

Irreversible 2-conversion set is NP-complete

Jan Kynčl, Bernard Lidický, and Tomáš Vyskočil *

Department of Applied Mathematics and Inst. for Theoretical Computer Science**
Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic
{kyncl,bernard,whisky}@kam.mff.cuni.cz

Abstract. An irreversible k -threshold process is a process on a graph where vertices change color from white to black if they have at least k black neighbors. An irreversible k -conversion set is a subset S of vertices of a graph G such that the irreversible k -threshold process changes all vertices of G to black if S is the initial set of black vertices. We show that deciding the existence of an irreversible 2-conversion set of a given size is NP-complete which answers a question of Dreyer and Roberts. Moreover, we show an optimal irreversible 3-conversion set for a toroidal grid, which simplifies constructions of Pike and Zou.

Keywords: threshold models; spread of disease; irreversible k -conversion set; NP-complete problem; toroidal grid

1 Introduction

Mathematical modelling of the spread of infectious diseases was recently studied by Roberts [6] and by Dreyer and Roberts [2]. They used the following model.

Let $G = (V, E)$ be a graph with vertices colored white and black. An *irreversible k -threshold process* is a process where vertices change color from white to black. More precisely, a white vertex becomes black at time $t + 1$ if at least k of its neighbors are black at time t .

An *irreversible k -conversion set* S is a subset of V such that the irreversible k -threshold process starting with vertices of S set to black and all other white will result in a graph G with all vertices black after finite number of steps.

More general models of spread of infectious diseases and the complexity of the related problems were studied by Boros and Gurwich [1].

A natural question to ask is what is the minimum size of an irreversible k -conversion set in a graph G .

Problem $IkCS(G, s)$:

Input: a graph G and a positive integer s

Output: YES if there exists an irreversible k -conversion set of size s in G
NO otherwise

* The results of this paper were obtained during the DIMACS/DIMATIA REU program at Rutgers University in June and July 2007, supported by the project ME 886 of the Ministry of Education of the Czech Republic.

** supported by the project 1M0545 of the Ministry of Education of the Czech Republic

It is proved in [2] that k CS is NP-complete for a fixed $k \geq 3$ by an easy reduction from the independent set problem. For $k = 1$ the problem is trivially polynomial since one black vertex per connected component is necessary and sufficient. Dreyer and Roberts [2] asked what is the complexity of the k CS problem if $k = 2$. As the first result of this paper we resolve this open question.

Theorem 1. *The problem I2CS is NP-complete even for graphs of maximum degree 4.*

A subset W of vertices of a graph $G = (V, E)$ is a *vertex feedback set* if $V \setminus W$ is acyclic. For 3-regular graphs, the I2CS problem is equivalent to finding a vertex feedback set, which can be solved in polynomial time [7].

Note that the problem $\text{I2CS}(G, s)$ is trivially polynomial if the maximum degree of G is at most two as a path of length l requires $\lceil \frac{l+1}{2} \rceil$ black vertices and a cycle of length l requires $\lceil \frac{l}{2} \rceil$ vertices. However, we do not know the complexity of I2CS for graphs of maximum degree three. Boros and Gurwich [1] proved that if every vertex has its own threshold, then determining the minimum size of the conversion set is NP-complete.

The second result presented in this paper is a construction of an optimal irreversible 3-conversion set for a toroidal grid $T(m, n)$ which is the Cartesian product of the cycles C_m and C_n . Previously known lower and upper bounds differ by a linear $O(m + n)$ term [2,3,4]. Recently, we have found that Pike and Zou [5] gave an optimal construction. We present a simpler optimal construction.

Theorem 2. *Let T be a toroidal grid of size $m \times n$, where $m, n \geq 3$. If $n = 4$ or $m = 4$ then T has an irreversible 3-conversion set of size at most $\frac{3mn+4}{8}$. Otherwise, T has an irreversible 3-conversion set of size at most $\frac{mn+4}{3}$.*

2 Irreversible 2-conversion set

In this section we give a proof of Theorem 1.

The problem is trivially in NP. A verification that $S \subseteq V$ is an irreversible 2-conversion set can be done by iterating the threshold process. It is enough to check only the first $|V|$ steps. Hence the verification can be done in a polynomial time.

In the rest of the proof we show that $\text{I2CS}(G, s)$ is NP-hard by a polynomial-time reduction from 3-SAT. We introduce a variable gadget, a clause gadget and a gadget which checks if all clause gadgets are satisfied.

