y]$. For the other possible configuration, where $v_3, v_5 \in X_2$, we use the down-right diagonal. It is essential that in either case we can proceed the diagonals to the right. In the sequel we refer to such a diagonal from a 6-copy as an *O-strip*. The O-strip contains only 5-copies. Note that the O-strip can be one-way infinite.

It may happen that two O-strips have different orientations and hence they cross. Assume that the partition contains appropriate 6-copies $O[x - 2i, y - 2i]$ and $O[x - 2j, y + 2j]$ for positive i, j such that $O[x, y]$ is in the intersection of the two corresponding O-strips.

Assume also that between $O[x, y]$ and $O[x - 2i, y - 2i]$ are only 5-copies as well as for the other O-strip. We reuse the invariant from the previous paragraph and get that X contains no v_5, v_6, v_7 or v_8 of $O[x, y]$. Moreover, at most three vertices of $v_1, \ldots, v_4$ may belong to X. Hence $O[x, y]$ contains at most three vertices of X. See Figure 6.5.

Now we are ready to prove the limit on the density of X. For every 6-copy C we traverse the diagonal while increasing the first coordinate. We either encounter a z-copy D where $z < 5$ or the diagonal consist only of 5-copies. The z-copy D is *a pairing copy* for C. Note that D can be in two pairs but then $z < 4$.

Let x be an arbitrary vertex. We use the fact that $\lim_{l\to\infty} \frac{|\Delta N_l(x)|}{|N_l(x)|} = 0$ on $\mathbb{Z}^2$. We denote by $O_l(x)$ the set of copies of O which are included in $N_l(x)$.

Now we show that $|X \cap N_l(x)| \leq 5|O_l| + c|\Delta N_l(x)|$. If a 6-copy and its pair copy are both in $O_l(x)$ then they contribute to $X \cap N_l(x)$ at most 10 vertices. Indeed, if the two copies are paired with a single copy of O then these three contain at

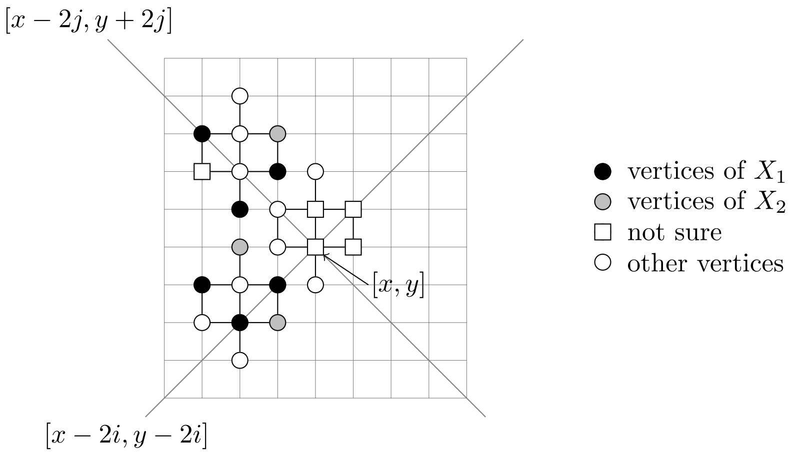


Figure 6.5: Intersection of two O-strips. In every possible intersection some vertices are forced to be in X_1, X_2, or they are not covered at all. The square vertices are not forced.

most 15 vertices of X.

Observe that the number of 6-copies which has no pair copy in O_l is linear in $|\Delta N_l(x)|$ since traversing a diagonal of a copy of O without its pair in $O_l(x)$ ends on the boundary. Note that $O_l(x)$ does not have to cover whole $N_l(x)$ but it can miss linearly many vertices of the boundary. See Figure 6.6.

