4-critical graphs on surfaces without contractible (≤ 4)-cycles

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Abstract

We show that if G is a 4-critical graph embedded in a fixed surface Σ so that every contractible cycle has length at least 5, then G can be expressed as $G = G' \cup G_1 \cup G_2 \cup \ldots \cup G_k$, where $|V(G')|$ and k are bounded by a constant (depending linearly on the genus of Σ) and $G_1, \ldots, G_k$ are graphs (of unbounded size) whose structure we describe exactly. The proof is computer-assisted—we use computer to enumerate all plane 4-critical graphs of girth 5 with a precolored cycle of length at most 16, that are used in the basic case of the inductive proof of the statement.

1 Introduction

The problem of 3-coloring triangle-free graphs embedded in a fixed surface, motivated by the celebrated Grötzsch theorem [7], has drawn much attention. Thomassen [10] showed that if a graph G is embedded in the torus or the projective plane so that every contractible cycle has length at least 5, then G

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is 3-colorable. Thomas and Walls [9] showed that graphs of girth at least 5 embedded in the Klein bottle are 3-colorable and gave a description of all 4-critical graph on the Klein bottle without contractible cycles of length at most 4. Gimbel and Thomassen [6] showed that graphs of girth 6 embedded in the double torus are 3-colorable and described triangle-free projective plane graphs that are not 3-colorable.

Recently, Dvořák, Král' and Thomas [5] gave a structural description of 4-critical (i.e., minimal non-3-colorable) triangle-free graphs embedded in a fixed surface, and used this result to give a linear-time algorithm to decide 3-colorability of such graphs. In particular, this description implies the following.

Theorem 1 (Dvořák, Král' and Thomas [5])**.** *There exists an absolute constant K such that every* 4*-critical graph of girth* 5 *embedded in a surface of genus g has at most Kg vertices.*

This improves a doubly-exponential bound by Thomassen [11]. Let us note that the linear bound was proved by Postle [8] also for girth 5 and 3-list coloring. Somewhat unsatisfactorily, the bound on K given by Dvořák et al. [5] is rather weak, proving that $K < 10^{28}$ (we are not aware of any non-trivial lower bound, and suspect that $K \approx 100$ should suffice). One of the reasons why this bound is so large is hidden in the handling of the basic case of the induction, where they prove that if G is a plane graph with exactly two faces C_1 and C_2 of length at most 4, all other cycles have length at least 5 and the distance between C_1 and C_2 is at least 1500000, then any precoloring of C_1 and C_2 extends to a proper coloring of G by three colors. In this paper, we give a computer-assisted proof showing that it suffices to assume that the distance between C_1 and C_2 is at least 4, which can be used to show that $K < 10^{21}$. We were originally hoping in a bigger improvement on K.

Theorem 2. *Let G be a plane graph and let C_1 and C_2 be faces of G of length at most* 4*, such that every cycle in G distinct from C_1 and C_2 has length at least* 5*. If the distance between C_1 and C_2 is at least* 4*, then every precoloring of $C_1 \cup C_2$ extends to a proper* 3*-coloring of G.*

Combining these results, we give a more precise description of the structure of the 4-critical graphs without contractible cycles of length at most 4 (a cycle is *contractible* if it separates the surface to two parts and at least one of them is homeomorphic to the open disc). Thomassen [10] showed that every graph of girth at least 5 embedded in the projective plane or in the torus is 3-colorable. Actually, he proved a stronger claim that enables him to apply induction: every graph embedded in the projective plane or in the torus so that all contractible cycles have length at least 5 (but there may be non-contractible triangles or 4-cycles) is 3-colorable. Thus, it might seem possible to strengthen Theorem 1 by allowing non-contractible triangles or 4-cycles. However, Thomas and Walls [9] exactly characterized 4-critical graphs embedded in the Klein bottle so that no contractible cycle has length at most 4, showing that there are infinitely many such graphs.

Let $\mathcal{C}$ be the class of plane graphs that can be obtained from a cycle of length 4 by a finite number of repetitions of the following operation: given a graph with the outer face $v_1v_2v_3v_4$ of length 4 such that v_1 and v_3 have degree two, add new vertices v_2', v_3' and v_4' and edges v_1v_2', $v_2'v_3'$, $v_3'v_4'$, $v_4'v_1$ and v_3v_3', and let $v_1v_2'v_3'v_4'$ be the outer face of the resulting graph. Examples of elements of $\mathcal{C}$ are the 4-cycle, and the graphs Z_3 in Figure 1, A_{11} in Figure 11 and $Z_3A_{11}a$ and $Z_3A_{11}b$ in Figure 15. Let $\mathcal{C}'$ be the class of graphs obtained from those in $\mathcal{C}$ by adding a chord joining the pair of vertices of degree two in both of the 4-faces (thus introducing 4 triangles). Each graph in $\mathcal{C}'$ can be embedded in the Klein bottle by putting crosscaps on both newly added chords; the 4-faces of a graph in $\mathcal{C}$ thus become 6-faces in such an embedding of the corresponding graph in $\mathcal{C}'$.

Thomas and Walls [9] proved that a graph embedded in the Klein bottle without contractible (≤ 4)-cycles is 4-critical if and only if it belongs to $\mathcal{C}'$. We extend this result to other surfaces.

Theorem 3. *There exists a function $f(g) = O(g)$ with the following property. Let G be a 4-critical graph embedded in a surface Σ of genus g so that every contractible cycle has length at least 5. Then G contains a subgraph H such that*

- *$|V(H)| \leq f(g)$, and*
- *if F is a face of H that is not equal to a face of G, then F has exactly two boundary walks, each of the walks has length 4, and the subgraph of G drawn in the closed region corresponding to F belongs to $\mathcal{C}$.*

2 Preliminaries

In order to state more technical results necessary to prove Theorems 2 and 3, we need a few definitions. The graphs considered in this paper are undirected and without loops and parallel edges. By a *coloring* of a graph we always mean a proper 3-coloring. By the *genus* $g(\Sigma)$ of a surface Σ we mean the Euler genus, i.e., $2h + c$, where h is the number of handles and c is the number of crosscaps attached to the sphere in order to create Σ. If G is a graph embedded in Σ, a *face* F of G is a maximal connected open subset of $\Sigma - G$. Sometimes, we also let F stand for the subgraph of G consisting of the edges of G contained in the closure of F. We let $\ell(F)$ be the sum of the lengths of the boundary walks of F in G.

A graph G is *k-critical* if G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable. A well-known result of Grötzsch [7] states that all triangle-free planar graphs are 3-colorable, i.e., there are no planar triangle-free 4-critical graphs. Since the cycles of length 4 can be easily eliminated, the main part of the proof of Grötzsch's theorem concerns graphs of girth 5. Generalizing this result, Thomassen [11] proved that there exists a function f such that every 4-critical graph of girth 5 and genus g has at most $f(g)$ vertices (where f is double-exponential in g), and thus the number of such graphs is finite. This was later improved by Dvořák et al. [5], by showing that the number of vertices of such a graph is at most linear in g (Theorem 1). Both the original result of Thomassen and its improvement allow a bounded number of vertices to be precolored. To state this generalization, we need to extend the notion of a 4-critical graph.

There are two natural ways one can define a critical graph with precolored vertices. Consider a graph G and a subgraph (not necessarily induced) $S \subseteq G$. We call G *strongly S-critical* if there exists a coloring of S that does not extend to a coloring of G, but extends to a coloring of every proper subgraph $G' \subset G$ such that $S \subseteq G'$. We say that G is *S-critical* if for every proper subgraph $G' \subset G$ such that $S \subseteq G'$, there exists a coloring of S that does not extend to a coloring of G, but extends to a coloring of G'. We call a (strongly) S-critical graph G *nontrivial* if $G \neq S$. Note that every strongly S-critical graph is also S-critical, but the converse is false (for example, if G is a cycle S with two chords, then G is S-critical, but not strongly S-critical). Also, G is $\emptyset$-critical (or strongly $\emptyset$-critical) if and only if G is 4-critical.

Dvořák et al. [5] bounded the size of critical graphs as follows:

Theorem 4 (Dvořák et al. [5]). *Let $K = 10^{28}$. Let G be a graph embedded in a surface Σ of genus g and let $\{F_1, F_2, \ldots, F_k\}$ be a set of faces of G such that the open region corresponding to F_i is homeomorphic to the open disk for $1 \le i \le k$. If G is $(F_1 \cup F_2 \cup \ldots \cup F_k)$-critical and every cycle of length of at most 4 in G is equal to F_i for some $1 \le i \le k$, then*

$$|V(G)| \le \ell(F_1) + \ldots + \ell(F_k) + K(g + k).$$

Let us note that such a claim does not hold without the restriction on the cycles of length 4, since Youngs [13] gave a construction of an infinite family of 4-critical triangle-free graphs that can be embedded in any surface distinct from the sphere.

Analogously, we will prove a generalization of Theorem 3 allowing a bounded number of precolored vertices (Theorem 25). It is easy to reduce the proof to the case that Σ is the sphere and exactly two cycles are precolored. In this case, we say that the graph is embedded in the *cylinder*, and we call the precolored cycles the *boundaries* of the cylinder. In fact, it suffices to consider the case that both boundaries have length at most 4. By cutting along cycles of length at most 4, such a graph decomposes to a possibly large number of graphs embedded in the cylinder such that the only cycles of length at most 4 are the boundaries. The main part of our proof is based on an enumeration of such graphs:

Theorem 5. *Let G be a connected graph embedded on the cylinder with distinct boundaries C_1 and C_2 such that $\ell(C_1), \ell(C_2) \le 4$ and every cycle in G distinct from C_1 and C_2 has length at least 5. If G is $(C_1 \cup C_2)$-critical, then G is isomorphic to one of the graphs drawn in Figures 1 and 2.*

It is straightforward to check that the distance between the boundaries in the described critical graphs is at most three; hence, Theorem 5 implies Theorem 2.

Theorem 5 is proved by the method of reducible configurations: Considering a graph G on the cylinder in that the distance between the boundaries C_1 and C_2 is at least 5, we find a *reducible configuration*—a subgraph that enables us to transform G to a smaller graph H that is nontrivial $(C_1 \cup C_2)$-critical if and only if G is nontrivial $(C_1 \cup C_2)$-critical. If H does not contain cycles of length at most 4 distinct from C_1 and C_2, then we argue that H is not one of the graphs enumerated in Theorem 5, thus showing that G is not $(C_1 \cup C_2)$-critical. Otherwise, we cut H along the cycles of length at most

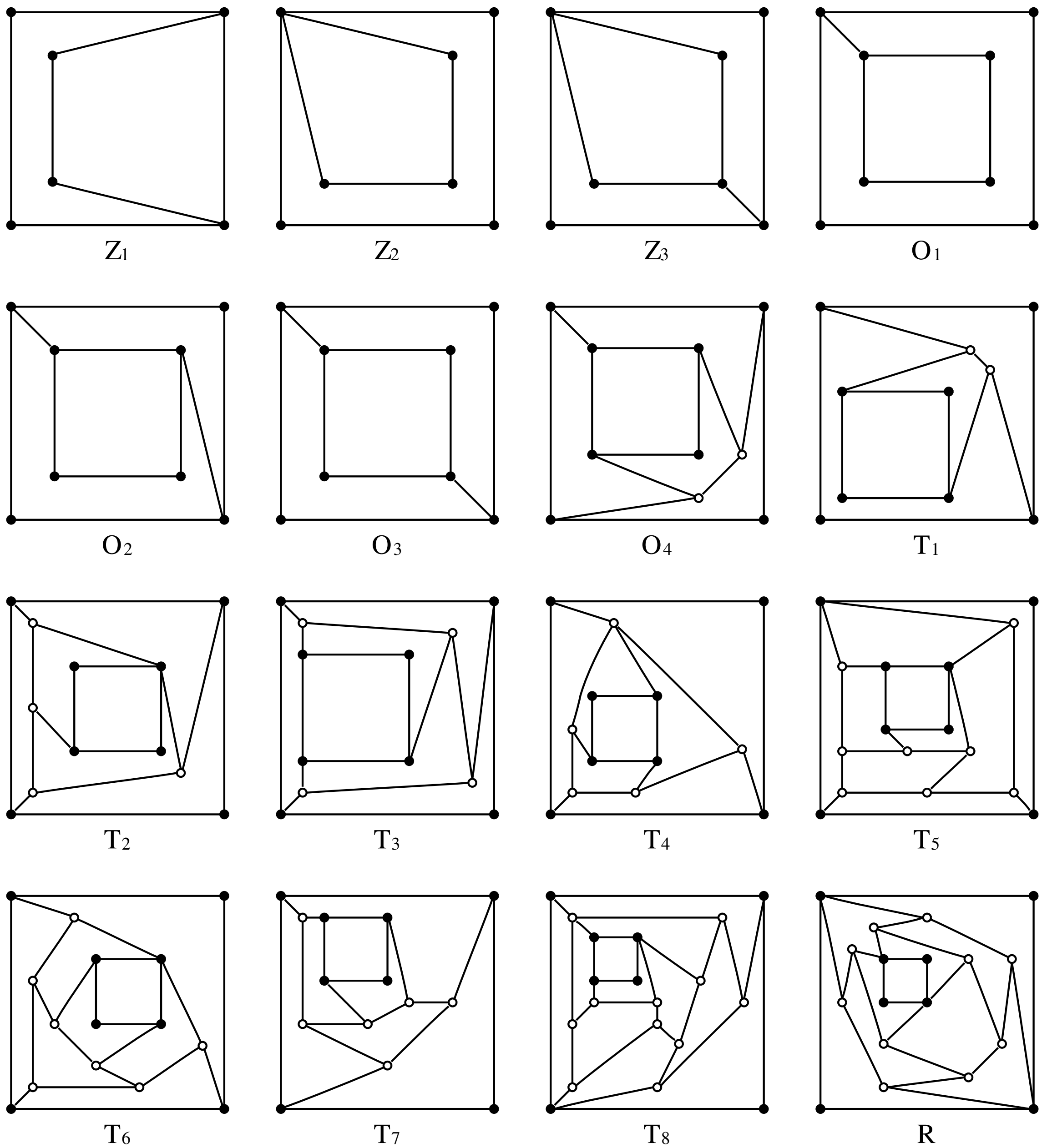


Figure 1: Critical graphs on the cylinder, bounded by 4-cycles.

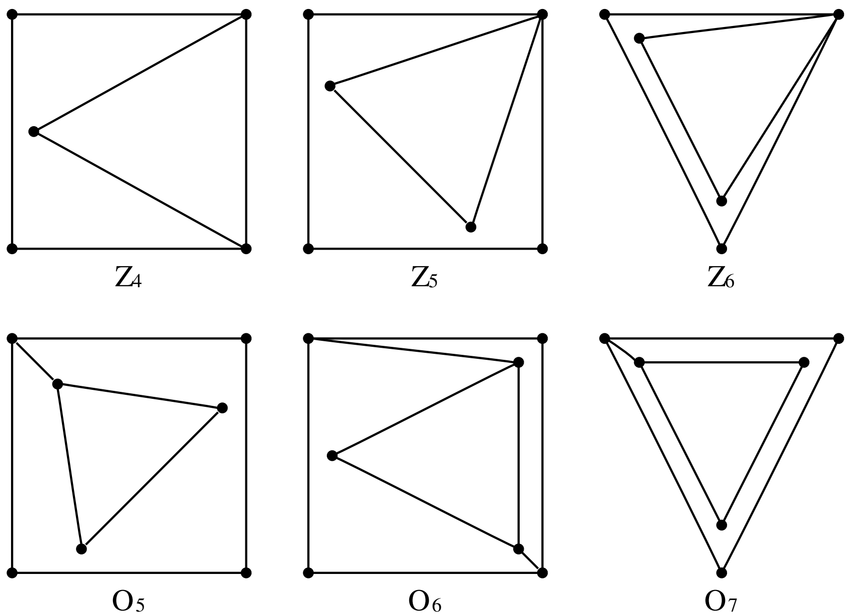
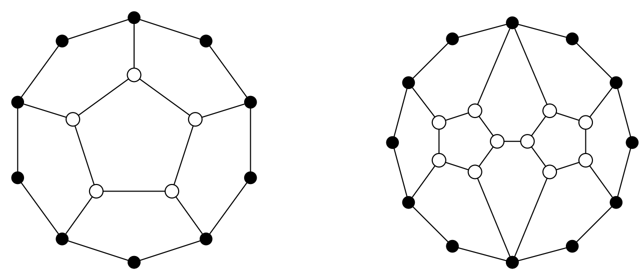


Figure 2: Critical graphs on the cylinder, with precolored triangle.



(a) (b)

Figure 3: Nontrivial critical graphs with precolored face of length at most 12.

4, use Theorem 5 to describe the resulting pieces, and conclude that every precoloring of C_1 and C_2 extends to a coloring of H, again implying that H (and thus also G) is not $(C_1 \cup C_2)$-critical.

This leaves us with the case that the distance between C_1 and C_2 in G is at most 4. In that case, we color the shortest path between C_1 and C_2 and cut the graph along it, obtaining a graph of girth 5 with a precolored face of length at most 16. Such critical graphs with a precolored face of length at most 11 were enumerated by Walls [12] and independently by Thomassen [11], who also gives some necessary conditions for graphs with a precolored face of length 12. The exact enumeration of graphs with a precolored face of length 12 appears in Dvořák and Kawarabayashi [4]. These results can be summarized as follows.