Since a white vertex needs two black neighbors to become black, we have the following observation.

Observation 3 *Every irreversible 2-conversion set contains all vertices of degree one.*

According to this observation, in the figures of the gadgets we draw vertices of degree one black.

Let \mathcal{F} be an instance of 3-SAT. We denote the number of variables by n and the number of clauses by m . We construct a graph $G_{\mathcal{F}}$ and give a number s such that \mathcal{F} is satisfiable if and only if $G_{\mathcal{F}}$ has an irreversible 2-conversion set of size s .

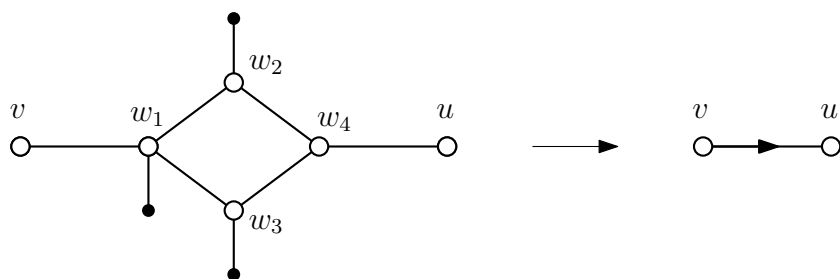


Fig. 1. A one-way gadget.

First we introduce a *one-way* gadget; see Figure 1. The gadget contains two vertices u and v which are called *start* and *end* of the one-way gadget. Vertices w_1, w_2, w_3 and w_4 are called *internal* vertices of the gadget.

Observation 4 *Let u and v be start and end of the one-way gadget. If internal vertices are white at the beginning then the following holds:*

1. *If v is black then u gets a black neighbor from the gadget in three steps.*
2. *The color of u has no influence on colors of the other vertices of the gadget.*

We refer to the one-way gadget by a directed edge in the following figures. Later, we set s such that S cannot contain any internal vertices of one-ways. Thus, in the rest of the proof we assume that all internal vertices are white at the beginning.

Variable gadget

A gadget $g(X_i)$ for a variable X_i , where $1 \leq i \leq n$, consists of a triangle $x_i y_i z_i$ and two *antennas*; see Figure 2. The length of the antenna connected to x_i (resp. y_i) is equal to the number of occurrences of X_i (resp. $\neg X_i$) in the clauses of \mathcal{F} . We call the white vertices of x_i antenna *positive outputs* and the vertices of y_i antenna *negative outputs*. One-way gadgets with starts in the outputs have ends in clause gadgets. The vertex z_i is adjacent to u_i lying on a distributing path, which we define later.

We show that exactly one of x_i and y_i is black at the beginning. This represents the value of the variable X_i . The vertex x_i corresponds to the true and y_i to the false evaluation of X_i . The purpose of the connection between u_i and z_i is to convert all vertices of the gadget to black if \mathcal{F} is satisfiable.

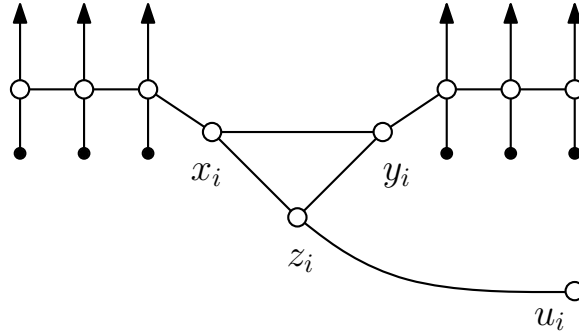


Fig. 2. A variable gadget $g(X_i)$ connected to a vertex u_i of a distributing path.

Observation 5 *Let S be an irreversible 2-conversion set. The gadget $g(X_i)$ has the following properties.*

- If x_i is black then all positive outputs will become black in the process. Similarly for y_i and negative outputs.*
- If two of x_i, y_i, z_i are black then all vertices of the gadget will become black in the process.*
- S must contain at least one vertex of x_i, y_i and z_i .*
- If S contains exactly one vertex of the gadget (except the vertices of degree one) then it must be x_i or y_i .*
- If S contains exactly one vertex of the gadget then z_i gets black only if u_i gets black.*

Proof. The first two properties are easy to check and hence we skip them.