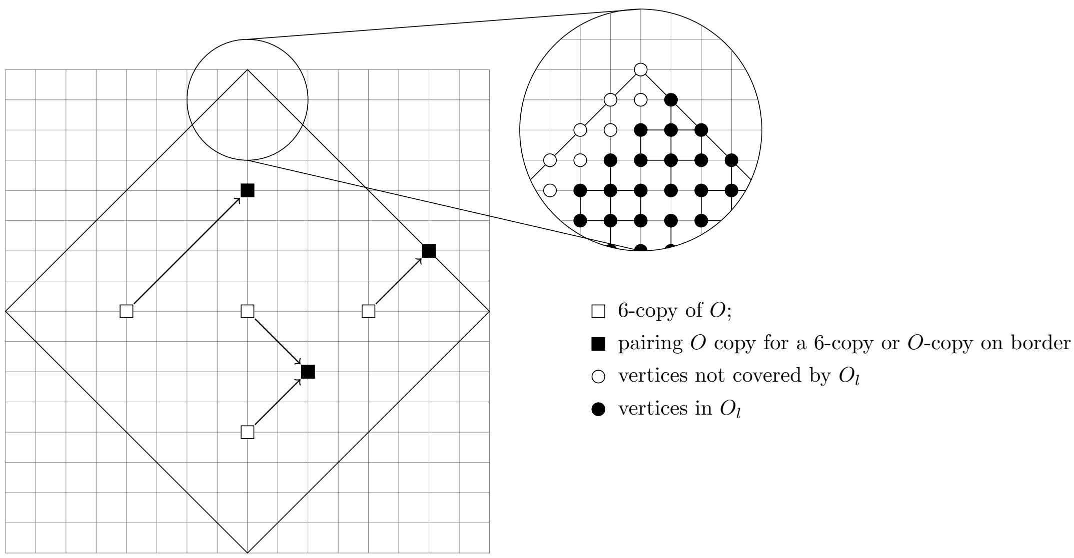


Figure 6.6: Bounding density of X in $N_l(x)$

Finally, the density of X is:

$$d(X) \leq \limsup_{l\to\infty} \left(\frac{5}{8} + \frac{c|\Delta N_l(x)|}{|N_l(x)|} \right) = \frac{5}{8}.$$

$\square$

Theorem 6.11. *For the infinite square lattice $\mathbb{Z}^2$ it holds that $10 \leq \chi_\rho(\mathbb{Z}^2)$.*

Proof. We compute an upper bound on the density of the union of packings X_i, $1 \leq i \leq 9$. The bound for the union of X_1 and X_2 is given in Lemma 6.10. The other packings are bounded separately by using Lemma 6.9.

$$d\Big(\bigcup_{i=1}^{9} X_i\Big) \leq \frac{5}{8} + \sum_{i=3}^{9} \frac{1}{A(i)} = \frac{3830381}{3837600} < 1.$$

Finally Lemma 6.5 implies that the packing chromatic number of $\mathbb{Z}^2$ is at least 10. □

6.3.3 Lower bound 12 (computer assisted)

By using a computer we were able to improve the lower bound on the square lattice from the previous subsection from 10 to 12.

Theorem 6.12. *The packing chromatic number for the square lattice is at least 12.*

Proof. The proof relies on computer. We describe the main idea of the algorithm, which proves the theorem. All necessary code for running the computation is available at `http://kam.mff.cuni.cz/~bernard/packing`.

The algorithm for proving Theorem 6.12 is a brute force search through all possible configurations on lattice 15×9. It is too time consuming to simply check every configuration. Hence we use the following observation to speed up the computation by avoiding several configurations.

Observation 6.13. *If there exists a coloring of the square lattice with 11 colors then it is possible to color lattice* 15×9 *where color 9 is at position* $[5, 5]$.

Any other color at any other position could be fixed instead of 9 at $[5, 5]$. Color 9 at $[5, 5]$ just sufficiently reduces the number of configurations to check. We do not claim that it is the optimal choice.

If there exists a coloring we simply find any vertex of color 9 and take a piece of the lattice in its neighborhood.

So in the search through the configurations we assume that at position $[5, 5]$ is precolored by 9. The coloring procedure gets a matrix and tries to color the vertices row by row. A pseudocode follows.

```
function boolean try_color(lattice, [x,y])
begin
  for color := 1 to 11 do
     if can use color on lattice at [x,y] then
        lattice[x,y] := color
        if [x,y] is the last point return true
        else if try_color(lattice, next([x,y]) then return true
     endif
  endfor
  return false
end
```

We have two implementations of this function. One is in the language C++ and the other is in Pascal. The first one is available online at `http://kam.mff.cuni.cz/~bernard/packing`. We include the full source code as well as descriptions of inputs and outputs. We were checking the outputs of both programs during the computation and we verified that they match. The total number of checked configurations was 43112312093324. The computation took about 120 days of computing time on a single core workstation in year 2009.

The procedure fails to color the matrix 15×9 with 9 at position $[5, 5]$. Hence we conclude that the packing chromatic number for the square lattice is at least 12. $\square$

6.4 Hexagonal lattice

Now we turn our attention to the infinite hexagonal lattice $\mathcal{H}$.

6.4.1 Upper bound 7

We first exhibit its packing coloring of $\mathcal{H}$ that uses only 7 colors.

