Rainbow triangles in three-colored graphs

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Abstract

Erdős and Sós proposed a problem of determining the maximum number $F(n)$ of rainbow triangles in 3-edge-colored complete graphs on n vertices. They conjectured that $F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd$, where $a + b + c + d = n$ and a, b, c, d are as equal as possible. We prove that the conjectured recurrence holds for sufficiently large n. We also prove the conjecture for $n = 4^k$ for all $k \geq 0$. These results imply that $\lim \frac{F(n)}{\binom{n}{3}} = 0.4$, and determine the unique limit object. In the proof we use flag algebras combined with stability arguments.

1 Introduction

An edge-coloring of a graph (or a subgraph of a graph) is *rainbow* if each of its edges has a different color. Let G be a 3-edge-colored K_n, we define $F(G)$ to be the number of rainbow triangles in G, and define

$$F(n) = \max_{G:\ 3\text{-edge-colored } K_n} F(G).$$

The following conjecture on $F(n)$ was mentioned in [8] as an older problem of Erdős and Sós and it was mentioned again in [17].

Conjecture 1.

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd, \tag{1}$$

where $a + b + c + d = n$ and a, b, c, d are as equal as possible.

This recursive formula arises from the following construction. Denote by RB1111 a 3-edge-colored K_4, if it has the - up to isomorphism - unique coloring that every triangle in it is rainbow.

Construction 2. Fix an RB1111, and blow up its four vertices into four classes, of sizes a, b, c, d. The edges between two classes should inherit the color of the edge from the starting RB1111. This way, each of the triangles having vertices in three different classes will be rainbow. Inside of each class place an extremal coloring of K_a, K_b, K_c, K_d, see Figure 1.

A slight strengthening of Conjecture 1 is as follows.

Conjecture 3. *For every n, all 3-colorings of K_n attaining $F(n)$ are attained via Construction 2.*

Up to a permutation of the colors in each iterative step, this construction gives a unique candidate for an extremal 3-coloring of all edges of K_n. Note that for $n = 4^k$, the allowed color permutations in each step are in fact isomorphisms, so in this case the extremal coloring is conjectured to be unique up to isomorphism. In this paper, we prove Conjecture 3 for large enough n,
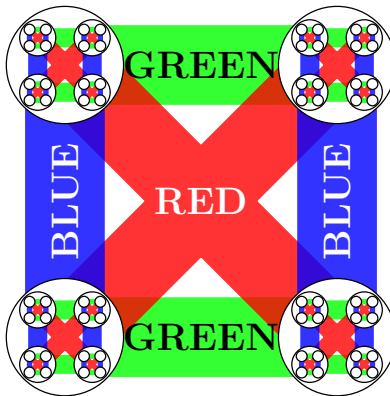
Figure 1: Sketch of conjectured extremal construction $G^{\square}$.

and for $n = 4^k$ for all $k \geq 0$.

Theorem 4. *There exists n_0 such that for every $n > n_0$*

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd, \tag{2}$$

where $a + b + c + d = n$ and a, b, c, d are as equal as possible.

Moreover, if G is a 3-edge-colored graph on n vertices containing $F(n)$ rainbow triangles, then $V(G)$ can be partitioned into four sets X_1, X_2, X_3 and X_4 of sizes a, b, c and d respectively, such that the edges containing vertices from different classes are colored like in a blow-up of a properly 3-edge-colored K_4, where vertices of the K_4 are blown-up by a, b, c and d vertices.

Theorem 5. *Conjecture 3 holds for $n = 4^k$, where $k \geq 1$. Moreover, the unique extremal example is the $(k - 1)$-times iterated blow-up of RB1111.*

We are not able to prove Conjecture 3 for all smaller n which are not powers of 4. Nevertheless, Theorem 4 is strong enough to directly imply the uniqueness of the extremal limit homomorphism (in the flag algebra sense), and thus the asymptotic density of rainbow triangles.

Theorem 6. *The unique limit homomorphism maximizing the density of rainbow triangles is given by the sequence of the iterated blow-ups of* RB1111. *This implies that*

$$\lim_{n \to \infty} \frac{F(n)}{\binom{n}{3}} = 0.4.$$

Counting the number of rainbow copies of given subgraphs was studied earlier, see for example [4] on a similar problem on hypercubes. Another natural question about triangles in 3-colored complete graphs, determining the *minimum* number of the monochromatic triangles, was solved in [7].

One of the tools we use to prove Theorem 6 are flag algebras. The tool was introduced by Razborov [20] as a general tool to approach problems from extremal combinatorics. Flag algebras have been successfully applied to various problems in extremal combinatorics. To name some of the applications, they were used for attacking the Caccetta-Häggkvist conjecture [14, 23], Turán-type problems in graphs [12, 13, 18, 19, 21, 24, 25], 3-graphs [2, 9, 11] and hypercubes [1, 5], extremal problems in a colored environment [3, 7], and also to problems in geometry [16] or extremal theory of permutations [6]. For more details on these applications, see a recent survey of Razborov [22].

In the case when flag algebras give a sharp bound on the density, usually the extremal structure is 'clean'. Even then, to obtain an exact result, it requires obtaining extra information from the flag algebra computations, and then apply some stability type method. In most cases, this last step uses results from the computation that certain small substructures appear with density $o(1)$.

For our problem, the conjectured extremal structure has an iterated structure, for which it is quite rare to obtain the precise density from flag algebra computations alone, see for example the problem on inducibility of small out-stars in oriented graphs [9] (note that the problem of inducibility of all out-stars was recently solved by Huang [15] using different techniques). In our case, a direct application of the semidefinite method gives only an upper bound on the limit value and shows that $\lim_{n \to \infty} \frac{F(n)}{\binom{n}{3}} < 0.40005$. However, using flag algebras to find bounds on densities of other substructures and combining them with other combinatorial arguments, we manage to obtain the precise result, at least when n is a power of 4, or when n is sufficiently large. We hope that our methods may give some insights on how to attack some other hard problems.