First we check the property (c). Every vertex of the triangle $x_i y_i z_i$ has only one neighbor outside the triangle. Hence if all three vertices are white, they remain white forever since each of them has at most one black neighbor. Hence S must contain at least one of them.

Now we check the property (d). If S is allowed to contain only one of $\{x_i, y_i, z_i\}$ then all positive and negative outputs are white at the beginning. Moreover, the positive outputs may become black only if x_i gets black. Similarly for negative outputs and y_i .

Suppose for contradiction that $z_i \in S$. Then both x_i and y_i have only one black neighbor (z_i) at the beginning. During the process the other black neighbor has to be some output vertex which is not possible. Hence S must contain x_i or y_i .

Finally, we check the property (e). By (d) we know that z_i is white at the beginning and assume without loss of generality that y_i is also white while x_i is black. The vertex z_i can get black if y_i or u_i gets black. So assume for contradiction that y_i gets black before z_i . The only possibility is that the vertex from the antenna adjacent to y_i gets black. But it is not possible since output vertices are white at the beginning and they are connected to the rest of the graph by one-ways. \square

Note that if x_i or y_i is in S then there is still a chance that the process converts all vertices of the gadget to black, as the vertex u_i may become black during the process.

Let L be a set of all degree one vertices in $G_{\mathcal{F}}$. We set parameter s to $|L| + n$. Thus every variable gadget has exactly one of x_i and y_i black at the beginning and all other vertices of $G_{\mathcal{F}}$ of degree at least two are white. We compute $|L|$ after we describe all the remaining gadgets.

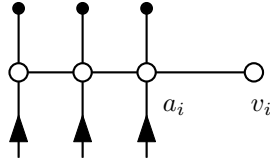


Fig. 3. A clause gadget $g(C_i)$ connected to a vertex v_i of a collecting path.

Clause gadget

The gadget $g(C_i)$ for a clause $C_i = (L_o \vee L_p \vee L_q)$, where $1 \leq i \leq m$ and L_o, L_p , and L_q are literals, is depicted in Figure 3. It consists of a path on three vertices which correspond to the literals in the clause. We call it the *spine* of the clause gadget. Each vertex of the spine has one neighbor of degree one and is connected to the gadget of the corresponding variable by a one-way. The vertex of a clause corresponding to a literal X_i is connected to a positive output of $g(X_i)$ and the vertex corresponding to a literal $\neg X_i$ is connected to a negative output of $g(X_i)$. Finally, one vertex of the spine denoted by a_i is connected to a vertex v_i of a collecting path, which is defined later.

Observation 6 *If one vertex of the spine is black, then all vertices of the clause gadget get black in the process.*

Collecting and distributing gadget

A *collecting path* is a path on m vertices v_1, \dots, v_m where each v_i is connected to a clause gadget. Moreover, the vertex v_1 is also connected to a vertex of degree one. A *distributing path* is a path on n vertices u_1, \dots, u_n . Each u_i is connected to a vertex of degree one and to the vertex z_i of the variable gadget $g(X_i)$. Finally, v_m is connected to u_1 ; see Figure 4.

Observation 7 *If the vertices of the distributing and collecting paths are white at the beginning they will become all black in the process only if all the clause gadgets get black during the process.*

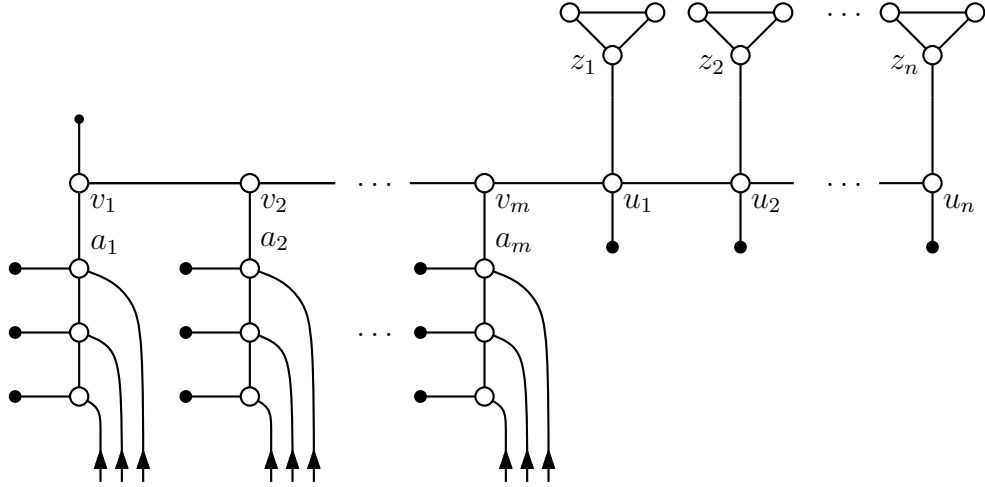


Fig. 4. A collecting path v_1, \dots, v_m and a distributing path u_1, \dots, u_n