Theorem 6.14. *For the hexagonal lattice $\mathcal{H}$, $\chi_\rho(\mathcal{H}) \leq 7$.*

Proof. We exhibit a tiling of $\mathcal{H}$; refer to Figure 6.7. One class of the bipartition of the lattice $\mathcal{H}$ is the first color class X_1. The other class of bipartition can be covered by packings $X_2, \ldots, X_7$. The pattern for filling the hexagonal lattice consists of 12 vertices. It is bordered by a bold line in the figure. $\square$

6.4.2 Coloring two layers

Next we turn our attention to layers of the hexagonal lattice. First we show that two layers of hexagonal lattice have finite packing chromatic number and in the subsection we show that six layers cannot be colored.

Theorem 6.15. *For two layers of the hexagonal lattice $P_2 \square \mathcal{H}$, $\chi_\rho(P_2 \square \mathcal{H}) \leq 537$.*

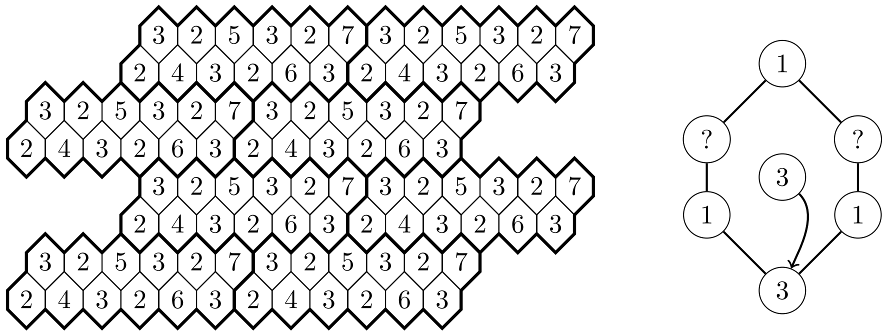


Figure 6.7: The pattern for partitioning hexagonal lattice using 7 packings of pairwise different width.

Proof. We have found a tiling of $P_2 \square \mathcal{H}$ by a method similar to the method Holub a Soukal [28]. As the resulting pattern on 1179648 vertices is a bit too large for printing on a paper it can be found on the enclosed CD. Here we only briefly described the method. The method requires a computer to succeed in a reasonable time.

We start by coloring half of the vertices by color one and we process color one by one in the increasing order. For every color c we take all already generated coloring using colors $1, \ldots, c-1$ and we extend them by using c on uncolored vertices. We store for further processing with color $c+1$ only patterns with minimum number of uncolored vertices and even from them we randomly chose just several as the total number of possible patterns is horribly large. $\square$

6.4.3 Six layers are not colorable

In this last subsection we show that six layers of the hexagonal lattice cannot be covered by a finite number of packings of pairwise different width. We follow the same approach as we have used for proving Theorem 6.8. We number the hexagonal layers of $P_6 \square \mathcal{H}$ by $1, 2, 3, 4, 5, 6$ where layer 1 and layer 6 are on the boundary. Every vertex is in one layer.

Lemma 6.16. *For every $l \geq 6$, the density of X_{2l} on $P_6 \square \mathcal{H}$ is at most $\frac{1}{9l^2-36l+66}$. The upper bounds on $d(X_2), d(X_4), \ldots, (X_{10})$ are given in the next table.*

l	1	2	3	4	5
$d(X_{2l}) \leq$	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{1}{34}$	$\frac{1}{65}$	$\frac{1}{111}$

Proof. We count the size of $N_l(x)$ and obtain an upper bound on the density due to Lemma 6.6. The size of $N_l(x)$ depends on the choice of x. More precisely it depends on the layer of x. The smallest size of $N_l(x)$ is for x in one of the boundary layers. On the other hand it is the largest for layers 3 and 4. Hence we bound the size $N_l(x)$ from below by the size of N_l of vertices in layer 1.

Let y be a vertex of $\mathcal{H}$. Then the number of vertices at distance l is $3l$. Hence the number of vertices at distance at most l including y is

$$|N_{\mathcal{H}l}| := 1 + \sum_{i=1}^{l} 3i = 1 + 3\frac{(l+1)l}{2}.$$

For a vertex x in the layer 1 we compute the size of $N_l(x)$ in the following way:

$$|N_l(x)| = \sum_{i=l-5}^{l} |N_{\mathcal{H}i}| = 9l^2 - 36l + 66.$$

Note that the last equality holds only for $l \geq 6$. The values of $N_l(x)$ for smaler values of l were computed explicitly.

$\square$

Lemma 6.17. *Any packings X_1, X_2, X_3, and X_4 on $P_3 \square \mathcal{H}$ staisfy that:*

- $d(X_3) \leq \frac{2}{18}$.
- $d(X_1 \cup X_2 \cup X_4) \leq \frac{12}{18}$.