Given a plane graph with the outer face B, a *chord* of B is an edge in $E(G) \setminus E(B)$ incident with two vertices of B. A t-*chord* of B is a path $Q = q_0q_1 \dots q_t$ of length t ($t \geq 2$) such that $q_0 \neq q_t$ and $V(Q) \cap V(B) = \{q_0, q_t\}$. Sometimes, we refer to a chord as a 1-*chord*. A *shortcut* is a t-chord of B such that t is smaller than the distance between u and v in B.

Theorem 6 (Dvořák and Kawarabayashi [4]). *Let G be a plane graph of girth at least 5 and B the outer face of G of length at most 12. If B is a cycle, G contains no shortcut of length at most two, no two vertices of G of degree two are adjacent and G is nontrivial B-critical, then G is isomorphic to one of the graphs in Figure 3.*

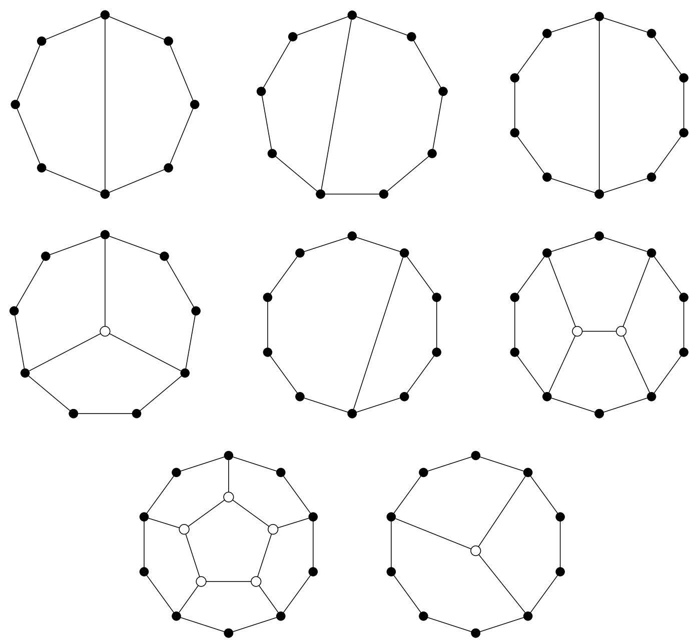


Figure 4: Nontrivial critical graphs with precolored face of length at most 10.

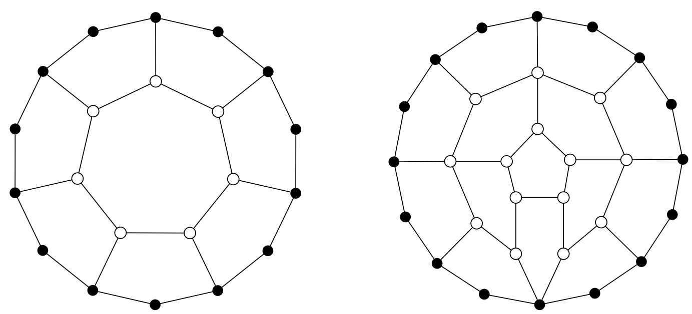


Figure 5: Nontrivial critical graphs with precolored faces of length 14 and 16, respectively.

Let us note that all other critical graphs with the precolored face of length at most 12 can be constructed from the graphs in Figure 3 and a 5-cycle by a sequence of subdividing the edges of the outer face and gluing pairs of graphs along paths of length at most two in their outer faces. For instance, all such nontrivial critical graphs with $\ell(B) \leq 10$ are drawn in Figure 4.

The number of critical graphs grows exponentially with the length of the precolored face, and enumerating all the graphs becomes increasingly difficult. We implemented an algorithm to generate such graphs based on the results of Dvořák and Kawarabayashi [4], and used the computer to enumerate the graphs with the outer face of length at most 16. There are 108 such graphs with the precolored face of length 13, 427 for length 14, 1746 for length 15 and 7969 for length 16, up to isomorphism (including the case that $G = B$). Even excluding the trivial cases that G has a shortcut of length at most 2 or contains two adjacent vertices of degree two as in Theorem 6, there still remain 8 graphs with the precolored face of length 14 (there are none with a precolored face with length 13), 13 with the length 15 and 76 with the length 16, thus we do not include their list in this paper. Here, let us point out only the following claim, which still makes it possible to enumerate all the graphs easily:

Theorem 7. *Let G be a plane graph of girth 5 and B the outer face of G of length at most 16. If G has no shortcut of length at most 4 and G is nontrivial B-critical, then G is isomorphic to the graph in Figure 3(a) or to the graphs in Figure 5.*

The complete list of the graphs, as well as programs used to generate them can be found at `http://arxiv.org/abs/1305.2670`. A description of the programs can be found in Section 7.

In Section 4 we give a proof of Theorem 7. Section 5 is devoted to Theorem 5. Finally, Section 6 contains a proof of Theorem 3.

3 Properties of the critical graphs

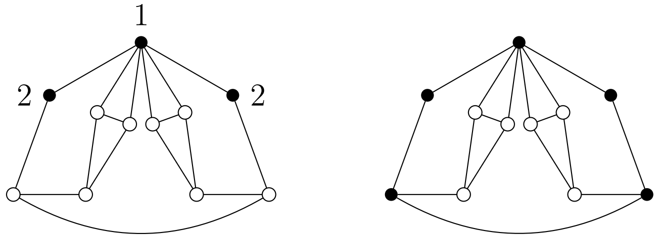
Let G be a T-critical graph, for some $T \subseteq G$. For $S \subseteq G$, a graph $G' \subseteq G$ is an *S-component* of G if $S \subseteq G'$, $T \cap G' \subseteq S$ and all edges of G incident with vertices of $V(G') \setminus V(S)$ belong to G'. When we use S-components, T will always be clear from the context. For example, if G is a plane graph with T contained in the boundary of its outer face and S is a cycle in G, then the subgraph of G consisting of the vertices and edges drawn in the closed disk bounded by S is an S-component of G.

Lemma 8. *Let G be a T-critical graph. If G' is an S-component of G, for some $S \subseteq G$, then G' is S-critical.*

Proof. Since G is T-critical, every isolated vertex of G belongs to T, and thus every isolated vertex of G' belongs to S. Suppose for a contradiction that G' is not S-critical. Then, there exists an edge $e \in E(G') \setminus E(S)$ such that every coloring of S that extends to $G' - e$ also extends to G'. Note that $e \notin E(T)$. Since G is T-critical, there exists a coloring ψ of T that extends to a coloring φ of $G - e$, but does not extend to a coloring of G. However, by the choice of e, the restriction of φ to S extends to a coloring φ' of G'. Let φ'' be the coloring that matches φ' on $V(G')$ and φ on $V(G) \setminus V(G')$. Observe that φ'' is a coloring of G extending ψ, which is a contradiction. □

Let us remark that Lemma 8 would not hold if we replaced "critical" with "strongly critical", see Figure 6 for an example. This is the main reason why we (unlike some previous works in the area, e.g. Thomassen [11]) consider critical rather than strongly critical graphs. However, since every strongly critical graph is also critical, all the characterizations and enumerations that we provide for critical graphs apply to strongly critical graphs as well.

Lemma 8 in conjunction with Theorem 6 describes the subgraphs drawn inside cycles in plane critical graphs. Let us state a few useful special cases of this claim explicitly:



(a) (b)

Figure 6: (a) A strongly critical graph, with a precolored path on three vertices; (b) not a strongly critical graph with a precolored 5-cycle.

Corollary 9. *Let G be a plane graph and T a subgraph of G such that G is T-critical. Suppose that every cycle in G that is not contained in T has length at least 5. Let C be a cycle in G and H the subgraph of G drawn in the closed disk bounded by C. Suppose that $H \cap T \subseteq C$. If $H \neq C$, then $\ell(C) \geq 8$. If $|V(H) \setminus V(C)| \geq 1$, then $\ell(C) \geq 9$. Finally, if $|V(H) \setminus V(C)| \geq 2$, then $\ell(C) \geq 10$.*

4 Graphs with one precolored face

In this section we describe an algorithm for enumerating all B-critical graphs of girth 5 with outer face B. First, we describe a previously know recursive description. Then we show that it can be turned into an algorithm for enumerating B-critical graphs. We implemented the resulting algorithm and we provide its source code.

Dvořák and Kawarabayashi [4] proved the following claim (in a more general setting of list-coloring):

Theorem 10 (Dvořák and Kawarabayashi [4])**.** *Let G be a plane graph of girth at least 5 with the outer face B bounded by a cycle of length at least 10. If G is B-critical, then $|E(G)| \leq 18\ell(B) - 160$ and $|V(G)| \leq \frac{37\ell(B)-320}{3}$.*

The obvious algorithm to enumerate the critical graphs by trying all the graphs of the size given by Theorem 10 is too slow. However, the proof of Theorem 10 identifies a list of configurations such that at least one of them must appear in each plane critical graph of girth at least 5 with the precolored outer face. For each such configuration, a reduction is provided that makes it possible to obtain G from critical graphs with a shorter precolored outer face. This leads to a practical algorithm to generate such graphs. For the algorithm, it turns out to be simpler to use the following easy corollary of the structural result of Dvořák and Kawarabayashi [4].

Theorem 11 (Dvořák and Kawarabayashi [4])**.** *Let G be a plane graph of girth at least 5 with the outer face B bounded by a cycle. If G is a B-critical graph, then G is 2-connected and at least one of the following holds:*

(a) G has a shortcut of length at most 4, or

(b) G contains two adjacent vertices of degree two (belonging to B), or

(c) there exists a path $P = v_0v_1v_2v_3v_4 \subseteq B$ and a 4-chord $Q = v_0w_1w_2w_3v_4$ of B such that $v_2w_2 \in E(G)$, or

(d) there exists a 4-chord $Q = w_0w_1w_2w_3w_4$ of B and 5-faces C_1 and C_2 such that a cycle $C \subseteq B \cup Q$ distinct from B bounds a face of G, $|V(C_1 \cap B)| = |V(C_2 \cap B)| = 3$, $C_1 \cap C = w_0w_1$ and $C_2 \cap C = w_3w_4$.

See Figure 7.

While these configurations are not sufficient to prove Theorem 10, each of the more complicated configurations considered in the proof of Theorem 10 contains one of the configurations of Theorem 11 as a subgraph. For the reduction in case (d), we also need the following result, which is shown for strongly critical graphs in Thomassen [11], and explicitly for critical graphs in Dvořák and Kawarabayashi [4]. For a plane graph G with the outer face B, let $m(G)$ be the length of the longest face of G distinct from B.

Theorem 12 ([4, 11])**.** *Let G be a plane graph of girth at least 5 with the outer face B bounded by a cycle. If G is a nontrivial B-critical graph, then $m(G) \leq \ell(B) - 3$.*

We now define several graph generating operations roughly corresponding to the cases (a)–(d) of Theorem 11. Let G_1 and G_2 be plane graphs with outer faces B_1 and B_2, respectively.

(a) Let $P_i = v_0^iv_1^i \dots v_t^i$ be paths such that $P_i \subseteq B_i$ for $i \in \{1, 2\}$ and some $t > 0$. We let $U(G_1, P_1, G_2, P_2)$ be the graph obtained from the disjoint union of G_1 and G_2 by identifying v_j^1 with v_j^2 for $j = 0, 1, \dots, t$ and suppressing the arising parallel edges.

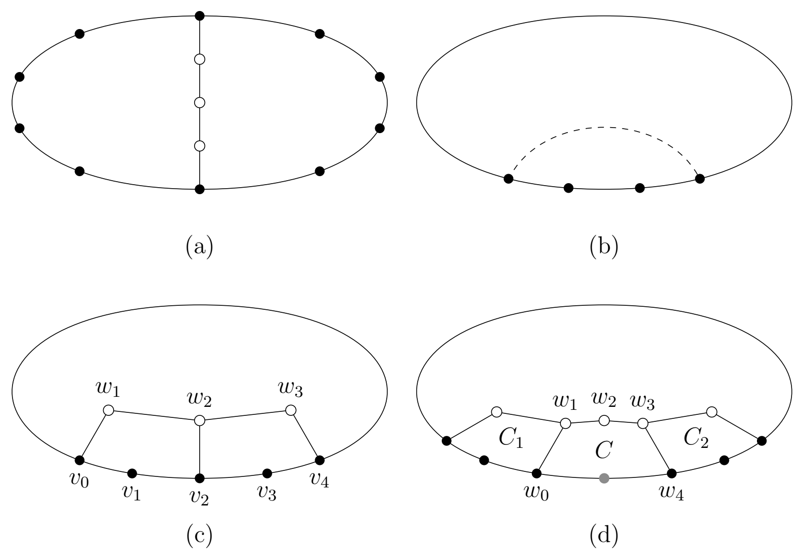


Figure 7: Cases of Theorem 11.

(b) For an edge $e \in E(G_1)$, let $S(G_1, e)$ be the graph obtained from G_1 by subdividing the edge e by one vertex.

(c) For a path $P = v_0w_1w_2w_3v_4 \subseteq B_1$, let $J(G_1, P)$ be the graph obtained from G_1 by adding new vertices v_1, v_2 and v_3 and edges v_0v_1, v_1v_2, v_2w_2, v_2v_3 and v_3v_4.

(d) Let $P = u_0u_1u_2u_3u_4 \subseteq B_1$ be a path, let $y_1 = u_1, y_2, \ldots, y_k = u_3$ be the vertices adjacent to u_2 in the cyclic order according to their drawing around u_2 and let f_i be the edge u_2y_i for $1 \le i \le k$. For $2 \le i \le k$ and $0 \le j \le 1$, let $X(G_1, P, f_i, j)$ be the plane graph obtained from G_1 by splitting u_2 to two vertices u_2' and u_2'' so that u_2' is adjacent to y_1, $y_2, \ldots, y_{i-1}$ and u_2'' is adjacent to $y_i, \ldots, y_k$, adding vertices x_1, x_2, $\ldots$, x_{4+j} and edges u_0x_1, x_1x_2, x_2x_3, $\ldots$, $x_{4+j}u_4$, $u_2'x_2$ and $u_2''x_{3+j}$. See Figure 8.

Note that $m(X(G_1, P, f_i, j)) \le m(G_1) + j + 3$ and the length ℓ of the outer face of $X(G_1, P, f_i, j)$ is equal to $\ell(B_1) + j + 1$. By Theorem 12, if G_1 is a nontrivial B_1-critical graph, then $m(X(G_1, P, f_i, j)) \le \ell(B_1) + j < \ell$.

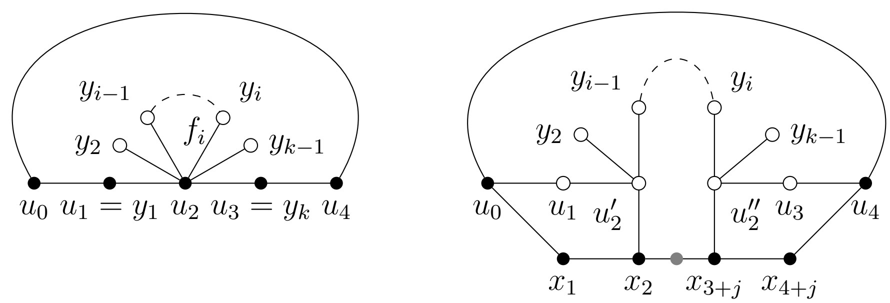


Figure 8: $X(G_1, P, f_i, j)$.

For $i \geq 5$, let $\mathcal{K}_i$ be the set of all (up to isomorphism) plane graphs G of girth 5 with the outer face B bounded by a cycle, such that G is B-critical and $\ell(B) = i$. By Theorem 10 and Theorem 6, $\mathcal{K}_i$ is finite. For a 2-connected plane graph G with the outer face B, let $\mathcal{K}(G)$ be the set of all graphs $H \supseteq G$ with the outer face B such that for every face C of G, the subgraph of H drawn in the closed disk bounded by C belongs to $\mathcal{K}_{\ell(C)}$. In other words, $\mathcal{K}(G)$ consists of graphs obtained from G by pasting a critical graph to each face distinct from the outer one. Let us remark that we do not exclude the case that the pasted graph is trivial, i.e., a face of G may also be a face of some graphs in $\mathcal{K}(G)$. Note that $\mathcal{K}(G)$ is finite, and can be constructed by a straightforward algorithm if the sets $\mathcal{K}_i$ are provided for $5 \leq i \leq m(G)$.

For some ℓ, suppose that $\mathcal{S}$ is a finite set of plane graphs G of girth at least 5 with the outer face $B(G)$ bounded by a cycle of length ℓ. Let $\mathcal{K}(\mathcal{S}) = \bigcup_{G \in \mathcal{S}} \mathcal{K}(G)$. Let $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{S}$ be the set consisting of all $B(G)$-critical graphs G in $\mathcal{S}$. Let $\mathcal{J}'(\mathcal{S}) = \{J(G, P) : G \in \mathcal{S}, P \subseteq B(G), \ell(P) = 4\}$. Note that the outer face of each graph G in $\mathcal{J}'(\mathcal{S})$ has length ℓ. Let $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, …, be the sequence of sets of graphs such that $\mathcal{S}_0 = \mathcal{T}(\mathcal{S})$ and $\mathcal{S}_{i+1} = \mathcal{T}(\mathcal{J}'(\mathcal{S}_i))$ for $i \geq 0$. Let $\mathcal{J}(\mathcal{S}) = \bigcup_{i \geq 0} \mathcal{S}_i$. Since $\mathcal{J}(\mathcal{S}) \subseteq \mathcal{K}_\ell$, $\mathcal{J}(\mathcal{S})$ is finite, and there exists k such that $\mathcal{S}_i = \emptyset$ for each $i \geq k$. Therefore, the set $\mathcal{J}(\mathcal{S})$ can be constructed algorithmically by finding the sets $\mathcal{S}_0$, $\mathcal{S}_1$, …, as long as they are non-empty.