Proof. If all spines of clause gadgets are black then it is easy to observe that the vertices of the collecting path get black in at most m steps from v_1 to v_m . Once v_m is black all the vertices of the distributing path get black in at most n steps from u_1 to u_n . It remains to check that v_i cannot get black before a neighboring vertex a_i gets black.

We start by checking the vertices of the distributing path. By Observation 5(e), the vertex z_n cannot get black before u_n . Thus u_n cannot get black before u_{n-1} because u_{n-1} is one of the two remaining neighbors which can be black before u_n . Similarly, for $0 < i < n$, the vertices z_i and u_{i+1} cannot get black before u_i . Thus u_i cannot get black before u_{i-1} . Similarly, u_0 cannot get black before v_m .

Analogously, no vertex v_i , $0 < i \leq m$, of the collecting path can get black before v_{i-1} and a_i are both black. For $i = 0$ we get that a_0 must get black before v_0 . \square

The graph $G_{\mathcal{F}} = (V, E)$ corresponding to the 3-SAT instance \mathcal{F} constructed from these gadgets has a linear size in the size of \mathcal{F} . The size of L is $15m + n + 1$. Thus s is set to $n + |L| = 15m + 2n + 1$.

Lemma 8. *If \mathcal{F} is satisfiable then there exists an irreversible 2-conversion set S of size $n + |L|$ in $G_{\mathcal{F}}$.*

Proof. The set S consists of $|L|$ leaves and from every variable gadget $g(X_i)$ we choose either x_i or y_i if X_i is evaluated true or false, respectively. Since \mathcal{F} is satisfiable then after a finite number of steps every gadget for a clause has at least one black vertex. Then in at most two steps all clause gadgets are completely black. Next the collecting path gets black in at most m steps and the distributing path gets black in next n steps. Now, for $0 \leq i \leq n$, the vertex z_i has two black

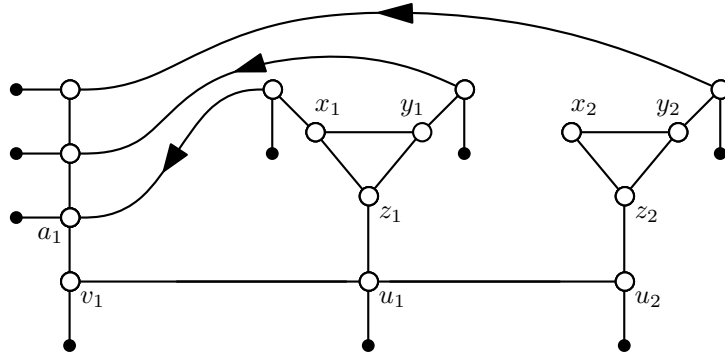


Fig. 5. A graph $G_{\mathcal{F}}$ for the formula $\mathcal{F} = (X_1 \vee \neg X_1 \vee \neg X_2)$.

neighbors and it gets black. The remaining white vertex of the pair x_i, y_i gets black in the next step. Finally, also the remaining antennas for every variable get black. Hence all vertices of $G_{\mathcal{F}}$ get black in the process. \square

Lemma 9. *If \mathcal{F} is not satisfiable then there is no irreversible 2-conversion set of size $n + |L|$.*

Proof. Assume for contradiction that there exists an irreversible 2-conversion set S of size $n + |L|$. By Observation 3, $L \subseteq S$. Moreover, due to Observation 5, S must contain one of $\{x_i, y_i\}$ for each $i \in [n]$. Hence there are no other black vertices. We derive the truth assignment of the variables in the following way. We set X_i true if $x_i \in S$ and false if $y_i \in S$.

Let $C = (L_o \vee L_p \vee L_q)$ be a clause of \mathcal{F} . The gadget corresponding to C gets black after finite number of steps of the process. By Observation 7, $g(C)$ got black because of one of $g(X_o)$, $g(X_p)$ or $g(X_q)$. Hence C is evaluated as true in \mathcal{F} . Therefore all clauses of \mathcal{F} are evaluated as true which is a contradiction with the assumption that \mathcal{F} is not satisfiable. \square

The proof of Theorem 1 is now finished.