Proof. We partition $P_3 \square \mathcal{H}$ into copies of $P_3 \square C_6$. The graph $P_3 \square C_6$ and partitioning of $\mathcal{H}$ into disjoint copies of C_6 are depicted in Figure 6.8.

The graph $P_3 \square C_6$ consists of three copies of C_6. We call them layer 1, layer 2, and layer 3 where layer 2 is the middle one.

The first claim of the lemma follows from the simple fact that $|X_3 \cap (P_3 \square C_6)| \leq 2$.

In the rest of the proof we abbreviate $X := X_1 \cup X_2 \cup X_4$.

Assume that it is possible to cover 13 vertices of $P_3 \square C_6$ by X. Then there is a copy C of C_6 such that $|X \cap C| = 5$. There are two possibilities of such a covering: either $|X_2 \cap C| = 1$ or $|X_2 \cap C| = 2$.

First we discuss the case that there are two layers with five vertices of X. The only possibility is that they are not neighbors because of vertices from X_4. Hence these layers are 1 and 3. Two cases of possible layer 1 are depicted in Figure 6.9. These two cases are compatible three cases for layer 3. We determined them by the position of a vertex from X_4 which is unique. It is not possible to cover more than one vertex in layer 2, therefore we get at most 11 covered vertices.

Now we know that one layer contains five vertices and the other two contain four vertices. We introduce two observations about X_2 and $X_1 \cup X_2$ which give us more information about possible structure of the layers.

The first observation is that if one of the layers contains two vertices of X_2 then the neighboring layer(s) does not contain any vertices of X_2. This holds since all vertices in the neighboring layers are at distance at most two from the vertices of X_2.

The second observation is that $P_3 \square C_6$ contains at most 11 vertices of $X_1 \cup X_2$. So let there be 12 such vertices. One layer may contain at most four vertices of

$X_1 \cup X_2$. Hence every layer contains four of them. Moreover, every layer contains exactly one vertex of X_2 since every layer must contain at least one. Take the middle layer and let v be the vertex from X_2. Since we want to cover four vertices of the middle layer, the vertices of X_1 are determined by the position of v. Then vertices of X_1 are also determined in the other two layers since there must be three of them in each; refer Figure 6.8. Now the only two vertices left for X_2 in layers one and three are too close to each other hence it is not possible to cover 12 vertices by $X_1 \cup X_2$.

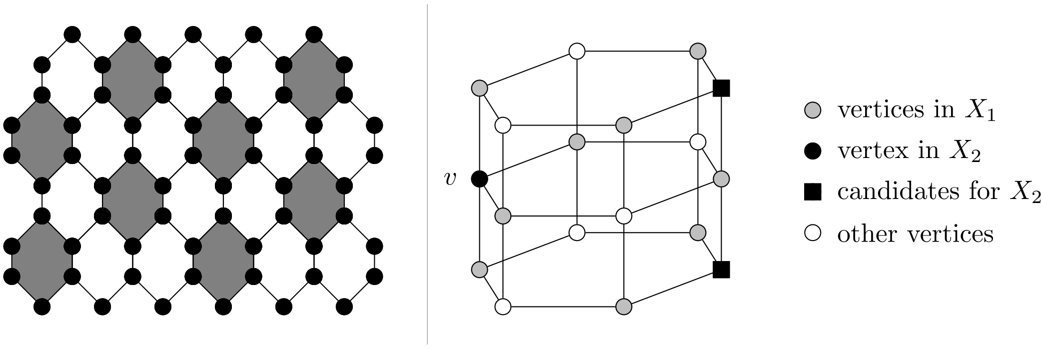


Figure 6.8: On the left-hand side is a possible tiling of the hexagonal lattice using C_6's. On the right-hand side is a coverage of $C_6 \,\square\, P_3$ by X_1 and X_2 which contains 9 vertices of X_1 and a vertex of X_2 in the middle layer. There are only two other candidate vertices for X_2, which are square vertices. But they are too close to be both in X_2.

Since $X_1 \cup X_2$ covers at most 11 vertices and we want to cover 13 vertices, we derived that two vertices must be from X_4. These two vertices must be in layer 1 and layer 3. Hence the layer containing five vertices of X must be layer 1 or layer 3. Assume without lost of generality that it is layer 1. The other two layers must each contain four vertices of X.

Hence the middle layer must contain one vertex from X_2 and three vertices of X_1. This implies that the first layer contains only one vertex from X_2. Hence we know the configurations for layer 1 and layer 2. See Figure 6.10. We observe that there are only three vertices in layer 3 which can be in X. Hence we failed to include 13 vertices of $P_3 \,\square\, C_6$ to X.