2   Notation

We say that a 3-edge-colored graph G on n vertices is *extremal* if G contains the maximum number of rainbow triangles among all 3-edge-colored graphs on n vertices.

Given a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set respectively, and write $v(G) = |V(G)|$.

Given two 3-edge-colored graphs G and G', an *isomorphism* between G and G' is a bijection $f : V(G) \to V(G')$ satisfying $\{f(v_1), f(v_2)\} \in E(G')$ if and only if $\{v_1, v_2\} \in E(G)$ and every pair of edges $\{v_1, v_2\} \in E(G)$ and $\{f(v_1), f(v_2)\} \in E(G')$ have the same color. Two 3-edge-colored graphs G and G' are *isomorphic*, which we denote by $G \cong G'$, if and only if there exists an isomorphism between G and G'.

In Section 4, we also use a coarser equivalence relation on 3-edge-colored graphs, the so-called *color-blind isomorphism*. We say that two 3-edge-colored graphs G and G' are color-blindly isomorphic if there exists a permutation $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ and a bijection $f : V(G) \to V(G')$ satisfying the following. A pair $\{f(v_1), f(v_2)\}$ is an edge in G' if and only if $\{v_1, v_2\} \in E(G)$, and for

3

every edge $\{v_1, v_2\} \in E(G)$ colored by c the corresponding edge $\{f(v_1), f(v_2)\} \in E(G')$ is colored by $\pi(c)$. In other words, G' becomes isomorphic to G (in the original sense) after renaming colors of all the edges in G' according to π.

For a 3-edge-colored graph G and a vertex set $U \subseteq V(G)$, denote by $G[U]$ the induced 3-edge-colored subgraph of G by the vertex set U. For a vertex v of G, we abbreviate $G[V \setminus \{v\}]$ to $G - v$.

Let H be a 3-edge-colored graph on t vertices and G be a 3-edge-colored graph on n vertices with $n \geq t$. Denote by $P(H, G)$ the number of t-subsets U of $V(G)$ such that $G[U] \cong H$, and define the *density* of H in G to be

$$p(H, G) = \frac{P(H, G)}{\binom{n}{t}}.$$

In other words, $p(H, G)$ is the probability that a random subset of $V(G)$ of size t induces a copy of H.

Fix a 3-edge-colored complete graph G. We denote by RBT the density of the properly 3-edge-colored triangles, i.e., the probability that random 3 vertices from G induce a 3-edge-colored triangle. Analogously, let TCT be the probability that random 3 vertices from G induce a triangle colored with exactly two colors, and MONOT the probability that random 3 vertices from G induce a monochromatic triangle. Note that both TCT and MONOT can be expressed as a linear combination of subgraph densities (in fact, each of them can be expressed as a combination of three subgraph densities). Also note that RBT + TCT + MONOT = 1.

By RB1111, we denote the density of properly 3-edge-colored K_4s. Similarly, let RB2111 be the probability that random 5 vertices from G induces a 3-edge-colored graph containing exactly two copies of RB1111. In other words, the vertices induces a 5-vertex blow-up of RB1111, where the edge inside the unique blob of size 2 can be colored arbitrarily. Next, we write RB1111$^+$ for the probability that random 5 vertices from G contains exactly one copy of RB1111. Again, the values of RB2111 and RB1111$^+$ can be expressed as a linear combination of subgraph densities, and it follows that RB1111 = $2/5 \cdot$ RB2111 + $1/5 \cdot$ RB1111$^+$.

Finally, we define RB3111 and RB2211 to be the probabilities that random 6 vertices from G induces the appropriate 6-vertex blow-up of RB1111. Specifically, RB3111 is the probability that the induced graph is obtained from RB1111 by blowing-up one of its vertices twice and coloring the three edges inside the blob arbitrarily. RB2211 denotes the other option – the probability that we choose two different vertices of RB1111 and blow-up both of them once. See Figure 2 for examples. As in all the previous cases, both RB3111 and RB2211 can be expressed as an appropriate linear combination of subgraph densities. Hence we call any of the probabilities defined in the last three paragraphs a *density expression*. With a slight abuse of notation, we will also use the same notation for the corresponding classes of subgraphs.

Let G be an extremal graph on n vertices and let D be some density expression. For any $X \subseteq V(G)$, we denote by $D(X)$ the density expression D restricted to subgraphs of G containing X, and we call $D(X)$ the *rooted density expression* of D at X in G. For example, for $X = \{x_1, x_2, x_3, x_4\}$, the rooted density expression RB2211(X) is the probability that random $6 - |X| = 2$ vertices from $V(G) \setminus X$ extends X to a subgraph from RB2211. Equivalently, it is the number of RB2211s containing the four vertices x_1, x_2, x_3, x_4 divided by $\binom{n-4}{2}$. For a fixed vertex $u \in V(G)$, we write $D(u)$ instead of $D(\{u\})$. Similarly, for a fixed edge vw, we write $D(vw)$ instead of $D(\{v, w\})$.
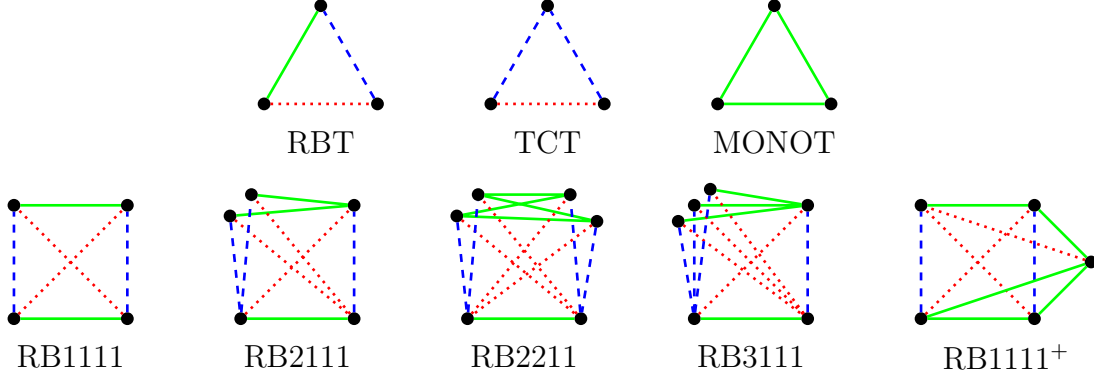
Figure 2: Examples of small configurations.