3 Irreversible 3-conversion set in toroidal grids

In this section we show a construction of an irreversible 3-conversion set S which proves Theorem 2. We denote the toroidal grid of size $n \times m$ by $T(n, m)$. When the dimensions of the grid are clear from the context or not important, we simply write T instead of $T(m, n)$. We assume that the entries of the grid are *squares* and two of them are neighboring if they share an edge. First we discuss the general case where $m \neq 4$ and $n \neq 4$.

We define a coordinate system on T such that the left bottom corner is $[0, 0]$. A *pattern* is a small and usually rectangular piece of a grid where squares are black and white. Placing a pattern P at position $[i, j]$ in T means that the left

bottom square of P is at $[i, j]$ in T . If a vertex of T has color defined by several patterns then it is white only if it is white in all the patterns. We describe a rectangle of a grid by the coordinates of its left bottom and right top squares. By tiling a rectangle R by a pattern P we mean placing several non-overlapping copies of P to R such that every square of R is covered.

Let $m = 3k + a$ and $n = 3l + b$, where $a, b \in \{0, 2, 4\}$. By g we denote the greatest common divisor of k and l .

For $0 \leq i \leq g - 2$ we place a pattern $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ at $[0, 3i]$. Next we tile the rest of the rectangle $[0, 0][3k - 1, 3l - 1]$ by a pattern $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$. The remaining part of the grid can be decomposed into three rectangles of dimensions $3k \times b$, $a \times 3l$ and $a \times b$ (some of them may be empty).

We distinguish several cases depending on a and b . They are depicted in Figure 6 and their description follows.

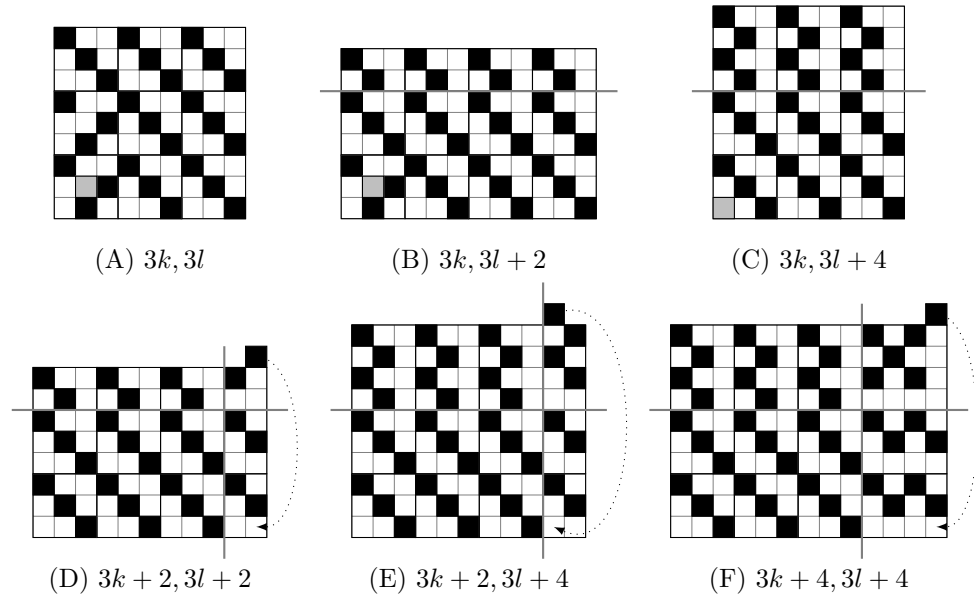


Fig. 6. The cases for $T(m, n)$.

- (A) $a = 0, b = 0$ We do not add anything now.
- (B) $a = 0, b = 2$ We tile the rectangle $3k \times 2$ with $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$.
- (C) $a = 0, b = 4$ We tile the rectangle $3k \times 4$ with $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$.
- (D) $a = 2, b = 2$ We tile the rectangle $3k \times 2$ with $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$, the rectangle $2 \times 3l$ with $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ and place $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ at $[3k, 3l]$.
- (E) $a = 2, b = 4$ We tile the rectangle $3k \times 4$ with $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$, the rectangle $2 \times 3l$ with $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ and place $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ at $[3k, 3l]$.