□

In the following lemma we estimate the density of X_5 on $P_6 \,\square\, \mathcal{H}$ by a simpler case study on $P_3 \,\square\, \mathcal{H}$.

Lemma 6.18. *The density of any packing X_5 on $P_3 \,\square\, \mathcal{H}$ is at most $\frac{1}{21.9}$.*

Proof. We bound the density using Lemma 6.6. We compute $A(x, 5)$ in $P_3 \,\square\, \mathcal{H}$ for a vertex x in one of two outer layers. Assume layer 1 for x. Then the area consists of vertices in $N_2(x)$ together with the part obtained from vertices at distance three from x. We distinguish several types of these vertices.

- six vertices from the layer 1 have one neighbor in $N_2(x)$,

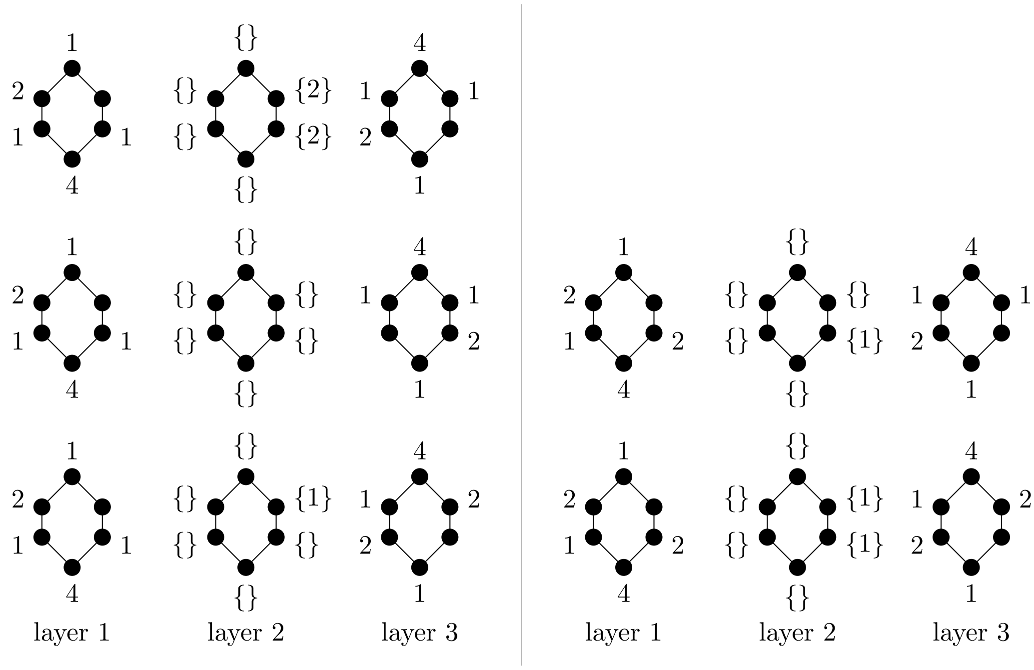


Figure 6.9: Layer 1 contains five vertices of X. There are two possibilities. The first one is on the left and the second one is on the right. Layer 3 contains also five vertices of X. Vertices from the middle layer are assigned lists of available packings.

- three vertices from the layer 1 have two neighbors in $N_2(x)$,
- six vertices from the layer 2 have two neighbors in $N_2(x)$,
- three vertices from the layer 3 have two neighbors in $N_2(x)$.

In total we have:

$$A(x, 5) = 15 + \frac{6}{4} + \frac{6}{4} + \frac{12}{5} + \frac{6}{4} = 21.9 \,.$$

For a vertex x from the middle layer the area $A(x, 5)$ is 25.4 hence we can estimate the area by 21.9 for any vertex of $P_3 \,\square\, \mathcal{H}$. Refer to Figure 6.11 for three hexagonal layers of $P_3 \,\square\, \mathcal{H}$ and $N_2(x)$. $\square$

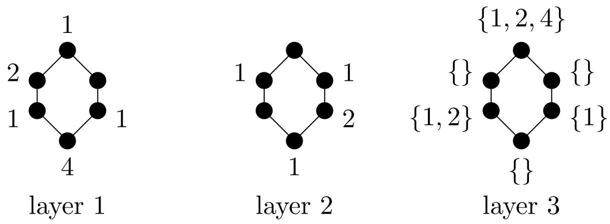


Figure 6.10: Let layer 1 contain five vertices of X and layer 2 contain four vertices of X. They must look as depicted. Vertices of the third layer have assigned lists of possible colors. But there are only three with nonempty list.