3   Outline of the proof of Theorem 4

The proof has some technical parts, so we give a thorough outline of the main ideas and motivations. Theorems 5 and 6 are consequences of Theorem 4, which we will prove in Section 5. Note that the first statement in Theorem 4 is a direct consequence of the second statement, so we only need to show the later one. We assume that G is a 3-edge-colored graph on n vertices maximizing the number of rainbow triangles.

Our first goal is to show that the vertices of G can be partitioned into four sets X_1, X_2, X_3, X_4 of almost equal size such that the edges between the sets look like in a blow-up of the properly 3-edge-colored K_4, see Figure 3. We start by carefully choosing a properly 3-edge-colored K_4 in G and use it to partition the vertices of G into sets $Z_1, \ldots, Z_4$ and a trash set Z_0. In this process we are guided by the conjectured extremal graph $G^{\square}$. In $G^{\square}$, most RB1111s contain one vertex in each X_i. We call an RB1111 Z *outer* if there are at least $n/2$ vertices v where $Z + v$ forms RB2111. Once we have found an outer RB1111 (call it Z), adding any other vertex will result in an RB2111 in $G^{\square}$. To recover the X_i from Z, we only have to check for every vertex in $G - Z$, which of the four vertices in Z is its twin.

Following this idea, we want to pick Z in G, such that Z lies in many RB2111s, and determine the Z_i accordingly. We can find such a Z through an averaging argument from bounds given to us from some standard flag algebra computations. But just knowing a bound on the number of RB2111s our set Z lies in will not tell us anything about the relative sizes of the Z_i, so this simple approach falls short of our goal. To remedy this problem, we look at subgraphs of size 6 instead. Adding two vertices to Z in the conjectured extremal graph gives us either an RB2211 or an RB3111. In G, the more RB2211s and the fewer RB3111s contain Z, the better the resulting sets Z_i will be balanced. Thus, we look for a Z maximizing

$$\mathrm{RB2211}(Z) - \tfrac{26}{9}\mathrm{RB3111}(Z), \tag{3}$$

where the value $\frac{26}{9}$ comes from our attempt to minimize[1] the gap in (15) from Section 5. Again, the best we can do is to find a Z which achieves at least the average of (3) over all RB1111.

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[1] If $\frac{26}{9}$ was replaced by 3, this function would be 0 in case all classes have the same sizes. Using a number a somewhat smaller than 3 forces the classes being more balanced.

Unfortunately, the bounds on the Z_i we get from this Z are not quite strong enough to later push through the whole proof, so we have to work yet a little harder. Notice that in $G^{\square}$, there are also RB1111s inside each of the four X_i. These *inner* RB1111s have much lower values in (3), so the average of that function is pushed down. On the other hand, if a vertex is added to an inner RB1111, in most cases it results in a copy of RB1111$^+$ and not RB2111 (which are always the result when starting from an outer RB1111). Following this observation, we consider the quantity

$$\text{RB2211}(Z) - \tfrac{26}{9}\text{RB3111}(Z) + \tfrac{27}{1000}\text{RB1111}^+(Z) \tag{4}$$

instead, where again $\frac{27}{1000}$ comes from optimizing (15) like $\frac{26}{9}$. The average of (4) over all RB1111 is a little higher than the average of (3) in $G^{\square}$, and the lower bound we get from flag algebra computations is improved as well. With this bound in hand, we can now find our Z by an averaging argument, and we can guarantee that the resulting $\{Z_i\}_{i=1}^4$ are fairly balanced, and contain most vertices of G. An edge between Z_i and Z_j is *funky* for $1 \le i < j \le 4$ if its color is different from what the RB1111 spanned by Z suggests. There are only few funky edges, as every such edge reduces RB2211(Z). We remove (very few) vertices incident to too many funky edges from Z_i, and obtain $X_1, \ldots, X_4$ and a trash set X_0 of all remaining vertices, while still maintaining fairly strong bounds on the sizes of X_is.

Using this structure, we can now step by step get closer to our goal. In Claim 12 we show that a vertex in X_i is not adjacent to almost all other vertices in X_i by edges of only one color. Otherwise, this vertex would lie in too few rainbow triangles, contradicting the simple Proposition 8 with the consequence that $\text{RBT}(v) = 0.4 + o(1)$ for every vertex v in G.

The remainder of the proof uses mostly recoloring arguments; we rule out certain scenarios by showing that recoloring some edges in these scenarios would increase the number of rainbow triangles.

If some edge uv is funky with $v \in X_i$, then the vast majority of the edges from v to other vertices in X_i must have the same color, as otherwise recoloring uv would increase the number of rainbow triangles. This is stated precisely in Claim 13.

The last two claims show that every vertex incident to funky edges must be incident to more than $0.4n$ edges of the same color. Using bounds from another flag algebra computation, we can show that this can occur only for very few vertices in Claim 15, and therefore the funky edges are incident to only a very small number of vertices. Using this knowledge, we can use a recoloring argument very similar to the one in Claim 13, yielding bounds contradicting Claim 12. This contradiction shows that in fact there are no funky edges.