Let

$$\mathcal{K}_i^{(a)} = \left\{ U(G_1, P_1, G_2, P_2) : \begin{array}{c} 1 \le t \le 4, i + 2t = i_1 + i_2, 5 \le i_1 \le i_2 \le i - 1, \\ G_1 \in \mathcal{K}_{i_1}, G_2 \in \mathcal{K}_{i_2}, \\ P_j \text{ is a path in the outer face of } G_j \\ \text{with } \ell(P_j) = t, \text{ for } j \in \{1, 2\} \end{array} \right\}.$$

Let

$$\mathcal{K}_i^{(b)} = \{S(G, e) : G \in \mathcal{K}_{i-1}, e \text{ is an edge of the outer face of } G\}.$$

Let

$$\mathcal{K}_i^{(d)} = \left\{ X(G, P, e, j) : \begin{array}{c} 0 \le j \le 1, G \in \mathcal{K}_{i-j-1}, \\ P = u_0u_1u_2u_3u_4 \text{ is a path in the outer face of } G, \\ e \ne u_1u_2 \text{ is incident with } u_2 \end{array} \right\}.$$

Let $\mathcal{K}_i''$ consist of all graphs in $\mathcal{K}_i^{(a)} \cup \mathcal{K}_i^{(b)} \cup \mathcal{K}(\mathcal{K}_i^{(d)})$ that have girth at least 5. Let $\mathcal{K}_i' = \mathcal{J}(\mathcal{K}_i'')$. Note that the set $\mathcal{K}_i'$ is finite and can be constructed algorithmically, given the sets $\mathcal{K}_j$ for $5 \le j < i$.

Theorem 13. *The following holds for $i > 5$: $\mathcal{K}_i = \mathcal{K}_i'$.*

Proof. Note that every graph $G \in \mathcal{K}_i'$ is a plane B-critical graph of girth at least 5, where B is the outer face of G and $\ell(B) = i$ and thus $\mathcal{K}_i' \subseteq \mathcal{K}_i$. Therefore, we only need to show that $\mathcal{K}_i \subseteq \mathcal{K}_i'$. By Theorem 10, there exists a constant N such that $|V(H)| \le N$ for every $H \in \mathcal{K}_i$.

Consider a graph $G \in \mathcal{K}_i$ with the outer face B. If there exists a path $P = v_0v_1v_2v_3v_4 \subseteq B$ and a 4-chord $Q = v_0w_1w_2w_3v_4$ of B such that $v_2w_2 \in E(G)$, then let B' be the cycle obtained from B by replacing P by Q. By Corollary 9, $v_0v_1v_2w_2w_1$ and $v_4v_3v_2w_2w_3$ are 5-faces, and by Lemma 8, $G' = G - \{v_1, v_2, v_3\}$ is B'-critical. It follows that $G = J(G', Q)$. We conclude that there exists a sequence (of length at most $N/3$) of plane graphs $G = G_0$, $G_1, \ldots, G_k$ of girth at least 5 with the outer faces $B = B_0, B_1, \ldots, B_k$, respectively, and paths $P_1, \ldots, P_k$ such that G_j is B_j-critical for $0 \le j \le k$, $P_j \subseteq B_j$ and $G_{j-1} = J(G_j, P_j)$ for $1 \le j \le k$, and G_k does not contain the configuration (c) of Theorem 11. In other words, as long as G_j contains the configuration (c), we keep reducing the graph and when there is no configuration (c), we stop. We claim that $G_k \in \mathcal{K}_i''$, implying that $G \in \mathcal{K}_i'$.

Since G_k is a plane B_k-critical graph of girth at least 5, Theorem 11 implies that it contains one of the configurations (a), (b) or (d). If it contains the configuration (a) (a shortcut Q of length at most 4), then let C_1 and C_2 be the cycles in $B_k \cup Q$ distinct from B_k, and let H_j be the subgraph of G_k drawn in the closed disk bounded by C_j for $j \in \{1,2\}$. By Lemma 8, H_j is C_j-critical, which implies that $H_j \in \mathcal{K}_{\ell(C_j)}$. Since Q is a shortcut, $\ell(C_j) < i$. We conclude that $G_k \in \mathcal{K}_i^{(a)} \subseteq \mathcal{K}_i''$. Therefore, we may assume that G_k has no shortcut of length at most 4.

Suppose that G_k contains two adjacent vertices u and v of degree two (the configuration (b)). Since G_k is B_k-critical, both u and v belong to $V(B_k)$. The edge uv is not contained in any cycle of length 5, since otherwise there would exist a shortcut of length at most two. Let H be the graph obtained from G_k by identifying the vertices u and v to a new vertex w, with the outer face B', and note that H has girth at least 5. Furthermore, H is B'-critical, since each precoloring of B corresponds to a precoloring of B' matching it on $V(B) \setminus \{u,v\}$. Observe that $G = S(H,e)$, where e is an edge incident with w, and thus $G_k \in \mathcal{K}_i^{(b)} \subseteq \mathcal{K}_i''$. Thus, we may assume that no two vertices of degree two are adjacent in G_k. In particular, G_k is a nontrivial B_k-critical graph, and there exists a precoloring φ of B_k that does not extend to a coloring of G_k.

Finally, consider the case that G_k contains the configuration (d). That is, there exists a 4-chord $Q = w_0w_1w_2w_3w_4$ of B_k and 5-faces C_1 and C_2 such that a cycle $C \subseteq B_k \cup Q$ distinct from B_k bounds a face of G, $|V(C_1 \cap B_k)| = |V(C_2 \cap B_k)| = 3$, $C_1 \cap C = w_0w_1$ and $C_1 \cap C = w_3w_4$. Since G_k does not contain adjacent vertices of degree two, we have $\ell(C) \leq 6$. Let $j = \ell(C) - 5$. Let H be the graph obtained from G_k by removing w_0, w_4 and their neighbors in $V(B)$ and by identifying w_1 with w_3 to a new vertex w, and let B' be the outer face of H. Since w_2 has degree at least three[1], $w_1w_2w_3$ is not a subpath of the boundary of a face $F \neq C$ in G_k; hence, Corollary 9 implies that the girth of H is at least 5. Indeed, if there is a cycle Z in H of length at most 4, it must contain w. We can replace w by w_1, w_2, w_3 and obtain a cycle Z' of length at most 6 in G. Since w has degree at least three, the cycle is not a face which contradicts Corollary 9.

Observe that the precoloring φ of B_k (which does not extend to G_k) extends to a coloring ψ of $(B_k \cup C_1 \cup C_2 \cup C) - \{w_2\}$ such that w_1 and w_3 have the same color. Since φ does not extend to a coloring of G_k, we conclude

[1]Note that vertices of degree one or two in C-critical graph G must be in C. For every vertex v of degree at most two, every coloring of $G - v$ extends to a coloring of G, which contradicts C-criticality if $v \notin V(C)$.

that the precoloring of B' given by ψ does not extend to a coloring of H. Therefore, H has a nontrivial B'-critical subgraph H'. Let $P \subseteq B'$ be the path of length 4 such that w is the middle vertex of P. Lemma 8 implies that $G_k \in \mathcal{K}(X(H', P, e, j))$ for some edge $e \in E(H')$ incident with w. Thus, $G_k \in \mathcal{K}(\mathcal{K}_i^{(d)}) \subseteq \mathcal{K}_i''$.

It follows that $G_k \in \mathcal{K}_i''$, and thus $G \in \mathcal{K}_i'$. Since the choice of G was arbitrary, this implies that $\mathcal{K}_i \subseteq \mathcal{K}_i'$ and hence $\mathcal{K}_i' = \mathcal{K}_i$. □

The sets $\mathcal{K}_5$, …, $\mathcal{K}_{12}$ are given by Theorem 6. Theorem 13 gives an algorithm that we used to construct the sets $\mathcal{K}_{13}$, …, $\mathcal{K}_{16}$ (we also used the program to generate the sets $\mathcal{K}_8, \ldots, \mathcal{K}_{12}$, to give it a better testing). Theorem 7 follows by the inspection of the graphs in $\mathcal{K}_5 \cup \ldots \cup \mathcal{K}_{16}$ (which was also computer assisted).

5 Graphs on the cylinder

Let us now turn our attention to graphs drawn in the cylinder. Our goal is to describe plane graphs that are critical for two precolored ($\leq$4)-faces, such that all other cycles have length at least 5. Such graphs can be thought of as embedded in the cylinder so that the two short faces are on the top and bottom of the cylinder.

First, in Lemma 14 we use $\mathcal{K}_5 \cup \ldots \cup \mathcal{K}_{16}$ to generate critical graphs on cylinder with two precolored (≤ 4)-cycles at distance at most 4 from each other and all other cycles of length at least 5. This part is computer assisted. Next, we glue pairs of these graphs together to obtain critical graphs with one non-precolored separating (≤ 4)-cycle, see Lemma 15. We discuss the outcomes of gluing three such graphs in Lemma 16. Finally, in Lemma 17 we give a general description of the critical graphs created by from those of Lemma 14 by gluing. We complete the description by Lemma 22, which shows that a plane graph with two precolored ($\leq$4)-faces at distance at least 5 and all other cycles of length at least 5 is never critical.

Lemma 14. *Let G be a connected graph embedded on the cylinder with distinct boundaries C_1 and C_2 such that $\ell(C_1), \ell(C_2) \leq 4$ and every cycle in G distinct from C_1 and C_2 has length at least 5. If G is $(C_1 \cup C_2)$-critical and the distance between C_1 and C_2 is at most 4, then G is isomorphic to one of the graphs drawn in Figures 1 or 2.*

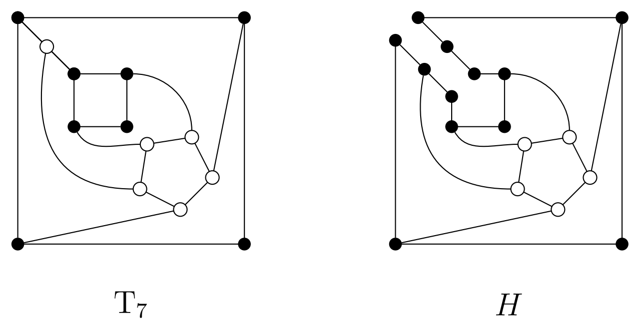


Figure 9: Splitting of T_7 into B-critical graph H where B is the outer face of H.

Proof. Let us first consider the case that $\ell(C_1) = \ell(C_2) = 4$. If C_1 and C_2 share an edge, then $C_1 \cup C_2$ contains a 6-cycle which bounds a face by Corollary 9. Thus G is graph Z_1 in Figure 1. Therefore, we may assume C_1 and C_2 share no edges.

Let P be a shortest path between C_1 and C_2. By Lemma 8, G is $(C_1 \cup C_2 \cup P)$-critical. Let H be the graph obtained from G by cutting along the path P, splitting the vertices of P into two and duplicating the edges of P, and let B be the resulting face. See Figure 9 for the splitting of T_7. Observe that H is B-critical and B is a cycle. Furthermore, $\ell(B) = \ell(C_1) + \ell(C_2) + 2\ell(P) \leq 16$, thus H is one of the graphs in $\mathcal{K}_5 \cup \ldots \cup \mathcal{K}_{16}$, which we enumerated using a computer in the previous section.

Note that G can be obtained from H by identifying appropriate paths in the face B. Using a computer, we checked all possible choices of H (as described in Section 4) and the paths, and checked whether the resulting graph satisfies the assumptions of this lemma. This way, we proved that G must be one of the graphs depicted in Figure 1.

If $\ell(C_1) = 3$ or $\ell(C_2) = 3$, then we subdivide edges of C_1 or C_2, so that the new precolored cycles C_1' and C_2' have length exactly 4. This does not change the distance between the cycles, and the resulting graph G' is $(C_1' \cup C_2')$-critical. Therefore, G' is one of the graphs depicted in Figure 1. Inspection of these graphs shows that G is one of the graphs in Figure 2. $\square$

Let us remark that there are no graphs satisfying the assumptions of Lemma 14 where the distance between C_1 and C_2 is exactly 4, and only one such graph R where the distance is exactly three. This is the last computer-based result of this paper (although we used the computer to check the correctness of the case analyses on several other places in the paper, all of them were also performed independently by hand). In particular, if someone proved Lemma 14 without the use of a computer, this would give computer-free proofs of Theorems 2 and 3.

For the graphs depicted in Figures 1 and 2, let B denote the outer face and T the other face of length at most 4. For a plane graph G with faces F_1 and F_2 and a precoloring ψ of F_1, let $c(G, F_1, \psi, F_2)$ be the number of colorings φ of F_2 such that $\psi \cup \varphi$ does not extend to a coloring of G (in case that F_1 and F_2 intersect, this includes the colorings that assign the common vertices colors different from those given by ψ; say in Z_2 if v is a common vertex of F_1 and F_2, we count colorings where $\psi(v) \neq \varphi(v)$). Let $c(G, F_1, F_2)$ be the maximum of $c(G, F_1, \psi, F_2)$ over all precolorings ψ of F_1. By a straightforward inspection of the listed graphs, we find that the values of $c(G, B, T)$ and $c(G, T, B)$ for the graphs in Figures 1 and 2 are as follows:

G	$c(G,B,T)$	$c(G,T,B)$	G	$c(G,B,T)$	$c(G,T,B)$
Z_1	15	15	O_6	4	11
Z_2	12	12	O_7	2	2
Z_3	16	16	T_1	8	8
Z_4	5	15	T_2	1	2
Z_5	4	12	T_3	4	4
Z_6	4	4	T_4	3	3
O_1	6	6	T_5	2	2
O_2	12	11	T_6	2	2
O_3	11	11	T_7	2	2
O_4	12	12	T_8	2	2
O_5	2	6	R	4	4

As an example, let us compute $c(T_1, B, T)$. Let $B = b_1b_2b_3b_4$ and $T = t_1t_2t_3t_4$ Denote the two not precolored vertices by z_1 and z_3, where z_i is adjacent to t_i and b_i for $i \in \{1, 3\}$. Observe that a precoloring ψ of B and φ of T do not extend to a coloring of T_1 if and only if $\{\psi(b_1), \varphi(t_1)\} = \{\psi(b_3), \varphi(t_3)\}$ and $\psi(b_1) \neq \varphi(t_1)$. Hence we need to consider only two cases for ψ: either $\psi(t_1) = \psi(t_3)$ or $\psi(t_1) \neq \psi(t_3)$. Assume first that $\psi(t_1) \neq \psi(t_3)$. Then $\varphi(b_1) = \psi(t_3)$ and $\varphi(b_3) = \psi(t_1)$. This leaves only one possibility for φ of b_2 and b_4. Hence there is one coloring φ such that $\phi \cup \varphi$ does not extend to T_1. For the second case assume that $\psi(t_1) = \psi(t_3)$. There are two possibilities for assigning $\varphi(b_1) = \varphi(b_3)$ such that $\psi(t_1) \neq \varphi(b_1)$. Each of these two possibilities can be extended to a coloring of T if four ways. Hence the total number of precolorings φ such that $\psi \cup \varphi$ does not extend is 8.

Based on these numbers, we characterize critical graphs obtained by pasting two such cylinders together. A cycle C in a plane graph *separates* subgraphs G_1 and G_2 if neither of the closed regions of the plane bounded by C contains both G_1 and G_2.

Let G be a plane graph and C_1 and C_2 be two cycles in G such that C_2 is drawn in the closed interior of C_1. A graph H *drawn between* C_1 and C_2 is a graph obtained from G by removing open exterior of C_1 and open interior of C_2. In particular, C_i bounds a face in H for $i \in \{1, 2\}$.

Lemma 15. *Let G be a connected graph embedded on the cylinder with distinct boundaries C_1 and C_2 such that $\ell(C_1), \ell(C_2) \leq 4$. Let $C \subseteq G$ be a cycle of length at most 4 separating C_1 from C_2. Assume that every cycle in G distinct from C, C_1 and C_2 has length at least 5, and that the distance between C_1 and C, as well as the distance between C and C_2, is at most 4. If G is $(C_1 \cup C_2)$-critical, then G is isomorphic to one of the graphs drawn in Figures 10, 11 and 12.*

Proof. Let G_i be the subgraph of G drawn between C_i and C, for $i \in \{1, 2\}$. By Lemma 8 and Lemma 14, G_i is equal to one of the graphs drawn in Figures 1 and 2. Let φ be a precoloring of $C_1 \cup C_2$ that does not extend to a coloring of G. Suppose first that $\ell(C) = 3$, i.e., the situation depicted in Figure 10. There exist 6 colorings of C by three colors. Observe that for every coloring ψ of C there exists $i \in \{1, 2\}$ such that the precoloring of C_i and C given by $\varphi \cup \psi$ does not extend to G_i, and thus $c(G_1, C_1, C) + c(G_2, C_2, C) \geq 6$. By symmetry, we may assume that $c(G_1, C_1, C) \geq 3$ and hence G_1 is one of Z_4, Z_5, Z_6 and O_6. If $G_1 \in \{Z_5, Z_6\}$, then C contains two vertices that have degree two in G_1, and since G is critical, they must either belong to C_2 or have degree at least three, implying that $G_2 \in \{Z_4, O_6\}$. Hence, G is one of the graphs D_1, D_2, D_3 or D_4. If $G_1 = Z_4$, then we conclude similarly that G is one of D_1, D_2, D_5, D_6, D_7 or D_8, and if $G_1 = O_6$, then G is one of D_3, D_4, D_7, D_9, D_{10} or D_{11}.