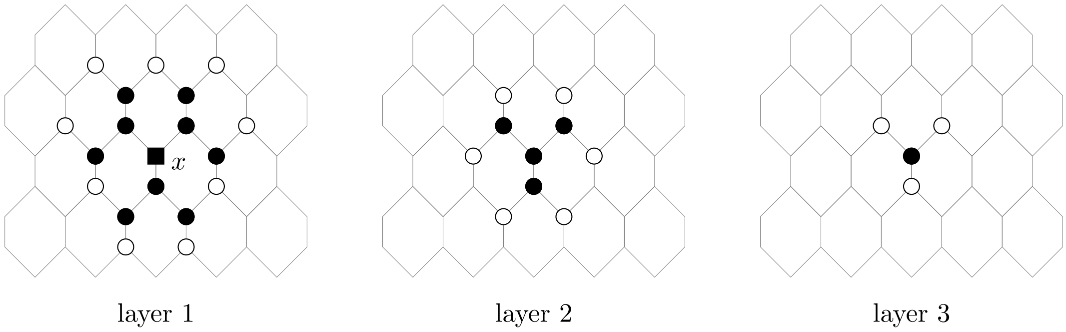


Figure 6.11: Three layers of hexagonal lattice. Black square corresponds to x. Black vertices correspond to vertices from $N_2(x)$ and white vertices are at distance 3 from x.

Theorem 6.19. *For any $m \geq 6$ it holds that $\chi_\rho(P_m \square \mathcal{H}) = \infty$.*

Proof. Assume $m = 6$. We show that the sum of densities of all k-packing is strictly less than 1 and we get a contradiction with Lemma 6.5.

The lattice $P_6 \square \mathcal{H}$ can be partitioned into two copies of $P_3 \square \mathcal{H}$. Hence we can use bound on $X_1 \cup X_2 \cup X_3 \cup X_4$ from Lemma 6.17. Also X_5 can be bounded using Lemma 6.18. Since a $(2l+1)$-packing is also a $2l$-packing we bound the density of X_{2l+1} by the density of X_{2l}. Note that the density of X_{2l} may be bounded by $\frac{1}{2l^2}$.

We get the contradiction by the following estimate that holds for any packing coloring $X_1, \ldots, X_k$:

$$\begin{aligned} d\Big(\bigcup_{i=1}^{k} X_i\Big) &\leq \frac{14}{18} + \frac{1}{21.9} + \sum_{i=6}^{\infty} d(X_i) \\ &\leq \frac{541}{657} + \sum_{i=6}^{59} d(X_i) + \sum_{i=30}^{\infty} \frac{2}{(2i)^2} \\ &\leq 0.982 + \frac{1}{2}\int_{i=29}^{\infty} \frac{di}{i^2} \quad \leq \quad 0.982 + \frac{1}{58} \quad < \quad 1\,. \end{aligned}$$

Again, the exact value of the sum of the first 59 summands was determined by a computer program. $\square$

Chapter 7

Conclusion

In this work we have presented results on three different variants of colorings. We have started with studying 6-critical graphs. We have disproved a conjecture of Thomassen about 6-critical graphs on the Klein bottle by enumerating all nine non-isomorphic 6-critical graphs on the Klein bottle in Theorem 4.1. We followed with disproving a conjecture of Oporowski and Zhao by introducing a 6-critical graph with five crossings different from K_6 and we have pushed their result further and have proved that K_6 is the only 6-critical graph with crossing number at most four.

The work in this area might follow by enumerating critical graphs on other surfaces or with higher crossing number. In particular, it would be interesting to know the complete list of 6-critical graphs with crossing number at most five. So far there are only two such graphs known and it remains open if there are any other such graphs.

In the second part of the work we have focused on list coloring of planar graphs with restrictions on short cycles. Coloring and list coloring of planar graphs are very popular topic with many recent results. We have presented a proof that planar graphs without triangles, 7- and 8-cycles are 3-choosable and a proof that planar graphs without triangles and constraints on 4-cycles are 3-choosable. The latter result is improving a result of Thomassen that every planar graph without triangles and 4-cycles is 3-choosable.

The current state of art in this area is monitored by a site maintained by Montassier [36]. In particular, Steinberg's conjecture that every planar graph without 4- and 5-cycles is 3-colorable is opened for more than thirty years. Another interesting opened problem, not mentioned on Montassier's page, is if there exists k such that every planar graph without odd cycles of lengths up to k is 3-choosable. So far it is only known that $k > 3$. An upper bound on k would strengthen a result of Alon and Tarsi [1] and show that locally bipartite planar graphs are 3-choosable.