Therefore, all the edges between X_i and X_j have the right color but we still need to deal with vertices in X_0. In Claims 17 and 18 we show that if we forcefully include a vertex from X_0 in any X_i, it will result in many funky edges. In other words, every vertex in X_0 looks very different from vertices in the other X_i. In fact, vertices in X_0 look so different from vertices in the X_i that we can show that they cannot lie in enough rainbow triangles, so X_0 must be empty. This last argument in Claim 19 relies on a massive case analysis handled by the computer, as we are maximizing a quadratic function over a 12-dimensional polytope with thousands of facets.

To complete the proof, we show in Claim 20 that the sizes of the X_i are almost balanced.

4   Flag algebras

The aim of this section is to establish the following statement.

Proposition 7. *There exists $n_0 \in \mathbb{N}$ such that every extremal 3-edge-colored complete graph G on at least n_0 vertices has the following properties:*

$$\frac{4}{15}\text{RB2211} - \frac{26}{45}\text{RB3111} + \frac{27}{5000}\text{RB1111}^+ > 0.002629395; \tag{5}$$

$$\text{RBT} < 0.40005; \tag{6}$$

$$\text{RB1111} < 0.09523837; \tag{7}$$

$$\tfrac{1}{3}\text{TCT} + \text{MONOT} < 0.33343492. \tag{8}$$

Let us give the related subgraph densities in Construction 2:

$$\text{RB2211} = 270/1023, \qquad\qquad \text{RB3111} = 120/1023,$$
$$\text{RB1111}^+ = 2/357, \qquad\qquad \text{RBT} = 0.4,$$
$$\text{RB1111} = 2/21, \qquad\qquad \text{TCT}/3 + \text{MONOT} = 1/3.$$

We also list the arithmetic values of (5) to (8) for Construction 2 below:

$$\frac{4}{15}\text{RB2211} - \frac{26}{45}\text{RB3111} + \frac{27}{5000}\text{RB1111}^+ \approx 0.002636964;$$

$$\text{RBT} = 0.4;$$

$$\text{RB1111} \approx 0.095238095;$$

$$\tfrac{1}{3}\text{TCT} + \text{MONOT} \approx 0.333333333.$$

The main tool used for the proof of Proposition 7 is flag algebras.

4.1 Flag algebra terminology

Let us now introduce the terminology related to flag algebras needed in this paper. Since we deal only with 3-edge-colored complete graphs, we restrict our attention just to this particular case. The central notions we are going to introduce are an algebra $\mathcal{A}$ and algebras $\mathcal{A}^\sigma$, where σ is a fixed 3-edge-coloring of a complete graph. Let us point out that we build flag algebras here with respect to the color-blind isomorphism instead of the standard isomorphism of 3-edge-colored graphs. This has been done only due to technical reasons, specifically, it decreased the computational effort needed for proving the inequalities in Proposition 7. Note that all the density expressions defined in Section 2 are invariant under permutations of the colors. Therefore, the values of the density expressions defined in Section 2 can be expressed as certain linear combinations of color-blind subgraph densities.

In order to precisely describe algebras $\mathcal{A}$ and $\mathcal{A}^\sigma$, we first need to introduce some additional notation. Let $\mathcal{F}$ be the set of all finite 3-edge-colored complete graphs modulo color-blind isomorphism. Next, for every $\ell \in \mathbb{N}$, let $\mathcal{F}_\ell \subset \mathcal{F}$ be the set of ℓ-vertex 3-edge-colored graphs from $\mathcal{F}$. For $H \in \mathcal{F}_\ell$ and $H' \in \mathcal{F}_{\ell'}$, recall that $p(H, H')$ is the probability that a randomly chosen subset of ℓ vertices in H' induces a subgraph isomorphic to H. Note that $p(H, H') = 0$ if $\ell' < \ell$. Let $\mathbb{R}\mathcal{F}$ be the set of all formal linear combinations of elements of $\mathcal{F}$ with real coefficients. Furthermore, let $\mathcal{K}$ be the linear subspace of $\mathbb{R}\mathcal{F}$ generated by all linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}_{v(H)+1}} p(H, H') \cdot H'.$$

Finally, we define $\mathcal{A}$ to be the space $\mathbb{R}\mathcal{F}$ factorized by $\mathcal{K}$.

The space $\mathcal{A}$ has naturally defined linear operations of an addition, and a multiplication by a real number. We now introduce a multiplication inside $\mathcal{A}$. We first define it on the elements of $\mathcal{F}$ in the following way. For $H_1, H_2 \in \mathcal{F}$, and $H \in \mathcal{F}_{v(H_1)+v(H_2)}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H)$ of size $v(H_1)$ and its complement induce in H subgraphs color-blindly isomorphic to H_1 and H_2, respectively. We set

$$H_1 \times H_2 = \sum_{H \in \mathcal{F}_{v(H_1)+v(H_2)}} p(H_1, H_2; H) \cdot H.$$

The multiplication on $\mathcal{F}$ has a unique linear extension to $\mathbb{R}\mathcal{F}$, which yields a well-defined multiplication also in the factor algebra $\mathcal{A}$. A formal proof of this can be found in [20, Lemma 2.4].

Let us now move to the definition of an algebra $\mathcal{A}^\sigma$, where $\sigma \in \mathcal{F}$ is an arbitrary 3-edge-colored complete graph with a fixed labelling of its vertex set. The labelled graph σ is usually called a *type* within the flag algebra framework. Without loss of generality, we will assume that the vertices of σ are labelled by $1, 2, \ldots, v(\sigma)$. Now we follow almost the same lines as in the definition of $\mathcal{A}$. We define $\mathcal{F}^\sigma$ to be the set of all finite 3-edge-colored complete graphs H with a fixed *embedding* of σ, i.e., an injective mapping θ from $V(\sigma)$ to $V(H)$ such that $\text{im}(\theta)$ induces in H a subgraph isomorphic to σ. Again, the graphs in $\mathcal{F}^\sigma$ are considered modulo color-blind isomorphism. The elements of $\mathcal{F}^\sigma$ are usually called σ-*flags* and the subgraph induced by $\text{im}(\theta)$ is called the *root* of a σ-flag.