Let us now consider the case that $\ell(C_1) = \ell(C_2) = \ell(C) = 4$, as depicted in Figure 11. Since C has 18 colorings, we have $c(G_1, C_1, C) + c(G_2, C_2, C) \geq 18$. We may assume that $c(G_1, C_1, C) \geq 9$, i.e., $G_1 \in \{Z_1, Z_2, Z_3, O_2, O_3, O_4\}$. If $G_1 \in \{Z_1, Z_2\}$ or $G_1 = O_2$ with C_1 being the outer face of O_2, then C contains two adjacent vertices whose degree is two in G_1. These vertices must either belong to C_2, or their degree must be at least three in

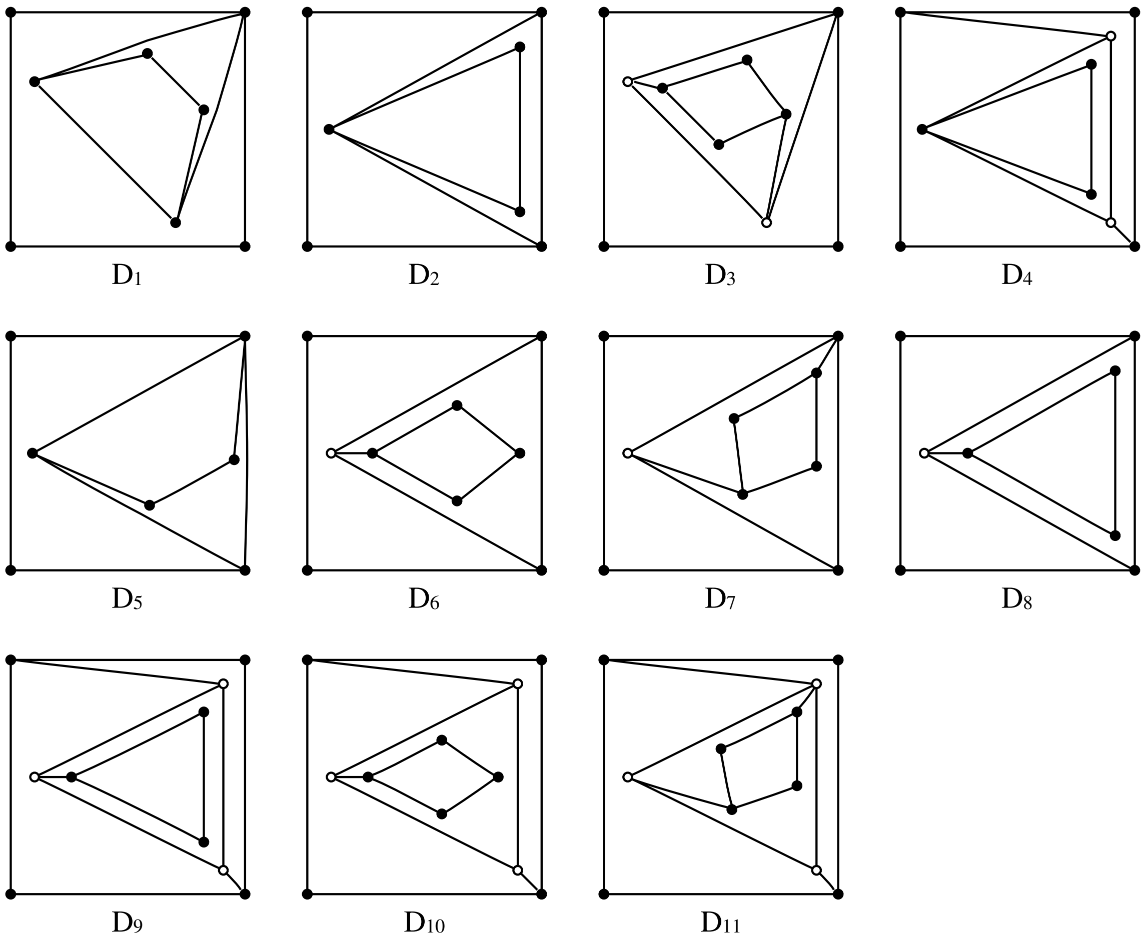


Figure 10: Critical graphs on the cylinder, one separating triangle.

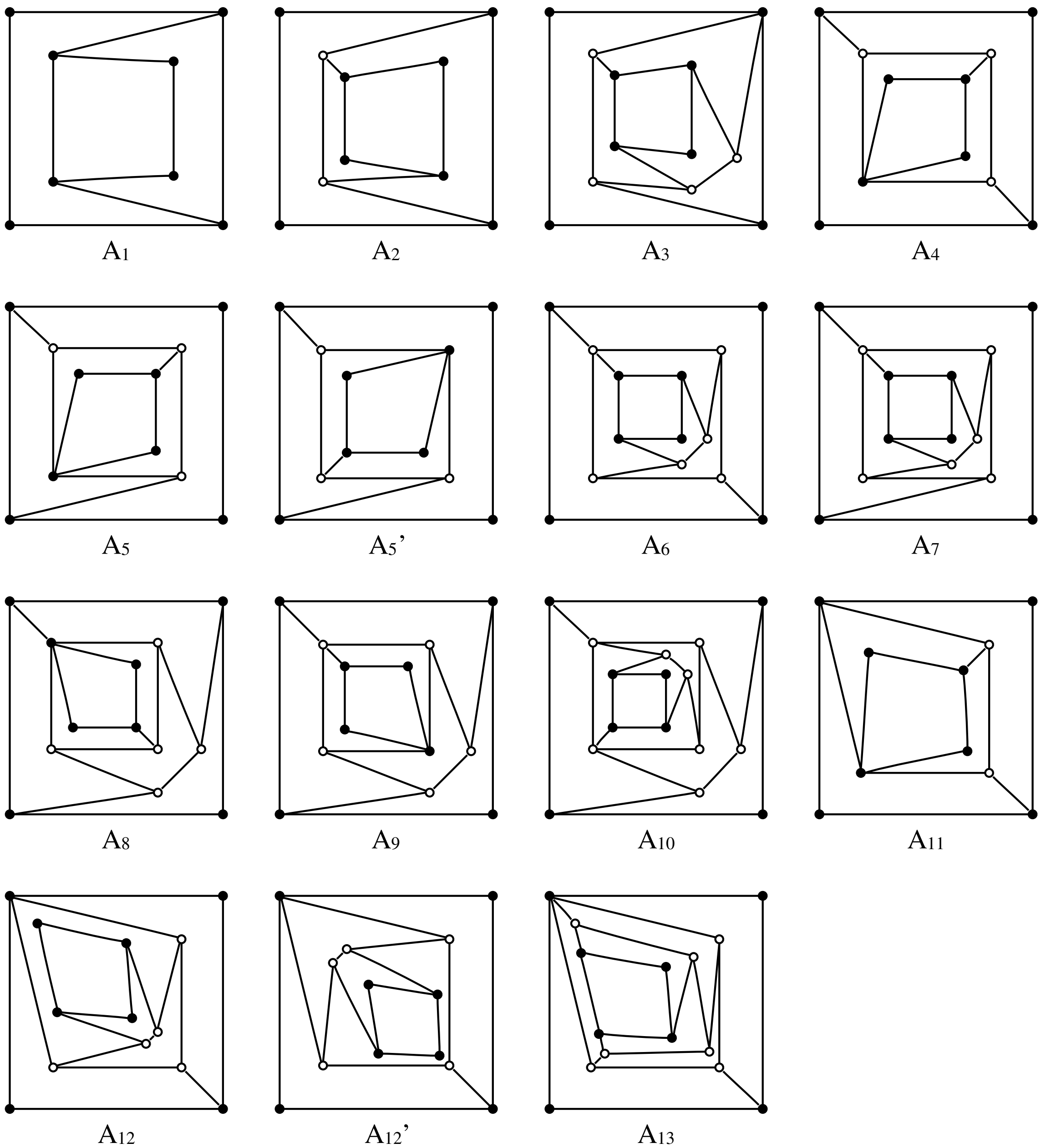


Figure 11: Critical graphs on the cylinder, one separating 4-cycle. Note that A_5 and A_5’ is the same graph but different embedding. We use primes to distinguish different embeddings.

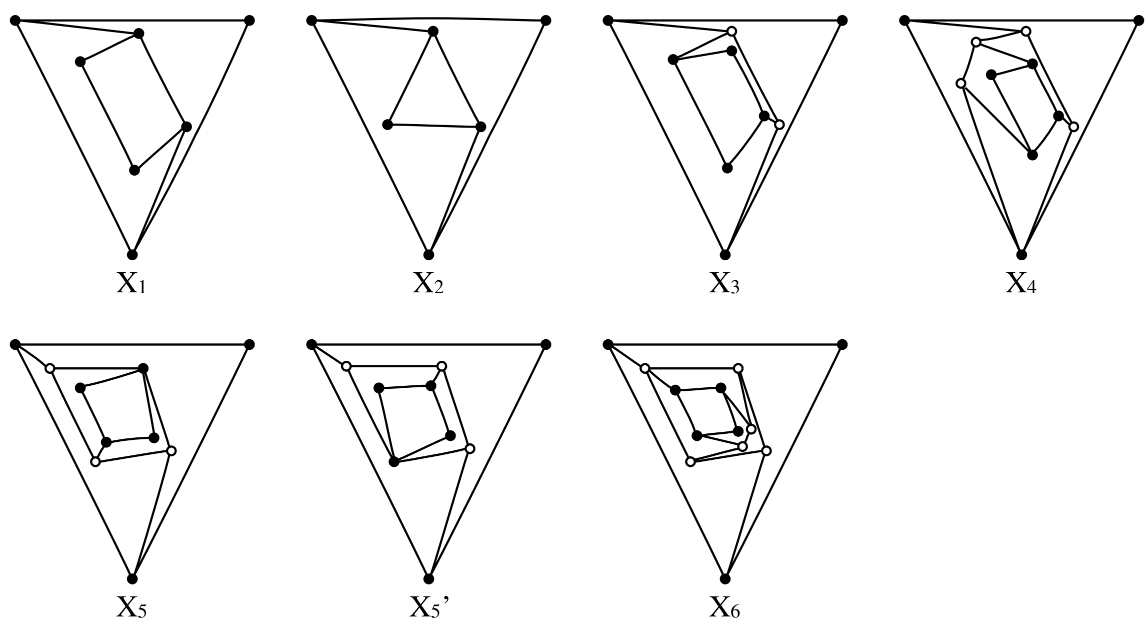


Figure 12: Critical graphs on the cylinder, one separating 4-cycle, precolored triangle.

G_2. We conclude (also taking into account that $c(G_2, C_2, C) \geq 3$) that $G_2 \in \{Z_1, O_2, O_4, T_3, T_4\}$, and (excluding the combinations that do not result in a critical graph), G is one of the graphs A_1, A_2 or A_3. From now on, assume that $G_1, G_2 \notin \{Z_1, Z_2\}$. If $G_1 = O_3$ or $G_1 = O_2$ with C being the outer face of O_2, then we similarly conclude that $G_2 \in \{Z_3, O_2, O_3, O_4, T_1\}$ and G is one of the graphs A_4, A_5, A'_5, A_6 or A_7. We may assume that $G_1, G_2 \notin \{O_2, O_3\}$. If $G_1 = O_4$, then $G_2 \in \{Z_3, O_1, O_4, T_1\}$, and G is A_8, A_9 or A_{10}. Finally, if $G_1 = Z_3$, then G is A_{11}, A_{12}, A'_{12} or A_{13}.

If $\ell(C) = 4$ and $\ell(C_1) = 3$ or $\ell(C_2) = 3$, then G is one of the graphs in Figure 12, obtained from those in Figure 11 by suppressing vertices of degree two. □

Again, let us summarize the values of $c(G, B, T)$ and $c(G, T, B)$ for these graphs:

G	$c(G,B,T)$	$c(G,T,B)$	G	$c(G,B,T)$	$c(G,T,B)$
D_1	12	12	A_5, A'_5	4	6
D_2	4	12	A_6	2	1
D_3	8	12	A_7	2	3
D_4	4	8	A_8	10	10
D_5	15	15	A_9	9	8
D_6	6	6	A_{10}	2	2
D_7	11	12	A_{11}	14	14
D_8	2	6	A_{12}, A'_{12}	4	4
D_9	2	4	A_{13}	4	4
D_{10}	6	4	X_1	9	3
D_{11}	6	6	X_2	3	3
A_1	9	9	X_3	1	1
A_2	1	3	X_4	2	1
A_3	2	3	X_5	4	2
A_4	4	2	X_6	2	1

We proceed by listing the graphs with two separating cycles of length at most 4.

Lemma 16. *Let G be a connected graph embedded on the cylinder with distinct boundaries C_1 and C_2 such that $\ell(C_1), \ell(C_2) \leq 4$. Let $C, C' \subseteq G$ be distinct cycles of length at most 4 separating C_1 from C_2, such that C separates C_1 from C'. Assume that every cycle in G distinct from C, C', C_1 and*

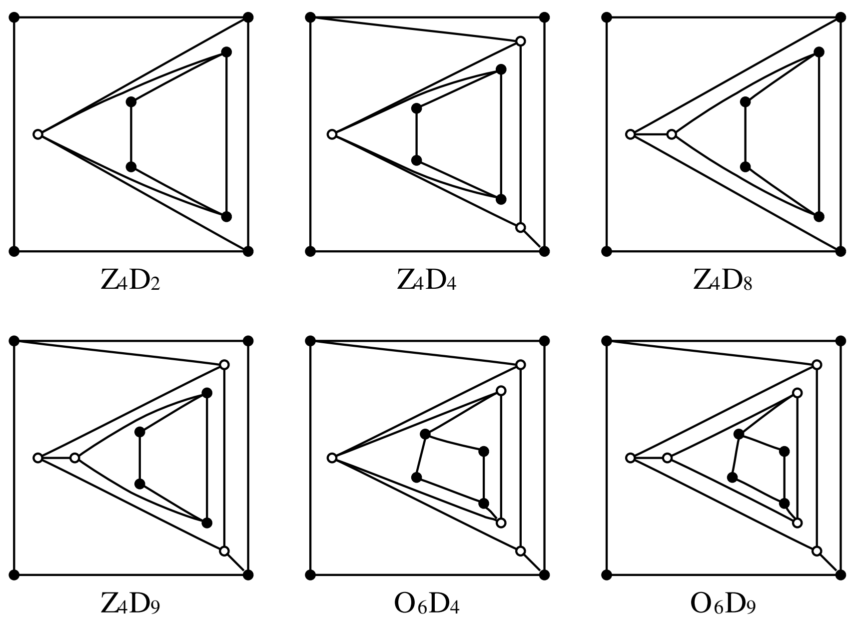


Figure 13: Critical graphs on the cylinder, two separating triangles.

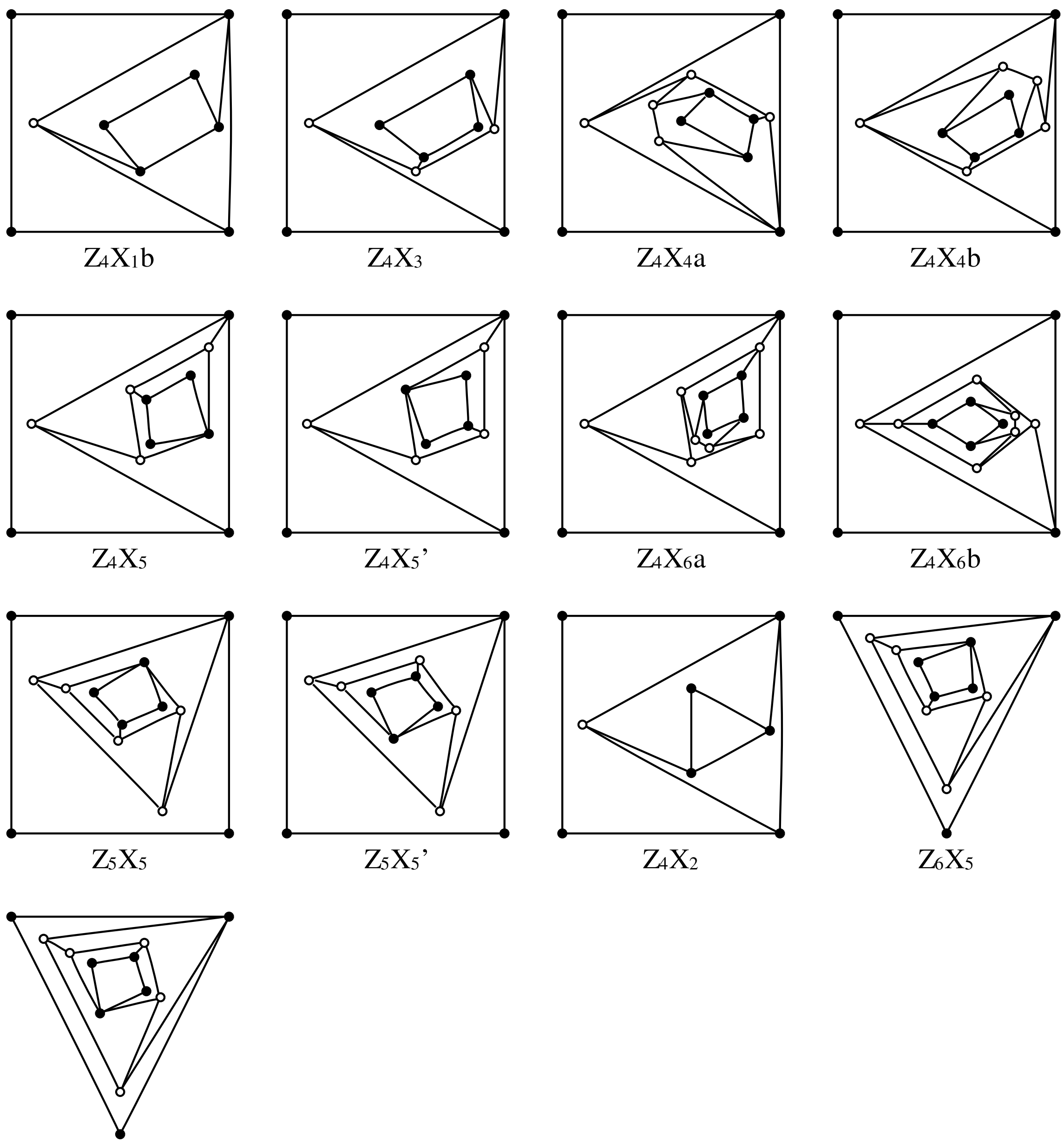


Figure 14: Critical graphs on the cylinder, separating triangle and a 4-cycle.