In the third part we have focused on packing coloring which is a very recent concept motivated by channel assignment. We have improved a lower bound on the packing chromatic number for the square lattice from 9 to 12 (the best upper bound is 17) and an upper bound of Brešar, Klavžar and Rall [5] for the hexagonal lattice from 8 to 7 and hence matched the lower bound 7. We have also improved a result of Finbow and Rall [20] about layers of the square lattice by showing that it is not possible to color even two layers, while they showed the same for infinite number of layers. We finished by showing that six layers of the hexagonal grid cannot be colored by a finite number of colors while two layers can be colored by a finite number of colors.

An open problem left on the hexagonal grid is to find the number of layers that can be colored by a finite number of colors. The number is known to be between two and five. It seems that three layers are difficult to color and hence it might be the case that three layers cannot be colored by a finite number of colors. Current results suggest that cubic planar graphs have bounded packing chromatic number. Hence it is an interesting problem for future research. The notion of density we have defined does not work on all graphs. In particular, it does not work on infinite 3-regular trees. Generalization of the definition of density to all graphs is also a possible direction of exploration.

Bibliography

[1] N. Alon and M. Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12:125–134, 1992.

[2] K. Appel and W. Haken. Every planar map is four colorable, part I: discharging. *Illinois J. of Math.*, 21:429–490, 1977.

[3] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable, part II: reducibility. *Illinois J. of Math.*, 21:491–567, 1977.

[4] B. Bollobás. *Modern Graph Theory*. Springer Science+Business Meda, Inc., USA, 3rd edition, 1998.

[5] B. Brešar, S. Klavžar, and D. F. Rall. On the packing chromatic number of Cartesian products, hexagonal lattice, and trees. *Discrete Appl. Math.*, 155:2303–2311, 2007.

[6] G. Chartrand and P. Zhang. *Chromatic Graph Theory*. Chapman & Hall/CRC, 2008.

[7] N. Chenette, L. Postle, N. Streib, R. Thomas, and C. Yerger. Five-coloring graphs in the Klein bottle. submitted.

[8] R. Diestel. *Graph Theory*. Springer-Verlag, Heidelberg, 4th edition, 2010.

[9] G. A. Dirac. Map colour theorems related to the Heawood colour formula. *J. London Math. Soc*, 31:460–471, 1956.

[10] G. A. Dirac. A theorem of R. L. Brooks and a conjecture of H. Hadwiger. *Proc. of the London Mathematical Society*, 17:161–195, 1957.

[11] Z. Dvořák, B. Lidický, and R. Škrekovski. On 3-choosability of plane graphs without 3-, 7- and 8-cycles. *Discrete Mathematics*, 309:5899–5904, 2009.

[12] Z. Dvořák, B. Lidický, and R. Škrekovski. 3-choosability of triangle-free planar graphs with constraints on 4-cycles. *SIAM J. on Discrete Mathematics*, 24:934–945, 2010.

[13] J. Ekstein, J. Fiala, P. Holub, and B. Lidický. The packing chromatic number of the square lattice is at least 12. *arXiv*, 2010. 1003.2291v1.

[14] D. Eppstein. Subgraph isomorphism in planar graphs and related problem. In *Proceedings of the 6th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 632–640. SIAM, 1995.

[15] D. Eppstein. Subgraph isomorphism in planar graphs and related problems. *J. Graph Algorithms and Applications*, 3:1–27, 1999.

[16] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. *Congr. Numer.*, 26:125–157, 1980.

[17] R. Erman, F. Havet, B. Lidický, and O. Pangrác. 5-colouring graphs with 4 crossings. *SIAM J. on Discrete Mathematics*, 2011. to appear.

[18] J. Fiala and P. A. Golovach. Complexity of the packing chromatic problem for trees. *Discrete Appl. Math.*, 158:771–778, 2010.

[19] J. Fiala, S. Klavžar, and B. Lidický. The packing chromatic number of infinite product graphs. *European J. of Combin.*, 30:1101–1113, 2009.

[20] A. Finbow and D. F. Rall. On the packing chromatic number of some lattices. *Discrete Appl. Math.*, 158:1224–1228, 2010. special issue LAGOS'07.

[21] S. Fisk. The non-existence of colorings. *J. of Combinatorial Theory, Series B*, 24:247–248, 1978.

[22] S. Fitzpatrick, J. Janssen, and R. Nowakowski. Distributive online channel assignment for hexagonal cellular networks with constraint. *Discrete Appl. Math.*, 143:84–91, 2004.