Again, for every $\ell \in \mathbb{N}$, we define $\mathcal{F}_\ell^\sigma \subset \mathcal{F}^\sigma$ to be the set of the σ-flags from $\mathcal{F}^\sigma$ that have size ℓ (i.e., the σ-flags with the underlying 3-edge-colored graph having ℓ vertices). Analogously to the case for $\mathcal{A}$, for two 3-edge-colored graphs $H, H' \in \mathcal{F}^\sigma$ with the embeddings of σ given by θ, θ', we set $p(H, H')$ to be the probability that a randomly chosen subset of $v(H) - v(\sigma)$ vertices in $V(H') \setminus \theta'(V(\sigma))$ together with $\theta'(V(\sigma))$ induces a subgraph that is color-blindly isomorphic to H through an isomorphism f that preserves the embedding of σ. In other words, the color-blind isomorphism f has to satisfy $f(\theta') = \theta$. Let $\mathbb{R}\mathcal{F}^\sigma$ be the set of all formal linear combinations of elements of $\mathcal{F}^\sigma$ with real coefficients, and let $\mathcal{K}^\sigma$ be the linear subspace of $\mathbb{R}\mathcal{F}^\sigma$ generated by all the linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}_{v(H)+1}^\sigma} p(H, H') \cdot H'.$$

We define $\mathcal{A}^\sigma$ to be $\mathbb{R}\mathcal{F}^\sigma$ factorised by $\mathcal{K}^\sigma$.

We now describe the multiplication of two elements from $\mathcal{F}^\sigma$. Let $H_1, H_2 \in \mathcal{F}^\sigma$, $H \in \mathcal{F}_{v(H_1)+v(H_2)-v(\sigma)}^\sigma$, and θ be the fixed embedding of σ in H. As in the definition of multiplication for $\mathcal{A}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H) \setminus \theta(V(\sigma))$ of size $v(H_1) - v(\sigma)$ and its complement in $V(H) \setminus \theta(V(\sigma))$ of size $v(H_2) - v(\sigma)$, extend $\theta(V(\sigma))$ in H to subgraphs color-blindly isomorphic to H_1 and H_2, respectively. Again, by isomorphic here we mean that there is a color-blind isomorphism that preserves the fixed embedding of σ. This definition naturally extends to $\mathcal{A}^\sigma$.

Now consider an infinite sequence $(G_n)_{n \in \mathbb{N}}$ of 3-edge-colored complete graphs of increasing orders. We say that the sequence $(G_n)_{n \in \mathbb{N}}$ is *convergent* if the probability $p(H, G_n)$ has a limit for every $H \in \mathcal{F}$. A standard compactness argument (e.g., using Tychonoff's theorem) yields that every such infinite sequence has a convergent subsequence. All the following results can be found in [20]. Fix a convergent increasing sequence $(G_n)_{n \in \mathbb{N}}$ of 3-edge-colored graphs. For every $H \in \mathcal{F}$, we set $\phi(H) = \lim_{n \to \infty} p(H, G_n)$ and linearly extend ϕ to $\mathcal{A}$. We usually refer to the mapping ϕ as

to the limit of the sequence. The obtained mapping ϕ is a homomorphism from $\mathcal{A}$ to $\mathbb{R}$. Moreover, for every $H \in \mathcal{F}$, we obtain $\phi(H) \geq 0$. Let $\text{Hom}^+(\mathcal{A}, \mathbb{R})$ be the set of all such homomorphisms, i.e., the set of all homomorphisms ψ from the algebra $\mathcal{A}$ to $\mathbb{R}$ such that $\psi(H) \geq 0$ for every $H \in \mathcal{F}$. It is interesting to see that this set is exactly the set of all limits of convergent sequences of 3-edge-colored complete graphs [20, Theorem 3.3].

Let $(G_n)_{n \in \mathbb{N}}$ be a convergent sequence of 3-edge-colored graphs and $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ be its limit. For $\sigma \in \mathcal{F}$ and an embedding θ of σ in G_n, we define G_n^θ to be the 3-edge-colored graph rooted on the copy of σ that corresponds to θ. For every $n \in \mathbb{N}$ and $H^\sigma \in \mathcal{F}^\sigma$, we define $p_n^\theta(H^\sigma) = p(H^\sigma, G_n^\sigma)$. Picking θ at random gives rise to a probability distribution $\mathbf{P}_\mathbf{n}^\sigma$ on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$, for every $n \in \mathbb{N}$. Since $p(H, G_n)$ converges (as n tends to infinity) for every $H \in \mathcal{F}$, the sequence of these probability distributions on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$ also converges [20, Theorems 3.12 and 3.13]. We denote the limit probability distribution by $\mathbf{P}^\sigma$. In fact, for any σ such that $\phi(\sigma) > 0$, the homomorphism ϕ itself fully determines the random distribution $\mathbf{P}^\sigma$ [20, Theorem 3.5]. Furthermore, any mapping ϕ^σ from the support of the distribution $\mathbf{P}^\sigma$ is in fact a homomorphism from $\mathcal{A}^\sigma$ to $\mathbb{R}$ such that $\phi^\sigma(H^\sigma) \geq 0$ for all $H^\sigma \in \mathcal{F}^\sigma$ [20, Proof of Theorem 3.5].