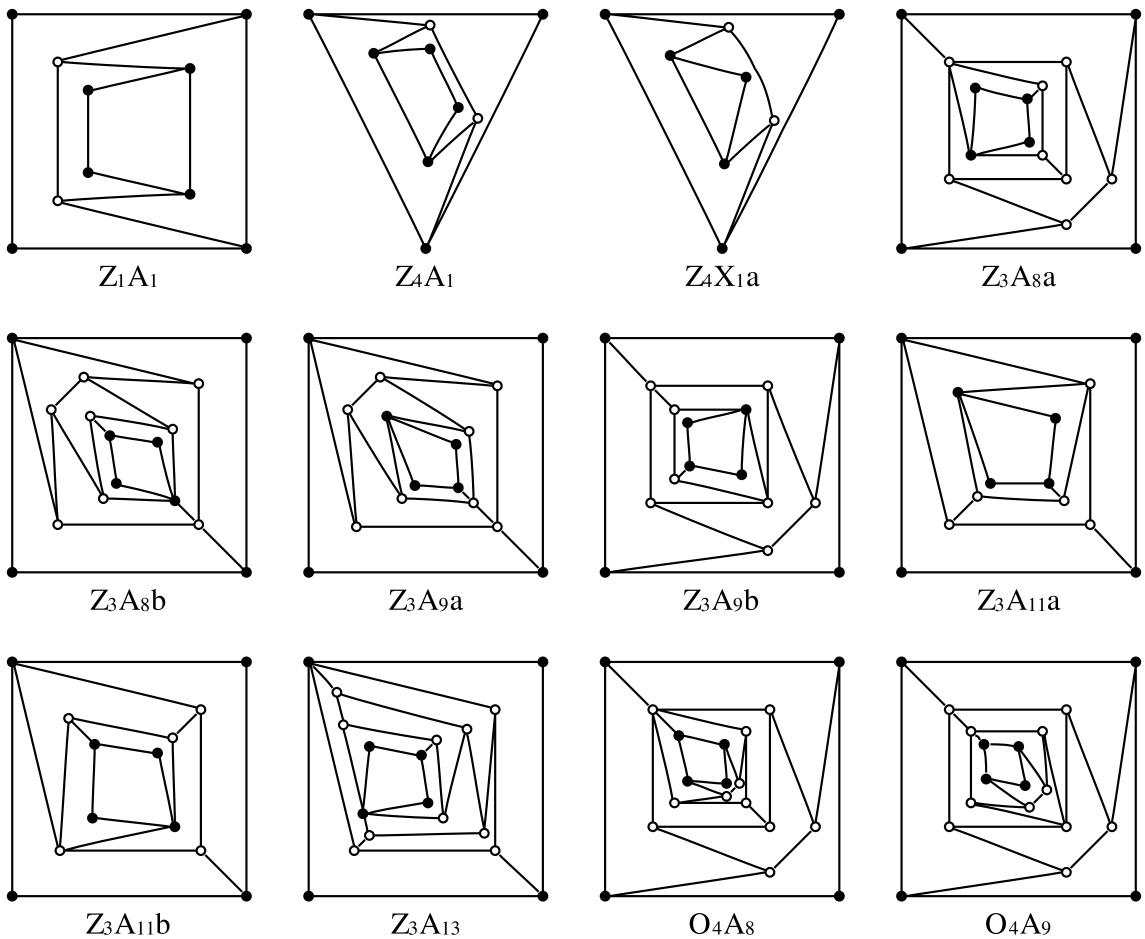


Figure 15: Critical graphs on the cylinder, two separating 4-cycles.

C_2 has length at least 5, and that the distances between C_1 and C, between C and C', and between C' and C_2 are at most 4. If G is $(C_1 \cup C_2)$-critical, then G is isomorphic to one of the graphs drawn in Figures 13, 14 and 15.

Proof. By symmetry between C_1, C and C_2, C, assume that $\ell(C) \leq \ell(C')$. Also, assume that $\ell(C_1) = \ell(C_2) = 4$—the graphs bounded by triangles follow by suppressing the precolored vertices of degree two. Let G_i be the subgraph of G drawn between C_i and C, for $i \in \{1, 2\}$. By Lemmas 8, 14 and 15, G_1 is equal to one of the graphs drawn in Figures 1 and 2 and G_2 is equal to one of the graphs in Figures 10, 11 and 12.

Suppose first that $\ell(C) = \ell(C') = 3$. It suffices to consider the graphs G_1 and G_2 such that G_1 is one of the graphs in Figure 2 and G_2 is one of the graphs in Figure 10, that is, $G_1 \in \{Z_4, Z_5, O_5, O_6\}$ and $G_2 \in \{D_2, D_4, D_8, D_9\}$. Furthermore, it suffices to consider the pairs satisfying $c(G_1, C_1, C) + c(G_2, C_2, C) \geq 6$. All critical graphs arising from these combinations are depicted in Figure 13. Let us remark that combination $G_1 = O_6$ and $G_2 = D_2$ is the same as $Z_4 D_4$ and combination $G_1 = O_6$ and $G_2 = D_8$ is the same as $Z_4 D_9$.

If $\ell(C) = 3$ and $\ell(C') = 4$, then we combine graphs G_1 from Figure 2 with graphs G_2 from Figure 12 such that $c(G_1, C_1, C) + c(G_2, C_2, C) \geq 6$, i.e., $G_1 = Z_4$ and $G_2 \in \{X_1, X_3, X_4, X_5, X_5', X_6\}$, or $G_1 \in \{Z_5, O_6\}$ and $G_2 \in \{X_1, X_5, X_5'\}$. All critical graphs arising from these combinations are depicted in Figure 14 (let us remark that the combination $G_1 = O_6$ and $G_2 = X_1$ is not critical, since the set of precolorings that extend to it is equal to that of D_{10}, which is its subgraph).

Finally, if $\ell(C) = \ell(C') = 4$, then we combine graphs G_1 from Figure 1 with graphs G_2 from Figure 11 such that $c(G_1, C_1, C) + c(G_2, C_2, C) \geq 18$ and C does not contain a non-precolored vertex of degree two. Furthermore, if $G_1 = Z_1$, we can exclude from consideration the graphs such that C is a cycle of non-precolored vertices of degree three, as an even cycle of vertices of degree three cannot appear in any critical graph. That is, for $G_1 = Z_1$ we need to consider $G_2 \in \{A_1, A_3, A_8, A_9, A_{13}\}$ (only $G_2 = A_1$ results in a critical graph). Almost all combinations need to be considered for $G_1 = Z_3$, where $G_2 \in \{A_8, A_9, A_{11}, A_{13}\}$ result in a critical graph. Once these combinations are considered, we may assume that $G_2 \notin \{A_1, A_{11}\}$ by symmetry, since in these graphs the subgraph drawn between C' and C_2 would be Z_1 or Z_3. Finally, we need to consider the combinations $G_1 \in \{O_2, O_3, O_4\}$ and $G_2 \in \{A_5, A_5', A_8, A_9\}$ or $G_1 = T_1$ and $G_2 = A_8$. All the critical graphs obtained by these combinations are in Figure 15. □

The numbers of non-extending colorings for these graphs are

G	$c(G,B,T)$	$c(G,T,B)$	G	$c(G,B,T)$	$c(G,T,B)$
Z_4D_2	12	12	Z_4X_2	3	9
Z_4Z_4	12	8	Z_6X_5, Z_6X_5'	4	2
Z_4D_8	6	6	Z_1A_1	3	3
Z_4Z_9	6	4	Z_4A_1	3	1
O_6D_4	8	8	Z_4X_1a	1	1
O_6D_9	4	4	Z_3A_8a	4	4
Z_4X_1b	9	9	Z_3A_8b	8	8
Z_4X_3	1	3	Z_3A_9a	4	4
Z_4X_4a	2	3	Z_3A_9b	4	4
Z_4X_4b	2	3	$Z_3A_{11}a$	12	12
Z_4X_5, Z_4X_5'	4	6	$Z_3A_{11}b$	12	12
Z_4X_6a	2	3	Z_3A_{13}	4	4
Z_4X_6b	2	3	O_4A_8	2	2
Z_5X_5, Z_5X_5'	4	6	O_4A_9	2	2

This rather tedious case analysis concludes with the next lemma.

Lemma 17. *Let G be a connected graph embedded in the cylinder with distinct boundaries C_1 and C_2 such that $\ell(C_1), \ell(C_2) \leq 4$. Let $C_1 = K_0, K_1, \ldots, K_k = C_2$ be a sequence of distinct cycles of length at most 4 in G such that K_i separates K_{i-1} from K_{i+1} for $1 \leq i \leq k-1$ and the distance between K_i and K_{i+1} is at most 4 for $0 \leq i \leq k-1$. Assume that every cycle of length at most 4 in G is equal to K_i for some $i \in \{0, \ldots, k\}$. If G is $(C_1 \cup C_2)$-critical, then one of the following holds:*

- *G is one of the graphs described by Lemmas 14, 15 or 16, or*
- *$G \in \mathcal{C}$, or*
- *G is one of the graphs drawn in Figure 16.*

Proof. By Lemmas 14, 15 or 16, we may assume that $k \geq 4$. The graphs described by Lemma 15 satisfy that if $\ell(C) = \ell(C') = 3$, then $\ell(C_1) = \ell(C_2) = 4$. Therefore, Lemma 8 implies that at least one of K_i, K_{i+1} and K_{i+2} has length 4, for $0 \leq i \leq k-2$. For $1 \leq i \leq k-1$, let P_i (respectively N_i) be the subgraphs of G drawn between K_i and C_1 (respectively C_2).

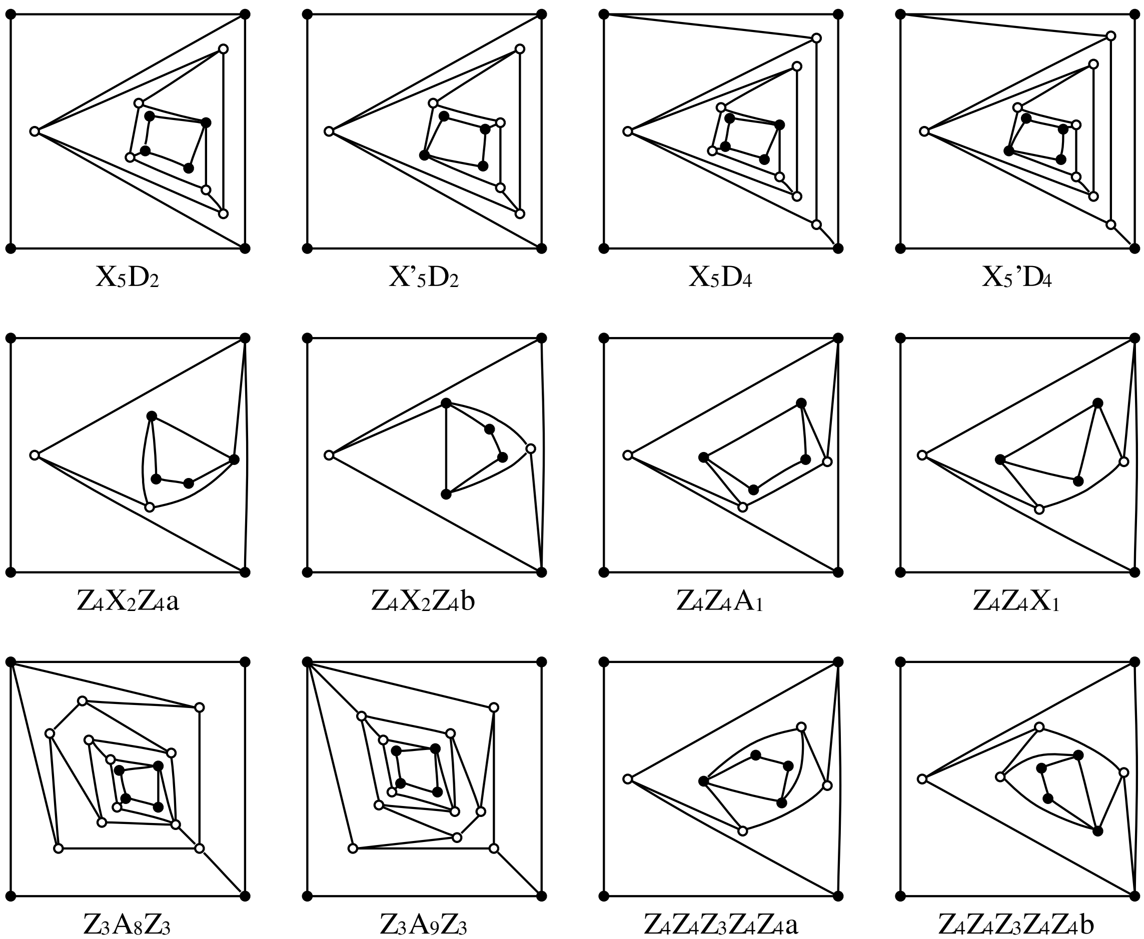


Figure 16: Other critical graphs on the cylinder.

Suppose first that $k = 4$, and assume that $\ell(C_1) = \ell(C_2) = 4$. If $\ell(K_2) = \ell(K_3) = 3$, then $P_2 \in \{X_1, X_3, X_4, X_5, X_5', X_6\}$ and $N_2 \in \{D_2, D_4, D_8, D_9\}$. Furthermore, $c(P_2, C_1, K_2)+c(N_2, C_2, K_2) \geq 6$, implying that $P_2 \in \{X_1, X_5, X_5'\}$ and $N_2 \in \{D_2, D_4\}$. The critical graphs arising this way are X_5D_2, $X_5'D_2$, X_5D_4 and $X_5'D_4$. The case that $\ell(K_1) = \ell(K_2) = 3$ is symmetric. If $\ell(K_1) = \ell(K_3) = 3$, then $P_3 = Z_4X_2$ and $N_1 = Z_4X_2$, and thus $N_3 = Z_4$. It follows that $G \in \{Z_4X_4Z_4a, Z_4X_4Z_4b\}$. If $\ell(K_2) = 3$ and $\ell(K_1) = \ell(K_3) = 4$, then $P_2, N_2 \in \{X_1, X_3, X_4, X_5, X_5', X_6\}$, and since $c(P_2, C_1, K_2) + c(N_2, C_2, K_2) \geq 6$, we conclude that $P_2 = N_2 = X_1$. However, the graph obtained by combining X_1 with itself is not critical. If $\ell(K_1) = 3$ and $\ell(K_2) = \ell(K_3) = 4$, then $N_1 = Z_4A_1$ and $P_1 \in \{Z_4, Z_5, O_5, O_6\}$. Since $c(P_1, C_1, K_1) + c(N_1, C_2, K_1) \geq 6$, it follows that $P_1 = Z_4$ and $G = Z_4Z_4A_1$. The case that $\ell(K_3) = 3$ and $\ell(K_1) = \ell(K_2) = 4$ is symmetric.

Finally, consider the case that $\ell(K_1) = \ell(K_2) = \ell(K_3) = 4$. Then P_3 is one of the graphs in Figure 15, implying that $P_1 \in \{Z_1, Z_3, O_4\}$, and by symmetry, $N_3 \in \{Z_1, Z_3, O_4\}$. If $N_3 \neq Z_3$, we have $c(P_3, C_1, K_3) \geq 6$, and thus $P_3 \in \{Z_3A_8b, Z_3A_{11}a, Z_3A_{11}b\}$. For all these choices of P_3, we have $P_1 = Z_3$. Therefore, by symmetry we may assume $N_3 = Z_3$. The combinations of Z_3 with the graphs in Figure 15 that result in a critical graph are $Z_3A_8Z_3$, $Z_3A_9Z_3$ and the graphs belonging to $\mathcal{C}$.

The only graph with $k = 4$ and $\ell(C_1) \leq 3$ or $\ell(C_2) \leq 3$ is $Z_4Z_4X_1$, obtained by suppressing a vertex of degree two in $Z_4Z_4A_1$. Thus, all the graphs with $k = 4$ satisfy the conclusion of this lemma.