(F) $a = 4, b = 4$ We tile the rectangle $3k \times 4$ with $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$, the rectangle $4 \times 3l$ with $\begin{smallmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix}$ and place $\begin{smallmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix}$ at $[3k, 3l]$.

The construction is finished for cases (D), (E) and (F). Cases (A), (B) and (C) require an extra black square. We place it at $[0,0]$ or $[1,1]$. It is colored grey in Figure 6.

Let S be the set of black squares in our construction. In the cases (A), (B) and (C) the size of S is $\frac{mn}{3} + 1 = \frac{mn+3}{3}$. In the cases (D) and (F) the size of S is $\frac{mn+2}{3}$ and in the last case (E) the size of S is $\frac{mn+4}{3}$.

Now we check the correctness of the construction. We start with the case (A) where $m = 3k$ and $n = 3l$.

By a *white cycle* we denote a connected set of white squares $W \subseteq T$ where every square in W has at least two neighbors in W . Note that T cannot contain any white cycle if the squares of an irreversible 3-conversion set are black.

Observation 10 *Let $T(3k, 3l)$ be filled with $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$. Then it contains g disjoint l -white cycles.*

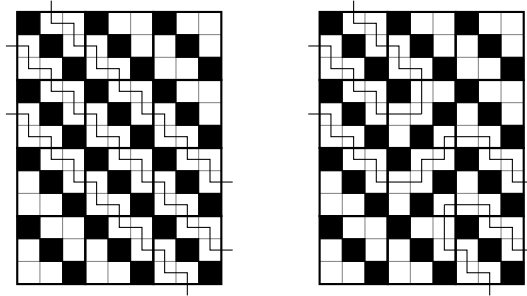


Fig. 7. Merging three white cycles into one.

Let the whole grid be filled by $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$. By Observation 10, there are g white cycles after the filling. First we merge the cycles into one long cycle by changing $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ to $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ in the first column and the first $g - 1$ rows; see Figure 7. Finally, we add one more black vertex to break the resulting cycle.

Observe that the small patterns used in (B) – (F) just extend the size of the toroidal grid but do not change the structure of white cycles from the $3k \times 3l$ rectangle. Thus the argument for the case (A) can be easily extended to all the other cases.

This finishes the construction for the general case.

Now we assume without loss of generality that $n = 4$. Let $m = 2k + a$, where $a \in \{1, 2\}$. We tile the rectangle $[0, 0][2k - 1, 3]$ by a pattern $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$. If $a = 1$ we place $\begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix}$ at $[2k - 1, 0]$ and if $a = 2$ we place $\begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix}$ at $[2k, 0]$. The resulting grids are depicted in Figure 8.

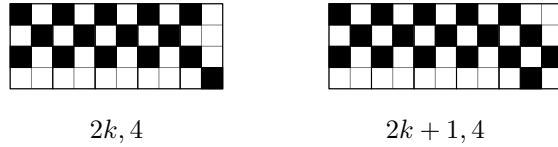


Fig. 8. The cases for $T(m, 4)$.

References

1. E. Boros and V. Gurvich, *On complexity of algorithms for modeling disease transmission and optimal vaccination strategy*, RUTCOR Research Report 16–2007 (2007).
2. P. A. Dreyer and F. S. Roberts, *Irreversible k -threshold processes: graph-theoretical threshold models of the spread of disease and of opinion*, *Discrete Appl. Math.* **157** (2009), no. 7, 1615–1627.
3. P. Flocchini, E. Lodi, F. Luccio, L. Pagli, and N. Santoro, *Dynamic monopolies in tori*, *Discrete Applied Mathematics* **137** (2004), no. 2, 197–212.
4. F. L. Luccio, *Almost exact minimum feedback vertex set in meshes and butterflies*, *Inform. Process. Lett.* **66** (1998), no. 2, 59–64.
5. D. A. Pike and Y. Zou, *Decycling cartesian products of two cycles*, *SIAM J. Discrete Math.* **19** (2005), no. 3, 651–663.
6. F. S. Roberts, *Challenges for Discrete Mathematics and Theoretical Computer Science in the Defense Against Bioterrorism*, DIMACS Technical Report 2002–40 (2002).
7. S. Ueno, Y. Kajitani, and S. Gotoh, *On the nonseparating independent set problem and feedback set problem for graphs with no vertex degree exceeding three*, *Discrete Math.* **72** (1988), 355–360.