The last notion we introduce is the *averaging* (or downward) operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}$. It is a linear operator defined on the elements of $H^\sigma \in \mathcal{F}^\sigma$ by $[\![H^\sigma]\!]_\sigma = p_H^\sigma \cdot H^\emptyset$, where $H^\emptyset$ is the (unlabeled) 3-edge-colored graph from $\mathcal{F}$ corresponding to H^σ, and p_H^σ is the probability that a random injective mapping from $V(\sigma)$ to $V(H^\emptyset)$ is an embedding of σ in $H^\emptyset$ yielding a σ-flag color-blindly isomorphic to H^σ. The key relation between ϕ and ϕ^σ is the following:

$$\forall H^\sigma \in \mathcal{A}^\sigma, \quad \phi\left([\![H^\sigma]\!]_\sigma\right) = \phi([\![\sigma]\!]_\sigma) \cdot \int \phi^\sigma(H^\sigma),$$

where the integration is over the probability space given by the random distribution $\mathbf{P}^\sigma$ on ϕ^σ. Therefore, if $\phi^\sigma(A^\sigma) \geq 0$ almost surely for some $A^\sigma \in \mathcal{A}^\sigma$, then $\phi\left([\![A^\sigma]\!]_\sigma\right) \geq 0$. In particular,

$$\forall A^\sigma \in \mathcal{A}^\sigma, \quad \phi\left([\![(A^\sigma)^2]\!]_\sigma\right) \geq 0. \tag{9}$$

The semidefinite method is a tool from the flag algebra framework that, for a given density problem of the form

$$\min_{\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})} \phi(A),$$

where $A \in \mathcal{A}$, systematically searches for 'best possible' inequalities of the form (9). If we fix in advance an upper bound on the size of graphs in the terms of inequalities we will be using, we can find the best inequalities of the form (9) using semidefinite programming. Furthermore, it is easy to extend this basic semidefinite method in such a way that together with inequalities (9), it uses also inequalities from a given finitely-dimensional linear subspace of $\mathcal{A}$.

4.2 Proof of Proposition 7

We start this section by showing that in an extremal graph, every two vertices participate in almost the same number of rainbow triangles.

Proposition 8. *In an extremal graph G on n vertices, for any pair of vertices $u, v \in V(G)$, we have $\binom{n-1}{2}(\text{RBT}(u) - \text{RBT}(v)) \leq n - 2$.*

Proof. Otherwise, we could delete v and duplicate u to u', i.e., for every vertex x we could color the edge xu' as xu. This implies that the color of uu' does not matter since uu' will not be in a rainbow triangle anyways. Let us call the new graph G'. Then

$$F(G') - F(G) \geq \binom{n-1}{2}(\text{RBT}(u) - \text{RBT}(v)) - \binom{n-2}{1}\text{RBT}(uv)$$

$$\geq \binom{n-1}{2}(\text{RBT}(u) - \text{RBT}(v)) - (n-2) > 0,$$

a contradiction. □

Combining this with the bound given by the iterative construction depicted in Figure 1 yields the following.

Corollary 9. *In an extremal graph G, $\text{RBT}(v) \geq 0.4 - o(1)$ for all vertices $v \in V(G)$.*

Let $(E_n)_{n \in \mathbb{N}}$ be any convergent sequence of extremal graphs of increasing orders with $e \in \text{Hom}^+(\mathcal{F}, \mathbb{R})$ being its limit. We call such e an extremal limit. We now look at the additional properties that every extremal limit needs to satisfy. We start with a "flag algebra version" of Corollary 9.

Corollary 10. *Let σ be the 1-vertex type, RBT^σ be the σ-flag of size three with all three edges colored differently (which is unique up to color-blind isomorphism), e be an extremal limit and e^σ be a random homomorphism drawn from $\mathbf{P}^\sigma$ of e. Then with probability 1,*

$$e^\sigma\left(\text{RBT}^\sigma - 1/4\right) \geq 0.$$

Furthermore, for any real $w \geq 0$ and $F^\sigma \in \mathcal{F}^\sigma$, it follows that

$$e\left(w \cdot [\![F^\sigma \times (\text{RBT}^\sigma - 1/4)]\!]_\sigma\right) \geq 0. \tag{10}$$

Next, we apply four times the semidefinite method that seeks for inequalities of the form (9) and (10) to conclude the following.

Lemma 11. *For every extremal limit e:*

$$e\left(\tfrac{4}{15}\text{RB2211} - \tfrac{26}{45}\text{RB3111} + \tfrac{27}{5000}\text{RB1111}^+\right) \geq \frac{1465936840976225933412082207134594049377 9}{5575186299632655785383929568162090376495104};$$

$$e\left(\text{RBT}\right) \leq \frac{11151645199111581268390153119301740786646069}{2787593149816327892691964784081045188247552 0};$$

$$e\left(\text{RB1111}\right) \leq \frac{2654858079423519437167848984032051438970 69}{2787593149816327892691964784081045188247552};$$

$$e\left(\tfrac{1}{3}\text{TCT} + \text{MONOT}\right) \leq \frac{5576885389284149539505627500589996258413877}{16725558898897967356151788704486271129485312}.$$

Proof. At the beginning, we express all four left-hand sides as a linear combination of densities of graphs on 6 vertices. Note that $|\mathcal{F}_6| = 4300$.

The first inequality can be obtained as the sum of the following inequalities:

- 163 inequalities of the form $e\left([\![(\sum_{F \in \mathcal{F}_5^\sigma} x_F \cdot F)^2]\!]_\sigma\right) \geq 0$, where σ is a (not always the same) type of on 4 vertices and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_5^\sigma$,

- 14 inequalities of the form $e\left(\left[\!\left[\left(\sum_{F\in\mathcal{F}_4^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \geq 0$, where σ is the only 2-vertex type (up to the blind-isomorphism) and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_4^\sigma$,

- one inequality of the form $e\left(\left(\sum_{F\in\mathcal{F}_3} x_F \cdot F\right)^2\right) \geq 0$, where $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_3$,

- 17 inequalities of the form $e\left(w \cdot [\![F \times (\mathrm{RBT}^\sigma - 1/4)]\!]_\sigma\right) \geq 0$, where σ is the 1-vertex type, $w \geq 0$ and $F \in \mathcal{F}_4^\sigma$,

- an inequality of the form $e\left(\sum_{F\in\mathcal{F}_6} y_F \cdot F\right) \geq 0$, where $y_F \geq 0$ for all $F \in \mathcal{F}_6$,

- the equation $e\left(z \cdot \sum_{F_i\in\mathcal{F}_6} F_i\right) = z$, where $z = \frac{1465936840976225933412082207134594049377 9}{5575186299632655785383929568162090376495104}$.