Suppose now that $k = 5$. The graphs P_4 and N_1 are among the graphs described by this lemma for $k = 4$. This implies that $P_1 \in \{Z_1, Z_3, Z_4\}$. If $\ell(K_1) = 3$, then $P_1 = Z_4$ and $N_1 = Z_4Z_4X_1$ and $G = Z_4Z_4Z_1Z_4Z_4a$ or $G = Z_4Z_4Z_1Z_4Z_4b$. The case that $\ell(K_4) = 3$ is symmetric. Therefore, assume that $\ell(K_1) = \ell(K_4) = 4$. This implies that $N_1 \notin \{Z_4X_2Z_4a, Z_4X_2Z_4b\}$. Neither Z_1 nor Z_4 can be combined with a graph from $\mathcal{C}$ to form a critical graph, as the resulting graph would contain a non-precolored vertex of degree two. The same argument shows that if $P_1 \in \{Z_1, Z_4\}$, then $N_1 \notin \{X_5D_4, X_5'D_4, Z_3A_8Z_3, Z_3A_9Z_3\}$. The combinations of Z_1 or Z_4 with X_5D_2, $X_5'D_2$. $Z_4Z_4A_1$ or $Z_4Z_4X_1$ are not critical. We conclude that $P_1 = Z_3$, and by symmetry, $N_4 = Z_3$. Since G does not contain non-precolored vertices of degree two, we have $N_1 \notin \{X_5D_2, X_5'D_2, Z_4Z_4A_1, Z_4Z_4X_1\}$. If $N_1 \in \mathcal{C}$, then $G \in \mathcal{C}$. Otherwise, $N_1 \in \{X_5D_4, X_5'D_4, Z_3A_8Z_3, Z_3A_9Z_3\}$. However, the combinations of Z_3 with these graphs are not critical.

Therefore, we may assume that $k \geq 6$. Let G_i be the subgraph of G drawn

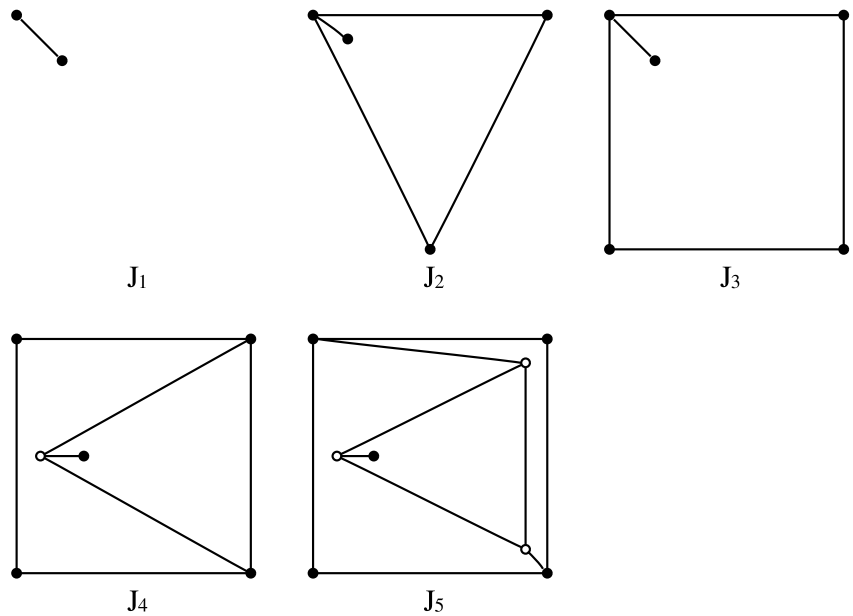


Figure 17: Critical graphs with a precolored vertex.

between K_i and K_{i+5}, for $0 \leq i \leq k-5$. By the previous paragraph, $G_i \in \{Z_4Z_4Z_1Z_4Z_4a, Z_4Z_4Z_1Z_4Z_4b\} \cup \mathcal{C}$, hence $\ell(K_i) = \ell(K_{i+5}) = 4$. Furthermore, considering G_{i-1} (if $i > 0$) or G_{i+1} (if $i < k-5$), we conclude that $\ell(K_{i+1}) = 4$ or $\ell(K_{i+4}) = 4$, implying that $G_i \in \mathcal{C}$. This implies that $G \in \mathcal{C}$. $\square$

Let us remark that if G is a graph in $\mathcal{C}$ with 4-faces C_1 and C_2, then G is $(C_1 \cup C_2)$-critical—to see this observe that that the precolorings of C_1 and C_2 in that the vertices of C_i of degree two have different colors for each $i \in \{1, 2\}$ do not extend to a coloring of G. Let us now point out some consequences of Lemma 17 that are useful in the proof of Theorem 5.

Lemma 18. *Let G be a connected plane graph, v a vertex of G and $C \subseteq G$ either a vertex of G, or a cycle bounding a face of length at most 4. Assume that every cycle of length at most 4 distinct from C separates v from C. Furthermore, assume that for every two subgraphs $K_1, K_2 \subseteq G$ such that $K_i \in \{v, C\}$ or K_i is a cycle of length at most 4 for $i \in \{1, 2\}$, either the distance between K_1 and K_2 is at most 4, or there exists a cycle of length at most 4 separating K_1 from K_2. If G is nontrivial $(v \cup C)$-critical, then G is* *one of the graphs J_1, J_2, $\ldots$, J_5 drawn in Figure 17.*

Proof. Let G' be the graph obtained from G in the following way: Add new vertices v' and v'' and edges of the triangle $C_1 = vv'v''$. If C is a single vertex, add also new vertices c' and c'' and edges of the triangle $C_2 = Cc'c''$, otherwise set $C_2 = C$. Observe that G' is $(C_1 \cup C_2)$-critical and satisfies assumptions of Lemma 17. The claim follows by the inspection of the graphs enumerated by Lemma 17, using the fact that C_1 and C_2 are disjoint, $\ell(C_1) = 3$ and v' and v'' have degree two. $\square$

The following claims follow by a straightforward inspection of the graphs listed in Lemmas 17 and 18:

Corollary 19. *Let G be a connected plane graph and C_1 and C_2 distinct subgraphs of G such that C_i is either a single vertex or a cycle of length at most 4 bounding a face, for $i \in \{1, 2\}$. Assume that G is nontrivial $(C_1 \cup C_2)$-critical and that every cycle of length at most 4 distinct from C separates C_1 from C_2. Furthermore, assume that for every two subgraphs $K_1, K_2 \subseteq G$ such that $K_i \in \{C_1, C_2\}$ or K_i is a cycle of length at most 4 for $i \in \{1, 2\}$, either the distance between K_1 and K_2 is at most 4, or there exists a cycle of length at most 4 separating K_1 from K_2.*

- *(a) If the distance between C_1 and C_2 is at least three and G has a face of length at least 7, then $G \in \{D_9, D_{10}, A_{12}, A'_{12}, Z_4D_8, Z_4D_9, O_6D_9, Z_5X_5, Z_5X'_5\}$.*
- *(b) If the distance between C_1 and C_2 is at least three and all cycles of length at most 4 in G distinct from C_1 and C_2 intersect in a non-precolored vertex, then $G \in \{R, J_5, D_9, D_{10}, A_{10}, A_{12}, A'_{12}, Z_4D_4, O_6D_4, Z_4X_4b, Z_3A_9b, O_4A_9\}$.*
- *(c) If the distance between C_1 and C_2 is at least three and all cycles of length at most 4 in G distinct from C_1 and C_2 intersect in a precolored vertex, then $G \in \{R, A_{12}, A'_{12}, Z_4X_4b\}$.*
- *(d) If the distance between C_1 and C_2 is at least two and G has a face of length at least 9, then $G \in \{D_6, D_{10}\}$.*
- *(e) If the distance between C_1 and C_2 is at least two and G is not 2-edge-connected, then $G \in \{J_4, J_5, D_6, D_8, D_9, D_{10}, Z_4D_8, Z_4D_9, O_6D_9\}$.*

(f) If the distance between C_1 and C_2 is at least two, G has a face M' of length at least 7, and there exists an edge e, a vertex x and a face $M \neq M'$ distinct from C_1 and C_2 such that

- *M and M' share the edge e, and x is incident with M,*
- *every path of length two between C_1 and C_2 contains the edge e, and*
- *every path of length at most 4 between C_1 and C_2 contains e or x or both, and*
- *every cycle of length at most 4 distinct from C_1 and C_2 contains e or x or both,*

then $G \in \{D_9, D_{10}, A_7, A_{12}, A'_{12}\}$.

Aksenov [1] proved that every planar graph with at most three triangles is 3-colorable. Let us note that the result was recently reproved with a significantly simpler proof [3] and the description of planar graphs with 4 triangles that are not 3-colorable is known [2]. In the origianl proof, Aksenov showed that for any plane graph G, if a face B of length at most 4 is precolored and G contains at most one triangle distinct from F, then the precoloring of F extends.

Theorem 20 (Aksenov [1])**.** *Let G be a plane graph with the outer face B of length at most 4. If G is nontrivial B-critical, then G contains at least two triangles distinct from B.*

The next lemma (following from Theorem 20) enables us to consider only connected graphs in the proof of Theorem 5:

Lemma 21. *Let G be a plane graph and C_1 and C_2 distinct subgraphs of G such that C_i is either a single vertex or a cycle of length at most 4 bounding a face, for $i \in \{1, 2\}$. Assume that every cycle in G of length at most 4 separates C_1 from C_2. If G is nontrivial $(C_1 \cup C_2)$-critical, then G is connected.*

Proof. We may assume that C_1 and C_2 are faces, since otherwise we can add a new cycle of length three to G to replace C_1 or C_2 if they were single vertices. If G were not connected, then there would exist a cycle K of length at most 4 and a nontrivial K-component G' of G such that G' contains at most one triangle distinct from K. By Lemma 8, G' is K-critical, contradicting Theorem 20. □

The following lemma finishes the proof of Theorem 5:

Lemma 22. *Let G be a connected plane graph and C_1 and C_2 distinct subgraphs of G such that C_i is either a single vertex or a cycle of length at most 4 bounding a face, for $i \in \{1,2\}$. Assume that every cycle in G distinct from C_1 and C_2 has length at least 5. If G is $(C_1 \cup C_2)$-critical, then the distance between C_1 and C_2 is at most 4.*

Proof. Suppose for a contradiction that G is a smallest counterexample to this claim, i.e., the distance between C_1 and C_2 is at least 5, and if H is a graph satisfying the assumptions of Lemma 22 with $|E(H)| < |E(G)|$, then the distance between its precolored cycles is at most 4. For future references, let us note that

the distance between C_1 and C_2 in G is at least 5, (1)

and

all cycles distinct from C_1 and C_2 in G have length at least 5. (2)

Let us now show some properties of G.

G is 2-connected. (3)

Proof. Suppose that v is a cut-vertex in G. Since G is $(C_1 \cup C_2)$-critical, Grötzsch's theorem implies that v separates C_1 from C_2. Let G_1 and G_2 be induced subgraphs of G such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v\}$, $C_1 \subseteq G_1$ and $C_2 \subseteq G_2$. By Lemma 8, G_i is $(C_i \cup v)$-critical, for $i \in \{1,2\}$. Furthermore, $|E(G_i)| < |E(G)|$, thus the distance between C_i and v is at most 4. By (1) and symmetry, we may assume that the distance between C_1 and v is at least three. However, since G_1 does not contain a cycle of length at most 4 distinct from C_1, this contradicts Lemma 18. □

No two vertices of degree two in G are adjacent. (4)

Proof. Suppose that vertices v_1 and v_2 of degree two in G are adjacent. Since G is critical, both v_1 and v_2 are precolored, and by symmetry, we may assume that they belong to C_1. Since G is 2-connected, it follows that $\ell(C_1) = 4$. Let $C_1 = v_1v_2v_3v_4$. By (2), v_3v_4 and $v_3v_2v_1v_4$ are the only paths of length at most three between v_3 and v_4. It follows that the graph G' obtained from G by identifying v_1 with v_2 to a new vertex v does not contain a cycle of length at most 4 distinct from vv_3v_4 and C_2. Furthermore, observe that G' is $(vv_3v_4 \cup C_2)$-critical, $|E(G')| < |E(G)|$ and the distance between vv_3v_4 and C_2 is at least 5. This contradicts the minimality of G. □

Let us fix a precoloring φ of $C_1 \cup C_2$ that does not extend to a coloring of G. By the minimality of G, φ extends to every proper subgraph of G that contains $C_1 \cup C_2$.

Let $v_1v_2v_3$ be a path in $C_1 \cup C_2$ such that v_2 has degree two and is incident with a face of length 5. *Then $\varphi(v_1) = \varphi(v_3)$.*

(5)

Proof. By symmetry, assume that $v_1v_2v_3 \subseteq C_1$. Since v_2 is incident with a 5-face and no cycle in G distinct from C_1 and C_2 has length 4, we conclude that $\ell(C_1) = 4$. Let $C_1 = v_1v_2v_3v_4$. Suppose for a contradiction that $\varphi(v_1) \neq \varphi(v_3)$. Let $v_1v_2v_3xy$ be a 5-face, and let G' be the graph obtained from $G - v_2$ by identifying v_1 and x to a new vertex z. Let $C_1' = v_3zv_4$. Note that the precoloring of $C_1' \cup C_2$ given by φ does not extend to a coloring of G', thus G' contains a nontrivial $(C_1' \cup C_2)$-critical subgraph G'' such that φ does not extend to a coloring of G''. The distance between C_1' and C_2 is at least 4. Observe that G'' contains a cycle C of length at most 4 distinct from C_1' and C_2, as otherwise we would obtain a contradiction with Lemmas 14 or 18 or with the minimality of G.

Since C does not exist in G, we have $z \in V(C)$. Let K be a cycle in G induced by $(V(C) \setminus \{z\}) \cup \{v_1yx\}$. Note that $V(K)$ indeed induces a cycle since it cannot have any chords. Moreover, K does not bound a face, since y has degree at least three. Suppose that the exterior of K contains both C_1 and C_2. Corollary 9 applied on K and its nonempty interior contradicts the criticality of G. Hence K separates C_1 from C_2, and thus C separates C_1' from C_2 in G''. Choose C among the cycles of length at most 4 in G'' distinct from C_1' and C_2 so that the subgraph $G_2'' \subseteq G''$ drawn between C and C_2 is as small as possible. This implies that all cycles in G_2'' distinct from C and

C_2 have length at least 5. By Lemma 8, G_2'' is $(C \cup C_2)$-critical, and by the minimality of G, the distance between C and C_2 is at most 4. Lemmas 14 and 18 imply that the distance between C and C_2 is at most three.

On the other hand, since $z \in V(C)$, by (1) the distance between C and C_2 is at least three. Therefore, the distance between C and C_2 is exactly three. By Lemma 14, $\ell(C) = \ell(C_2) = 4$ and $G_2'' = R$. The graph G'' contradicts Lemma 17. $\square$

We call a vertex v *light* if v is not precolored and the degree of v is exactly three.

The graph G does not contain the following configuration: A 5-face $F = v_1v_2v_3v_4v_5$ such that v_1, v_3, v_4 and v_5 are light, and either v_2 is light, or both v_4 and v_5 have a precolored neighbor.

(6)

Proof. If v_i is a light vertex of F, then let x_i be the neighbor of v_i that is not incident with F, for $1 \leq i \leq 5$. By (2), the vertices $x_1, \ldots, x_5$ are distinct. By (1), we may assume that all precolored neighbors of the vertices of F belong to C_1.

If both x_4 and x_1 are precolored, then there exists a path $P \subseteq C_1$ joining x_1 and x_4 and a closed region Δ of the plane bounded by the cycle K formed by P and $x_1v_1v_5v_4x_4$ such that Δ contains neither C_1 nor C_2. Since $\ell(K) \leq 7$, Corollary 9 implies that the open interior of Δ is a face. However, $\Delta \neq F$, which implies that v_5 has degree two. This is a contradiction, thus at most one of x_1 and x_4 is precolored. Similarly, at most one of x_3 and x_5 is precolored.

If v_2 is light, then by the symmetry of F we may assume that either no vertex of F has a precolored neighbor, or that x_4 is precolored and x_3 is not. By the previous paragraph, this also implies that x_1 is not precolored. If v_2 is not light, then both x_4 and x_5 are precolored, and thus neither x_1 nor x_3 are precolored.

Let G' be the graph obtained from G by removing the light vertices of F and adding the edge x_1x_3. Since $x_1 \neq x_3$, G' has no loops. Suppose that φ extends to a coloring ψ of G'. If v_2 is light, then each vertex of F has one precolored neighbor, thus it has two available colors. Furthermore, the lists of colors available at v_1 and v_3 are not the same, thus ψ extends to a coloring of F, giving a coloring of G that extends φ. Suppose that v_2 is not light. If $\psi(x_1) = \psi(v_2)$, then we can color vertices of F in order v_3, v_4, v_5, v_1. Similarly, ψ extends to F if $\psi(x_3) = \psi(v_2)$. Therefore, assume that $\psi(x_1) = 1$, $\psi(v_2) = 2$ and $\psi(x_3) = 3$. Then, color v_1 by 3, v_3 by 1 and extend the coloring to v_4 and v_5. This is possible, since by (5), $\varphi(x_4) = \varphi(x_5)$. We conclude that φ extends to a coloring of G, which is a contradiction.

Therefore, φ does not extend to a coloring of G', and G' has a nontrivial $(C_1 \cup C_2)$-critical subgraph G''. By the minimality of G, we have $x_1x_3 \in E(G'')$. Note that the distance between C_1 and C_2 in G'' is at least three, since neither x_1 nor x_3 is precolored. Also, every cycle in G'' of length at most 4 distinct from C_1 and C_3 contains the edge x_1x_3. If x_1x_3 were an edge-cut, then G'' contains no such cycle, and thus G'' would contradict Lemmas 14 and 18 or the minimality of G. It follows that x_1x_3 is incident with two distinct faces in G''. For a cycle M in G'' containing x_1x_3, let $\overline{M}$ be the closed walk in G obtained from M by replacing x_1x_3 by the path $x_1v_1v_2v_3x_3$. Let F' be the face of G'' incident with x_1x_3 such that the interior of the corresponding region bounded by $\overline{F'}$ in G contains v_4 and v_5. By Corrolary 9, $\ell(\overline{F'}) \geq 10$, and thus $\ell(F') \geq 7$.