[23] T. Gallai. Kritische graphen I. *Publ. Math. Inst. Hungar. Acad. Sci*, 8:165–192, 1963.

[24] T. Gallai. Kritische graphen II. *Publ. Math. Inst. Hungar. Acad. Sci*, 8:373–395, 1963.

[25] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris, and D. F. Rall. Broadcast chromatic numbers of graphs. *Ars Combin.*, 86:33–49, 2008.

[26] J.-L. Guo and Y.-L. Wang. 3-list-coloring planar graphs of girth 4. *Discrete Mathematics*, 311:413–417, 2011.

[27] P. J. Heawood. Map colour theorem. *Quart. J. Pure Appl. Math.*, 24:332–338, 1890.

[28] P. Holub and R. Soukal. A note on packing chromatic number of the square lattice. *Electronic Journal of Combinatorics*, 17, 2010. Note 17.

[29] T. R. Jensen and B. Toft. *Graph Coloring Problems.* Wiley-Interscience, New York, 1995.

[30] R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, New York, 1972.

[31] K. Kawarabayashi, D. Král', J. Kynčl, and B. Lidický. 6-critical graphs on the Klein bottle. *SIAM J. on Discrete Mathematics*, 23:372–383, 2009.

[32] P. C. B. Lam, W. C. Shiu, and Z. M. Song. The 3-choosability of plane graphs of girth 4. *Discrete Math.*, 294:297–301, 2005.

[33] X. Li. On 3-choosable planar graphs of girth at least 4. *Discrete Math.*, 309:2424–2431, 2009.

[34] B. Lidický. On 3-choosability of plane graphs without 6-, 7- and 8-cycles. *Australasian J. Combin.*, 44:77–86, 2009.

[35] J. Matoušek and J. Nešetřil. *Invitation to Discrete Mathematics*. Oxford University Press, USA, 1998.

[36] M. Montassier. 3-color-problem. `http:/www.labri.fr/perso/montassi/pmwiki/index.php?n=ThreeColorProblem.ThreeColorProblem`.

[37] B. Oporowski and D. Zhao. Coloring graphs with crossings. *Discrete Mathematics*, 309(9):2948–2951, 2009.

[38] N. Robertson, D. P. Sanders, P. D. Seymour, and R. Thomas. The four colour theorem. *J. Combinatorial Theory, Series B*, 70:2–44, 1997.

[39] B. Mohar S. Fisk. Coloring graphs without short non-bounding cycles. *J. of Combinatorial Theory, Series B*, 60:268–276, 1994.

[40] N. Sasanuma. Chromatic numbers of 6-regular graphs on the Klein bottle. to appear in Discrete Mathematics.

[41] C. Sloper. An eccentric coloring of trees. *Australas. J. Combin.*, 29:309–321, 2004.

[42] C. Thomassen. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, 62:180–181, 1994.

[43] C. Thomassen. Five-coloring graphs on the torus. *J. Combin. Theory Ser. B*, 62:11–33, 1994.

[44] C. Thomassen. 3-list-coloring planar graphs of girth 5. *J. Combin. Theory Ser. B*, 64:101–107, 1995.

[45] C. Thomassen. Color-critical graphs on a fixed surface. *J. Combin. Theory Ser. B*, 70:67–100, 1997.

[46] A. Vesel. private communication. 2007.

[47] V. G. Vizing. Vertex colorings with given colors (in russian). *Metody Diskret. Analiz, Novosibirsk*, 29:3–10, 1976.

[48] M. Voigt. List colourings of planar graphs. *Discrete Math.*, 120:215–219, 1993.

[49] M. Voigt. A not 3-choosable planar graph without 3-cycles. *Discrete Math.*, 146:325–328, 1995.

[50] R. A. Wilson. *Graphs, Colourings and the Four-Colour Theorem*. Oxford University Press, USA, 2002.

[51] H. Zhang. On 3-choosability of plane graphs without 5-, 8- and 9-cycles. *J. Lanzhou Univ., Nat. Sci.*, 41:93–97, 2005.

[52] H. Zhang and B. Xu. On 3-choosability of plane graphs without 6-, 7- and 9-cycles. *Appl. Math. J. Chinese Univ., Ser. B (Engl. Ed.)*, 19:109–115, 2004.

[53] H. Zhang, B. Xu, and Z. Sun. Every plane graph with girth at least 4 without 8- and 9-circuits is 3-choosable. *Ars Comb.*, 80:247–257, 2006.

[54] X. Zhu, M. Lianying, and C. Wang. On 3-choosability of plane graphs without 3-, 8- and 9-cycles. *Australas. J. Comb.*, 38:249–254, 2007.

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