The second inequality can be obtained as the sum of the following inequalities:

- 884 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_5^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is a (not always the same) type of on 4 vertices and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_5^\sigma$,

- 30 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_4^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is the only 2-vertex type (up to the blind-isomorphism) and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_4^\sigma$,

- an inequality of the form $e\left(-\sum_{F\in\mathcal{F}_6} y_F \cdot F\right) \leq 0$, where $y_F \geq 0$ for all $F \in \mathcal{F}_6$,

- the equation $e\left(z \cdot \sum_{F_i\in\mathcal{F}_6} F_i\right) = z$, where $z = \frac{1115164519911158126839015311930174078664606 9}{2787593149816327892691964784081045188247552 0}$.

The third inequality can be obtained as the sum of the following inequalities:

- 948 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_5^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is a (not always the same) type of on 4 vertices and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_5^\sigma$,

- 38 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_4^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is the only 2-vertex type (up to the blind-isomorphism) and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_4^\sigma$,

- 15 inequalities of the form $e\left(-w \cdot [\![F \times (\mathrm{RBT}^\sigma - 1/4)]\!]_\sigma\right) \leq 0$, where σ is the 1-vertex type, $w \geq 0$ and $F \in \mathcal{F}_4^\sigma$,

- an inequality of the form $e\left(-\sum_{F\in\mathcal{F}_6} y_F \cdot F\right) \leq 0$, where $y_F \geq 0$ for all $F \in \mathcal{F}_6$,

- the equation $e\left(z \cdot \sum_{F_i\in\mathcal{F}_6} F_i\right) = z$, where $z = \frac{2654858079423519437167848984032051438970 69}{2787593149816327892691964784081045188247552}$.

Finally, the last inequality can obtained as the sum of the following inequalities:

- 876 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_5^\sigma} x_F \cdot F\right)^2\right]\!\right]_\sigma\right) \leq 0$, where σ is a (not always the same) type of on 4 vertices and $x_F \in \mathbb{Q}$ for all $F \in \mathcal{F}_5^\sigma$,

- 34 inequalities of the form $e\left(-\left[\!\left[\left(\sum_{F\in\mathcal{F}_4^\sigma}x_F\cdot F\right)^2\right]\!\right]_\sigma\right)\le 0$, where σ is the only 2-vertex type (up to the blind-isomorphism) and $x_F\in\mathbb{Q}$ for all $F\in\mathcal{F}_4^\sigma$,

- 21 inequalities of the form $e\left(-w\cdot[\![F\times(\mathrm{RBT}^\sigma-1/4)]\!]_\sigma\right)\le 0$, where σ is the 1-vertex type, $w\ge 0$ and $F\in\mathcal{F}_4^\sigma$,

- an inequality of the form $e\left(-\sum_{F\in\mathcal{F}_6}y_F\cdot F\right)\le 0$, where $y_F\ge 0$ for all $F\in\mathcal{F}_6$,

- the equation $e\left(z\cdot\sum_{F_i\in\mathcal{F}_6}F_i\right)=z$, where $z=\frac{55768853892841495395056275005899962584413877}{16725558898897967356151788704486271129485312}$.

The exact rational values of all the coefficients x_F, y_F and w that appears in the inequalities above were obtained with computer assistance. They are available at `http://www.math.uiuc.edu/~jobal/cikk/rbt`, as well as a small Sage script that computes the corresponding sums. □

In order to prove Proposition 7, we just translate the previous statement back to the finite setting.

Proof of Proposition 7. Suppose one of the inequalities from the statement of Proposition 7 is false. For example, suppose that the inequality (6) is false. Therefore, for every $k\in\mathbb{N}$ we can find an extremal graph E_k on at least k vertices such that $\mathrm{RBT}\ge 0.40005$. By compactness, the sequence $(E_k)_{k\in\mathbb{N}}$ has a convergent subsequence and this subsequence converges to some extremal limit e. However, $e(\mathrm{RBT})\ge 0.40005$, which contradicts Lemma 11. □

5 Proof of Theorem 4

Let G be an extremal graph on n vertices, where n is sufficiently large. Let $Z=\{z_1,z_2,z_3,z_4\}$ be a subset of $V(G)$ such that Z induces an RB1111, and

$$\mathrm{RB2211}(Z)-\tfrac{26}{9}\mathrm{RB3111}(Z)+\tfrac{27}{1000}\mathrm{RB1111}^+(Z) \tag{11}$$

is maximized over all choices of Z.