Consider a cycle C of length at most 4 in G'' distinct from C_1 and C_2. If the cycle $\overline{C}$ of length at most 7 does not separate C_1 from C_2, then by Corollary 9 it bounds a face. Since $\overline{C} \neq F$, we conclude that v_2 has degree two. This is a contradiction, since v_2 is not precolored. We conclude that C separates C_1 from C_2. By Lemma 21, G'' is connected. Furthermore, by the minimality of G, the graph G'' satisfies the assumptions of Corollary 19. Since the distance between C_1 and C_2 in G'' is at least three and G'' has a face F' of length at least 7, Corollary 19(a) implies that $G'' \in \{D_9, D_{10}, A_{12}, A'_{12}, Z_4D_8, Z_4D_9, O_6D_9, Z_5X_5, Z_5X'_5\}$. Furthermore, since all cycles of length at most 4 distinct from C_1 and C_2 in G'' contain a common edge x_1x_3, we conclude that $G'' \in \{D_9, D_{10}, A_{12}, A'_{12}\}$. Since neither x_1 nor x_3 is precolored, the inspection of the possible choices for G'' shows that G'' contains a path Q of length at most 3 joining a vertex of C_1 with a vertex of C_2, such that $x_1x_3 \notin E(Q)$. However, Q is a subgraph of G, contradicting (1). $\square$

The graph G does not have any face F of length at least 7.

(7)

Proof. Suppose for a contradiction that $F = v_1v_2 \ldots v_k$ is a face of length

$k \geq 7$ in G. Since the distance between C_1 and C_2 is at least 5, we may assume that v_1, v_2 and v_3 are not precolored. Let G' be the graph obtained from G by identifying v_1 with v_3 to a new vertex v. Observe that φ does not extend to a coloring of G', thus G' has a nontrivial $(C_1 \cup C_2)$-critical subgraph G''. The distance between C_1 and C_2 in G'' is at least three. Furthermore, since v_2 has degree at least three and every cycle C in G'' of length at most 4 distinct from C_1 and C_2 corresponds to a cycle of length at most 6 in G containing the path $v_1v_2v_3$, Corollary 9 implies that each such cycle C separates C_1 from C_2 and satisfies $v \in V(C)$. By the minimality of G, the graph G'' satisfies the assumptions of Corollary 19(b). By (1), all paths of length at most 4 in G'' between C_1 and C_2 contain v. This implies that $G'' \in \{J_5, D_9, D_{10}, Z_4D_4, O_6D_4\}$. However, in these graphs, it is not possible to split v to two vertices (v_1 and v_3) in such a way that the resulting graph contains neither a path of length at most 4 between C_1 and C_2 nor a cycle of length at most 4 distinct from C_1 and C_2, which is a contradiction. □

All faces of G distinct from C_1 and C_2 have length 5.

(8)

Proof. Let $F = v_1v_2 \dots v_k$ be a face of G distinct from C_1 and C_2. By (7), $k \leq 6$. Suppose for a contradiction that $k = 6$. Let us first consider the case that v_1, v_3 and v_5 are not precolored. Then consider the graph G' obtained from G by identifying v_1, v_3 and v_5 to a single vertex v. Note that φ does not extend to a coloring of G', thus G' has a non-trivial $(C_1 \cup C_2)$-critical subgraph G''.

Consider now a cycle C of length at most 4 in G'' distinct from C_1 and C_2. Note that $v \in V(C)$, and by symmetry, we may assume that a cycle $K \subseteq G$ can be obtained from C by replacing v by $v_1v_2v_3$. Suppose that C does not separate C_1 from C_2. Then K does not separate C_1 from C_2, and by Corollary 9, K bounds a face distinct from F, hence v_2 has degree two. Since neither v_1 nor v_3 is precolored, we conclude that $C_1 = v_2$ or $C_2 = v_2$. But that implies that C separates C_1 from C_2 in G'', which is a contradiction. Therefore, every cycle of length at most 4 distinct from C_1 and C_2 separates C_1 from C_2 in G''.

As in the proof of (7), we conclude that $G'' \in \{J_5, D_9, D_{10}, Z_4D_4, O_6D_4\}$. Furthermore, since it is possible to split v to three vertices v_1, v_3 and v_5 so that the resulting graph contains neither a cycle of length at most 4 distinct

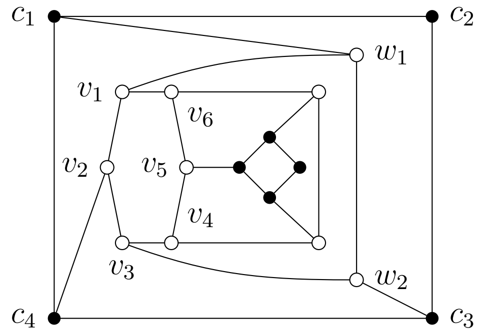


Figure 18: A graph with a 6-face.

from C_1 and C_2 nor a path between C_1 and C_2 of length at most 4, we have $G'' \in \{J_5, D_9, D_{10}\}$. Furthermore, we may assume that $C_1 = c_1c_2c_3c_4$ has length 4, there exists a path $c_1w_1w_2c_3$, v_1 is adjacent to w_1, v_3 is adjacent to w_2 and v_5 is adjacent to a vertex of C_2 in G. We choose the labels of c_2 and c_4 so that the 8-cycle $c_1w_1v_1v_2v_3w_2c_3c_4$ does not separate C_1 from C_2. Since v_2 cannot be a non-precolored vertex of degree two, Corollary 9 implies that v_2 is adjacent to c_4, and it is not precolored. Corollary 9 also implies that $c_1c_2c_3w_2w_1$ is a face. By (1), v_4 and v_6 are not precolored. Therefore, we may identify v_2, v_4 and v_6 instead, and by a symmetric argument, we conclude that $\ell(C_2) = 4$ and G is the graph depicted in Figure 18. However, this graph is not $(C_1 \cup C_2)$-critical.

It follows that at least one of v_1, v_3 and v_5 is precolored, and by symmetry, at least one of v_2, v_4 and v_6 is precolored. If v_1 and v_4 were precolored and the rest of the vertices of F were internal, then Corollary 9 implies that $v_1v_2v_3v_4$ or $v_1v_6v_5v_4$ together with a path in $C_1 \cup C_2$ bounds a face, implying that v_2 or v_6 have degree two. This is a contradiction, thus by symmetry, we may assume that $v_1, v_2 \in V(C_1)$. Since G does not contain a cycle of length at most 4 distinct from C_1 and C_2, at least one of v_3 and v_6, say v_3, is not precolored. Also, Corollary 9 implies that v_4 and v_5 are not precolored. Let us consider the graph G' obtained by identifying v_1, v_3 and v_5 to a single vertex v and its $(C_1 \cup C_2)$-critical subgraph G''. By Corollary 19(c), we have $G'' \in \{R, A_{12}, A'_{12}, Z_4X_4b\}$. However, all these graphs contain a path of length at most 4 joining C_1 and C_2 that does not contain v, contradicting (1). $\square$

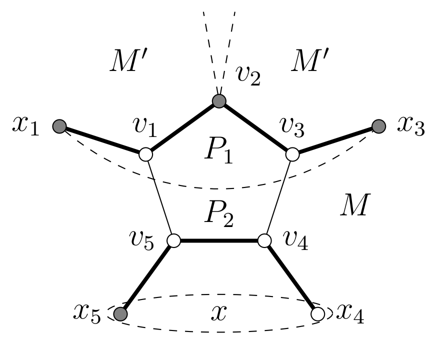


Figure 19: Configuration from (9).

No face of G is incident with 4 light vertices.

(9)

Proof. Suppose for a contradiction that $F = v_1v_2v_3v_4v_5$ is a face of G such that v_1, v_3, v_4 and v_5 are light. For $i \in \{1, 3, 4, 5\}$, let x_i be the neighbor of v_i that is not incident with F. By (2), the vertices x_i are distinct and x_4 is not adjacent to x_5. Also, by (6), we may assume that x_4 is not precolored and that v_2 is not light. See Figure 19.

Let G' be the graph obtained from G by removing v_1, v_3, v_4 and v_5, identifying x_4 with x_5 to a new vertex x, and adding the edge x_1x_3. Consider a coloring ψ of G'. We show that ψ extends to a coloring of G: Color both x_4 and x_5 by $\psi(x)$. If $\psi(v_2) = \psi(x_1)$, then color v_3, v_4, v_5 and v_1 in this order; each vertex has at least one available color. The case that $\psi(v_2) = \psi(x_3)$ is symmetric. Finally, if $\psi(x_1)$, $\psi(x_3)$ and $\psi(v_2)$ are pairwise different, then color v_1 by $\psi(x_3)$ and v_3 by $\psi(x_1)$, and extend this coloring to v_4 and v_5 (this is possible, since x_4 and x_5 are both colored by the same color $\psi(x)$). We conclude that φ does not extend to a coloring of G', and thus G' has a nontrivial $(C_1 \cup C_2)$-critical subgraph G''.

Let C be a cycle of length at most 4 in G'' distinct from C_1 and C_2, and let K be the corresponding cycle in G, obtained by replacing the edge x_1x_3 by the path $P_1 = x_1v_1v_2v_3x_3$ or the vertex x by the path $P_2 = x_4v_4v_5x_5$ (or both). Suppose that C does not separate C_1 from C_2. If $\ell(K) \leq 7$, then Corollary 9 implies that K bounds a face, and by (8), $\ell(K) = 5$. However, that implies $\ell(C) \leq \ell(K) - 3 \leq 2$, which is a contradiction. Therefore,

$\ell(K) \geq 8$, and thus $P_1, P_2 \subseteq K$. By planarity, $K - (P_1 \cup P_2)$ consists of paths Q_1 between x_1 and x_5 and Q_2 between x_3 and x_4. However, since $\ell(C) \leq 4$, at least one of Q_1 and Q_2 has length one, contradicting (2). We conclude that C separates C_1 from C_2. By Lemma 21, G'' is connected.

By (1), if $x_1 \in V(C_i)$, then $x_3 \notin V(C_{3-i})$ for $i \in \{1, 2\}$. Also, if $x_5 \in V(C_i)$, then x_4 has no neighbor in $V(C_{3-i})$. It follows that the distance between C_1 and C_2 in G'' is at least two. Let us also note that by (8), the distance between x_4 and x_5 in G is two, thus if $x_1x_3 \notin E(G'')$, then the distance between C_1 and C_2 in G'' is at least three.

Suppose now that v_2 has degree at most three in G. Since v_2 is not light and v_1 and v_3 are light, this implies that $v_2 = C_1$ or $v_2 = C_2$. We assume the former. In G'', v_2 has degree at most one. By Lemma 18, $G'' \in \{J_4, J_5\}$. Let x_2 be the neighbor of v_2 in G''. Note that $x_2 \notin \{x_1, x_3, x\}$ by (2). Let x_2' and x_2'' be the neighbors of x_2 distinct from v_2. Similarly, we conclude that $\{x_2', x_2''\} \cap \{x_1, x_3\} = \emptyset$. Since $x_2x_2'x_2''$ is a triangle, we have $x \in \{x_2', x_2''\}$, say $x = x_2''$. Then a path starting with $v_2x_2x_2'$ shows that the distance between C_1 and C_2 in G is at most three, which is a contradiction. We conclude that v_2 has degree at least 4.

For a face M of G'', let G_M be the subgraph of G drawn in the region of the plane corresponding to M, bounded by the closed walk $\overline{M}$ obtained from the boundary walk of M by replacing x_1x_3 by P_1 or x by P_2 (or both). Let us note that the open interior of this region is either an open disk, or a union of two open disks (the latter is the case when both x_1x_3 and v_2 are incident with M).

Suppose first that $x_1x_3 \notin E(G'')$. Let us recall that in this case, the distance between C_1 and C_2 in G'' is at least three. Since φ extends to every proper subgraph of G, we have $x \in V(G'')$. Let M be the face of G'' such that $v_1v_5, v_3v_4 \in E(G_M)$. If $\ell(\overline{M}) = \ell(M) + 6$, then x forms a cut in G'' and G'' contains no cycle of length at most 4 distinct from C_1 and C_2. We conclude that $G'' = R$ because the distance between C_1 and C_2 in G'' is at least three. However, R is 2-connected, which is a contradiction. Therefore, $\ell(\overline{M}) = \ell(M) + 3$. If $v_2 \in V(G'')$, then G_M contains two 2-chords $v_5v_1v_2$ and $v_4v_3v_2$. If $v_2 \notin V(G'')$, then G_M contains a vertex v_2 of degree at least 4 not contained in $\overline{M}$. In both cases, Theorem 6 implies that $\ell(\overline{M}) \geq 12$, and thus $\ell(M) \geq 9$. By Corollary 19(d), $G'' \in \{D_6, D_{10}\}$. The former is not possible, since the distance between C_1 and C_2 in G'' is at least three, thus $G'' = D_{10}$. Note that x must lie in the triangle in D_{10}. Observe that it is not possible to split such a vertex to two vertices x_4 and x_5 so that the distance between C_1 and C_2 is more than three, contradicting (1). This implies that $x_1x_3 \in E(G'')$.

Let M be the face of G'' incident with x_1x_3 such that $v_1v_5, v_3v_4 \in E(G_M)$, and M' the other face incident with x_1x_3. Suppose first that $M = M'$. Then by Corollary 19(e), $G'' \in \{J_4, J_5, D_6, D_8, D_9, D_{10}, Z_4D_8, Z_4D_9, O_6D_9\}$. Since x_1x_3 is not contained in any cycle, any cycle of length at most 4 in G'' distinct from C_1 and C_2 contains x. This implies that $G'' \notin \{Z_4D_8, Z_4D_9, O_6D_9\}$. By inspection of the remaining choices for G'' we conclude that x_1 or x_3 is precolored, and belongs to say C_2. Since the distance between C_1 and C_2 in G is at least 5, neither x_4 nor x_5 belongs to C_1, thus $G'' \notin \{J_4, D_6, D_8\}$ and x is not precolored. By symmetry, assume that $x_3 \in V(C_2)$. Then x_1 and x both belong to a triangle in G''. Note that x_1 and x_5 are not adjacent, thus x_1 is adjacent to x_4. By (1), x_4 is not adjacent to a vertex in C_1. The 5-cycle $x_1v_1v_5v_4x_4$ in G does not bound a face, thus by Corollary 9 it separates C_1 from C_2. Let y be the common neighbor of x_1 and x_5 in G. Since $G'' \in \{J_5, D_9, D_{10}\}$, x_5 and y have neighbors in C_1. By (8), x_4 and x_5 have a unique common neighbor z in G; also, the distance between x_3 and x_4 is two. By (1), $z \notin V(C_1)$. Note that $z \neq x_1$, since otherwise the 4-cycle x_1yx_5z would contradict (2). Observe that G contains an 8-cycle K consisting of $x_5zx_4x_1y$, edges between y and C_1 and between x_5 and C_1 and a path in C_1 such that K does not separate C_1 from C_2. Since z has degree at least three, Corollary 9 implies that z is adjacent to a vertex of C_1. However, this implies that the distance of x_4 is two both from $x_3 \in V(C_2)$ and from C_1, contradicting (1).

Therefore, $M \neq M'$. Let us note that every path of length two between C_1 and C_2 in G'' contains x_1x_3, and every path of length at most 4 contains x_1x_3 or x, and both x_1x_3 and x are incident with M. Suppose now that $\ell(M') \geq 7$. Corollary 19(f) implies that $G'' \in \{D_9, D_{10}, A_7, A_{12}, A'_{12}\}$. If $G'' \in \{D_9, D_{10}\}$, then x is adjacent both to x_1 and x_3, contradicting (2) or planarity. Otherwise, let $C \subseteq G''$ be the cycle of length 4 distinct from C_1 and C_2. Note that $x_1x_3 \notin E(C)$, thus $x \in V(C)$. But if x is split to two vertices (x_4 and x_5) so that C is not a cycle on the resulting graph, then the resulting graph contains a path of length at most 4 between C_1 and C_2 that does not contain x_1x_3, contradicting (1). It follows that $\ell(M') \leq 6$.

By (2), every path between x_1 and v_2 other than $x_1v_1v_2$ and every path between x_3 and v_2 other than $x_3v_3v_2$ has length at least three. Since $\ell(M') \leq 6$, we conclude that v_2 is not incident with M', and thus $v_2 \notin V(G'')$. Therefore, $G_{M'}$ is not a union of two cycles intersecting in v_2. Since v_2 has degree at least

4, Theorem 6 implies that $\ell(\overline{M'}) \geq 10$. It follows that $\ell(M') < \ell(\overline{M'}) - 3$, and thus $P_2 \subseteq \overline{M'}$.

Let us again consider a cycle C of length at most 4 in G'' distinct from C_1 and C_2, and let K be the corresponding cycle in G, obtained by replacing x_1x_3 by P_1 or x by P_2 or both. Since M and M' are both incident with both x and x_1x_3, there is a cut in G'' formed by x and x_1x_3. Thus $P_1 \cup P_2 \subseteq K$. However, this contradicts (2) or planarity. It follows that G'' does not contain a cycle of length at most 4 distinct from C_1 and C_2. Since M and M' are incident with both x_1x_3 and x, the minimality of G and Lemma 14 imply that $G'' = T_1$. But, T_1 contains two edge-disjoint paths of length two between C_1 and C_2, and at most one of them contains x_1x_3. It follows that the distance between C_1 and C_2 in G is at most 4, contradicting (1). □

Let us assign the *initial charge* $c_0(v) = \deg(v) - 4$ to each vertex and $c_0(F) = \ell(F) - 4$ to each face of G (including C_1 and C_2). By Euler's formula, the sum of these charges is -8. Now, each face of G distinct from C_1 and C_2 sends a charge of $1/3$ to each incident light vertex. This way we obtain the final charge c. Clearly, $c(v) \geq 0$ for each non-precolored vertex v, and $c(v) > 0$ if $\deg(v) > 4$.

The final charge of each face F of G distinct from C_1 and C_2 is non-negative. Furthermore, if F is incident with less than three light vertices, then $c(F) > 0$.

(10)

Proof. By (8), $\ell(F) = 5$, and thus $c_0(F) = 1$. If F is incident with k light vertices, then $c(F) = 1 - k/3$. Furthermore, (6) and (9) imply that $k \leq 3$, hence $c(F) \geq 0$, and if $k < 3$, then $c(F) > 0$. □

A face F distinct from C_1 and C_2 is C_i*-close* (for $i \in \{1, 2\}$) if F shares an edge with C_i. By (1) and (8), a C_1-close face cannot share a vertex with a C_2-close face.

For $i \in \{1, 2\}$, the sum S_i of the final charges of C_i (if C_i bounds a face), the vertices of C_i and the C_i-close faces is at least -4, and if it is equal to -4, then $V(G) \setminus V(C_1 \cup C_2)$ contains a vertex of degree at least 5.

(11)

Proof. If C_i is equal to a single vertex v, then by (3) its degree is at least 2, thus $c(v) = -2 > -4$.

Assume now that C_i is a triangle $v_1v_2v_3$. Then $c(C_i) = -1$. For $1 \le j < k \le 3$, let F_{jk} be the C_i-close face that shares the edge v_jv_k with C_i. If all vertices of C_i have degree at least three, then the final charge of each of them is at least -1, and by (10), $S_i \ge -4$. Furthermore, if $S_i = -4$, then all vertices of C_i have degree exactly three, and all non-precolored vertices of C_i-near faces are light. However, this implies that $V(F_{12} \cup F_{23} \cup F_{13}) \setminus V(C_i)$ induces a cycle C of length 6 consisting of light vertices. Observe that every coloring of $G - V(C)$ extends to a coloring of G, contradicting the criticality of G.

Let us consider the case that say v_1 has degree two. By (8), $F_{12} = F_{13}$ is a 5-face. Replacing the path $v_2v_1v_3$ in F_{12} by v_2v_3 results in a 4-cycle, contradicting (2).

Finally, assume that $C_i = v_1v_2v_3v_4$ has length 4, and thus $c(C_i) = 0$. For $1 \le j \le 4$, let F_j be the C_i-close face that shares the edge v_jv_{j+1} with C_i (where $v_5 = v_1$). If all vertices of C_i have degree at least three, then $c(v_j) \ge -1$ for $1 \le j \le 4$, and $S_i \ge -4$. Furthermore, $S_i = -4$ only if $\deg(v_j) = 3$ for $1 \le j \le 4$ and all non-precolored vertices of C_i-close faces are light. However, then $(F_1 \cup F_2 \cup F_3 \cup F_4) - V(C_i)$ is a cycle C of 8 light vertices. Observe that any coloring of $G - V(C)$ extends to a coloring of G, contradicting the criticality of G.

Therefore, we may assume that $\deg(v_1) = 2$. By (4), we have $\deg(v_2), \deg(v_4) \ge 3$. Suppose now that $\deg(v_3) \ge 3$. If at least one vertex of C_i has degree greater than 3, then $S_i \ge c(v_1) + c(v_2) + c(v_3) + c(v_4) + c(F_1) > -4$. Let us consider the case that $\deg(v_2) = \deg(v_3) = \deg(v_4) = 3$. By (8), $(F_1 \cup F_2 \cup F_3) - V(C_i)$ is a 5-cycle $w_1w_2w_3w_4w_5$, where w_1 is adjacent to v_2, w_3 is adjacent to v_3 and w_5 is adjacent to v_4. If w_1 and w_5 are light, then (8) implies that w_2 and w_4 have a common neighbor x such that $w_4w_5w_1w_2x$ is a 5-face. Since w_2 has degree at least three, $x \ne w_3$, and the 4-cycle $w_2w_3w_4x$ contradicts (2). Therefore, assume that say $\deg(w_1) \ge 4$. Then $S_i \ge -5 + c(F_1) + c(F_2) \ge -4$, and $S_i = -4$ only if w_2, w_3, w_4 and w_5 are light. If that were the case and all vertices of $V(G) \setminus V(C_1 \cup C_2)$ had degree at most 4, then $\deg(w_1) = 4$. Let x be the neighbor of w_1 distinct from w_2, w_5 and v_2. Let $G' = G - V(C_i) - \{w_1, w_2, w_3, w_4, w_5\}$, and let G'' be a $(C_{3-i} \cup x)$-critical subgraph of G' such that every precoloring of $C_{3-i} \cup x$ that extends to G'' also extends to G'. Note that the distance between x and C_{3-i} is at least three and G'' does not contain a cycle of length at most 4 distinct from C_{3-i}, and thus by the minimality of G and Lemma 18, G'' is trivial. It follows that every precoloring of x and C_{3-i} extends to G'. Let φ' be a coloring of G' that matches φ on C_{3-i}, such that $\varphi'(x) = \varphi(v_2)$. Then $\varphi' \cup \varphi$ extends to a coloring of G, since every vertex of the 5-cycle $w_1w_2w_3w_4w_5$ has two available colors, and the lists of colors available at w_3 and w_5 are not the same. This is a contradiction.

Finally, consider the case that $\deg(v_3) = 2$. If $\deg(v_2) = 3$, then by (8), $(F_1 \cup F_3) - \{v_1, v_2, v_3\}$ is a 4-cycle, contradicting (2). We conclude that $\deg(v_2) \geq 4$, and by symmetry, $\deg(v_4) \geq 4$. It follows that $S_i \geq c(v_1) + c(v_2) + c(v_3) + c(v_4) + c(F_1) + c(F_2) > -4$. $\square$

By (10) and (11), we have

$$-8 = \sum_{v \in V(G)} c(v) + \sum_{F \in F(G)} c(F) \geq c(w) + S_1 + S_2 > -8,$$

where w is the vertex of $V(G) \setminus V(C_1 \cup C_2)$ of maximum degree. This is a contradiction. $\square$

Theorem 5 follows from Lemmas 14 and 22. Together with Lemmas 17 and 21 have the following corollary:

Corollary 23. *Let G be a graph embedded in the cylinder with boundaries C_1 and C_2 such that $\ell(C_1), \ell(C_2) \leq 4$. If G is nontrivial $(C_1 \cup C_2)$-critical and every cycle of length at most 4 distinct from C_1 and C_2 separates C_1 from C_2, then $G \in \mathcal{C}$ or G is one of the graphs drawn in Figures 1, 2, 10, 11, 12, 13, 14, 15 or 16.*

6 The main result

Theorem 3 follows easily from Corollary 23 and Theorem 4. Let G be a graph embedded in a surface Σ, and let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a subset of faces of G. We say that a subgraph H of G is *$\mathcal{F}$-contractible* if $H \notin \mathcal{F}$ and there exists a closed disk $\Delta \subseteq \Sigma$ such that Δ contains H, but Δ does not contain any face of $\mathcal{F}$. For $F \in \mathcal{F}$, we say that H *surrounds* F if H is not $\mathcal{F}$-contractible and there exists a closed disk $\Delta \subseteq \Sigma$ such that Δ contains H and F, but no other face of $\mathcal{F}$. We say that a subgraph $H \subseteq G$ is *$\mathcal{F}$-good* if $F_1 \cup \ldots \cup F_k \subseteq H$ and if F is a face of H that is not equal to a face of G, then F has exactly two boundary walks, each of the walks has length 4, and the subgraph of G drawn in the closed region corresponding to F belongs to $\mathcal{C}$.

Let K be the constant from Theorem 4. Let us note that $K > 8$. Theorem 3 follows trivially from Grötzsch's theorem if $g = 0$. The following holds for graphs embedded in the cylinder:

Lemma 24. *Let G be a plane graph and F_1 and F_2 faces of G. If G is $(F_1 \cup F_2)$-critical and every cycle of length at most 4 separates F_1 from F_2, then G contains an $\{F_1, F_2\}$-good subgraph with at most $\ell(F_1)+\ell(F_2)+4K+20$ vertices.*

Proof. If G does not contain a cycle of length at most 4 distinct from F_1 and F_2, then by Theorem 4 we have $|V(G)| \leq \ell(F_1) + \ell(F_2) + 2K$, and we may set $H = G$. Otherwise, let C_i be the cycle of length at most 4 in G such that the subgraph $G_i \subseteq G$ drawn between F_i and C_i is as small as possible, for $i \in \{1, 2\}$. By Theorem 4, $|V(G_i)| \leq \ell(F_i) + \ell(C_i) + 2K$. Let M be the subgraph of G drawn between C_1 and C_2. If $|V(M)| \leq 20$, then $|V(G)| \leq \ell(F_1) + \ell(F_2) + 4K + 20$, and we set $H = G$. Suppose that $|V(M)| > 20$. If C_1 and C_2 are not vertex-disjoint, then there exists a subset Δ of the plane, disjoint with F_1 and F_2 and homeomorphic to an open disk, such that the boundary of Δ is formed by a closed walk (of length at most 8) in $C_1 \cup C_2$ and all vertices of M are contained in the closure of Δ. By a variant of Corollary 9, we would conclude that $V(M) = V(C_1 \cup C_2)$, contrary to the assumption that $|V(M)| > 20$. If C_1 and C_2 are vertex-disjoint, then Corollary 23 implies that $M \in \mathcal{C}$, and we set $H = G_1 \cup G_2$. $\square$

Let $\alpha = 21K + 104$ and $\beta = 15K + 76$. For other surfaces, we prove the following generalization of Theorem 3:

Theorem 25. *Let G be a graph embedded in a surface Σ of genus g and let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a set of faces of G such that the open region corresponding to F_i is homeomorphic to the open disk for $1 \leq i \leq k$. Assume that $g \geq 1$ or $k \geq 3$. If G is $(F_1 \cup F_2 \ldots \cup F_k)$-critical and every $\mathcal{F}$-contractible cycle has length at least 5, then G has an $\mathcal{F}$-good subgraph H with at most $\ell(F_1) + \ldots + \ell(F_k) + \alpha g + \beta(k-2) - 4$ vertices.*

Proof. Let us prove the claim by the induction. Let us assume that the claim is true for all graphs embedded in surfaces of genus smaller than g, or embedded in Σ with fewer than k precolored faces. Let $\ell = \ell(F_1) + \ldots + \ell(F_k)$.

Suppose first that G contains a cycle $C \notin \mathcal{F}$ of length at most 4 that does not surround any face in $\mathcal{F}$. Cut Σ along C and cap the resulting hole(s) by disk(s); the vertices and edges of C are duplicated, resulting in a graph G'. Let us now discuss several cases:

- If the curve given by the drawing of C in Σ is one-sided, then G' is embedded in a surface Σ' of genus $g-1$. Let C_1 be the face of G' corresponding to C; note that $\ell(C_1) = 2\ell(C)$. Observe that G' is $(\mathcal{F} \cup \{C_1\})$-critical. Thomassen [10] proved that every graph embedded in the projective plane without contractible cycles of length at most 4 is 3-colorable, and thus if $g = 1$, then $k \geq 1$. We conclude that if $g(\Sigma') = 0$, then $|\mathcal{F} \cup \{C_1\}| \geq 2$.

  If $g(\Sigma') = 0$ and $|\mathcal{F} \cup \{C_1\}| = 2$, then by Lemma 24, G' has an $(\mathcal{F} \cup \{C_1\})$-good subgraph H' with at most $\ell + \ell(C_1) + 4K + 20 \leq \ell + \alpha g + \beta(k-2) - 4$ vertices.

  Otherwise, we may apply the induction hypothesis, hence G' has an $(\mathcal{F} \cup \{C_1\})$-good subgraph H' with at most $\ell + \ell(C_1) + \alpha(g-1) + \beta(k-1) - 4 \leq \ell + \alpha g + \beta(k-2) - 4$ vertices.

  In both cases, the graph $H \subseteq G$ obtained from H' by identifying the corresponding vertices of C_1 is $\mathcal{F}$-good, and has at most $\ell + \alpha g + \beta(k-2) - 4$ vertices.

- If C is two-sided, then let C_1 and C_2 be the faces of G' corresponding to C. If C is not separating, then G' is embedded in a surface of genus $g-2$. If $g = 2$ and $k = 0$, then by Lemma 24, G' has a $(\{C_1, C_2\})$-good subgraph H' with at most $\ell(C_1) + \ell(C_2) + 4K + 20 \leq \ell + \alpha g + \beta(k-2) - 4$ vertices. Otherwise, we can apply induction hypothesis to G' and conclude that it has a $(\mathcal{F} \cup \{C_1, C_2\})$-good subgraph H' with at most $\ell + \ell(C_1) + \ell(C_2) + \alpha(g-2) + \beta k - 4 \leq \ell + \alpha g + \beta(k-2) - 4$ vertices. The graph $H \subseteq G$ obtained from H' by identifying the corresponding vertices of C_1 and C_2 is $\mathcal{F}$-good and has at most $\ell + \alpha g + \beta(k-2) - 4$ vertices.

- Finally, if C is two-sided and separating, then G' consists of subgraphs G_1 and G_2 embedded in surfaces Σ_1 and Σ_2, respectively, such that $g = g(\Sigma_1) + g(\Sigma_2)$. Let $\mathcal{F}_i$ be the subset of $\mathcal{F}$ contained in Σ_i and $k_i = |\mathcal{F}_i|$, for $i \in \{1, 2\}$. Let $\ell_i = \sum_{F \in \mathcal{F}_i} \ell(F)$. Since C is not $\mathcal{F}$-contractible and does not surround a face of $\mathcal{F}$, we have either $g(\Sigma_i) <$

g, or $|\mathcal{F}_i \cup \{C_i\}| < k$ for $i \in \{1,2\}$, and furthermore, if $g(\Sigma_i) = 0$, then $|\mathcal{F}_i \cup \{C_i\}| \geq 3$. By the induction hypothesis, G_i has an $(\mathcal{F}_i \cup \{C_i\})$-good subgraph H_i with at most $\ell_i + \ell(C_i) + \alpha g(\Sigma_i) + \beta(k_i - 1) - 4$ vertices. The graph $H \subseteq G$ obtained from H_1 and H_2 by identifying the corresponding vertices of C_1 and C_2 has at most $\ell + \ell(C_1) + \ell(C_2) - \ell(C) + \alpha g + \beta(k-2) - 8 \leq \ell + \alpha g + \beta(k-2) - 4$ vertices.

Therefore, we may assume that every cycle of length at most 4 in G surrounds a face in $\mathcal{F}$. Then, there exist cycles $C_1, \ldots, C_k \subseteq G$ such that

- for $1 \leq i \leq k$, either $C_i = F_i$, or $\ell(C_i) \leq 4$ and C_i surrounds F_i,
- if Δ_i is the open disk bounded by C_i that contains the face F_i, then $\Delta_i \cap \Delta_j = \emptyset$ for $1 \leq i < j \leq k$, and
- the graph G' obtained from G by removing all vertices and edges contained in $\Delta_1 \cup \ldots \cup \Delta_k$ contains no cycle of length at most 4 distinct from C_1, C_2, $\ldots$, C_k.

Let G_i be the subgraph of G drawn in the closure of Δ_i, for $1 \leq i \leq k$. Note that G_i is $(F_i \cup C_i)$-critical, and by Lemma 24, G_i has an $(F_i \cup C_i)$-good subgraph H_i with at most $\ell(F_i) + \ell(C_i) + 4K + 20$ vertices. By Theorem 4, $|V(G')| \leq \sum_{i=1}^{k} \ell(C_i) + K(g+k)$. Note that $H = G' \cup H_1 \cup \ldots \cup H_k$ is $\mathcal{F}$-good, and it has at most $\ell + (4K+24)k + K(g+k) \leq \ell + \alpha g + \beta(k-2) - 4$ vertices. The previous inequality does not hold for $k = 0$ and $g = 1$. However, in this case G is a projective planar graph without contractible cycles of length at most 4 and hence G is 3-colorable by a result of Thomassen [10]. □

Proof of Theorem 3. Follows from Grötzsch's theorem and Theorem 25, with $f(g) = \alpha g$. □

7 Programs

Both authors of the paper wrote independent programs implementing the algorithm following from Theorem 13, as well as the programs to verify the claims of Theorem 7 and Lemma 14. The complete lists of the graphs, as well as programs used to generate them can be found at `http://arxiv.org/abs/1305.2670`. For the technical details describing the programs and their usage, see `README` files in the subdirectories. The subdirectory `dvorak` also contains the programs used to verify the claims of Section 5, which were first derived manually without computer.

The most time-consuming part of the graph generation is criticality testing. We applied the straightforward algorithm following from the definition of the critical graph: given a planar graph G with the outer face B, for each edge e not incident with B we tested whether there exists a precoloring of B that does not extend to G, but extends to $G - e$. We augmented this algorithm with a few simple heuristics to speed it up (e.g., all vertices in $V(G) \setminus V(B)$ must have degree at least three). Generating the set $\mathcal{K}_{16}$ took about 10 minutes on a 2.67GHz machine. We believe that by parallelization and possibly using a more clever criticality testing algorithm, it would be possible to generate the graphs at least up to $\mathcal{K}_{20}$, if someone would need them.

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