Note that in every RB2211, four of the 15 vertex subsets of size 4 induce copies of RB1111, three in every RB3111, and one of the five sets in every RB1111$^+$. Denote by $\mathcal{Z}$ the set of all properly 3-edge-colored K_4s. Since (11) is maximized, we can lower bound it by the average over all $Y\in\mathcal{Z}$ and we obtain

$$\left(\mathrm{RB2211}(Z)-\tfrac{26}{9}\mathrm{RB3111}(Z)+\tfrac{27}{1000}\mathrm{RB1111}^+(Z)\right)\binom{n-4}{2}$$

$$\ge\frac{1}{|\mathcal{Z}|}\sum_{Y\in\mathcal{Z}}\left(\left(\mathrm{RB2211}(Y)-\tfrac{26}{9}\mathrm{RB3111}(Y)\right)\binom{n-4}{2}+\tfrac{27}{2000}\mathrm{RB1111}^+(Y)\binom{n-4}{1}(n-5)\right)$$

$$\ge\frac{\left(4\,\mathrm{RB2211}-3\cdot\tfrac{26}{9}\mathrm{RB3111}\right)\binom{n}{6}+\tfrac{27}{2000}\mathrm{RB1111}^+\binom{n}{5}(n-5)}{\mathrm{RB1111}\binom{n}{4}}$$

$$=\frac{\tfrac{4}{15}\mathrm{RB2211}-\tfrac{26}{45}\mathrm{RB3111}+\tfrac{27}{5000}\mathrm{RB1111}^+}{\mathrm{RB1111}}\binom{n-4}{2}.$$

Using (5) and (7), this gives

$$\text{RB2211}(Z) - \tfrac{26}{9}\text{RB3111}(Z) + \tfrac{27}{1000}\text{RB1111}^+(Z) > 0.02760856. \tag{12}$$

For $1 \le i \le 4$, we define sets of vertices Z_i which look like z_i to the other vertices of Z. Formally,

$$Z_i := \{v \in V(G) : G[(Z \setminus z_i) \cup v] \cong \text{RB1111}\} \text{ for } 1 \le i \le 4.$$

Note that $Z_i \cap Z_j = \emptyset$ for $i \ne j$. We call an edge $v_i v_j$ *funky*, if the color of $v_i v_j$ is different from the color of $z_i z_j$, where $v_i \in Z_i$, $v_j \in Z_j$, $1 \le i < j \le 4$. In other words, $G[Z \cup \{v_i, v_j\}] \not\cong \text{RB2211}$, i.e., every funky edge destroys a potential copy of $\text{RB2211}(Z)$. Denote by E_f the set of funky edges. With this notation, for sufficiently large n (12) implies that

$$2 \sum_{1 \le i < j \le 4} |Z_i||Z_j| - 2|E_f| - \tfrac{26}{9} \sum_{1 \le i \le 4} |Z_i|^2 + \tfrac{27n}{1000} \left(n - \sum_{1 \le i \le 4} |Z_i| \right) > 0.02760856 \times 2 \binom{n-4}{2}.$$

For $X_i \subseteq Z_i$, where $1 \le i \le 4$, let $X_0 := V(G) \setminus \bigcup X_i$. Let f be the number of funky edges not incident to vertices in X_0, divided by n^2 for normalization, and denote $x_i = \tfrac{1}{n}|X_i|$ for $0 \le i \le 4$. Choose X_is such that the left hand side of

$$2 \sum_{1 \le i < j \le 4} x_i x_j - 2f - \tfrac{26}{9} \sum_{1 \le i \le 4} x_i^2 + \tfrac{27}{1000} x_0 > 0.02760856 \tag{13}$$

is maximized.

From this, it is not difficult to check (see Appendix A) that

$$x_0 < 0.0059605; \tag{14}$$
$$0.244287 < \ x_i < 0.255713 \quad \text{for} \quad 1 \le i \le 4; \tag{15}$$
$$0.493403 < \ x_i + x_j < 0.506597 \quad \text{for} \quad 1 \le i < j \le 4; \tag{16}$$
$$f < 0.000084609; \tag{17}$$
$$-\tfrac{25}{27} x_1 + 2x_i - \tfrac{1}{3} x_2 - \tfrac{1}{3} x_3 - \tfrac{1}{3} x_4 < 0.0315 \quad \text{for} \quad 2 \le i \le 4; \tag{18}$$
$$2x_1 - x_2 + x_3 - x_0 > 0.484987; \tag{19}$$
$$x_i + x_0 < 0.2563 \quad \text{for} \quad 1 \le i \le 4. \tag{20}$$

By symmetry, (18) and (19) hold also after permuting the variables. However, we use them explicitly only in this permutation. Furthermore, for any vertex $v \in X_i$ we use $d_f(v)$ to denote the number of funky edges from v to $(X_1 \cup X_2 \cup X_3 \cup X_4) \setminus X_i$ after normalizing by n. The contribution of $v \in X_1$ to (13) is

$$\frac{1}{n} \left(2(x_2 + x_3 + x_4) - 2d_f(v) - 2 \cdot \tfrac{26}{9} x_1 + o(1) \right).$$

If this quantity was negative, (13) could be increased by moving v to X_0, contradicting our choice of X_i. This and (15) imply that

$$d_f(v) \le x_2 + x_3 + x_4 - \tfrac{26}{9} x_1 + o(1) \le 1 - \tfrac{35}{9} x_1 + o(1) < 0.049995, \tag{21}$$

and symmetric statements hold for $v \in X_2, X_3, X_4$.

By symmetry, we may assume that the non-funky edges are colored as in Figure 3.
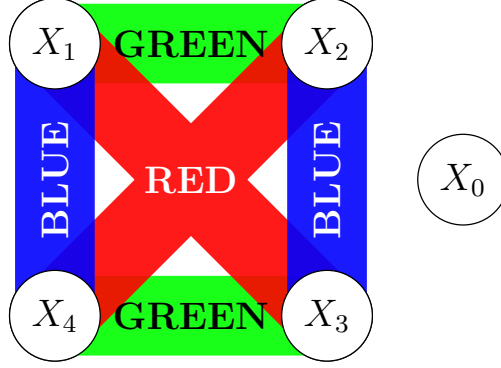
Figure 3: Coloring of the non-funky edges.

Next, we will prove that a vertex $v \in X_i$ cannot be adjacent to almost all vertices of X_i by edges of only one color. For a vertex $v \in V(G)$, we denote by $r_i(v)$, $b_i(v)$ and $g_i(v)$ the numbers of red, blue and green edges from v to X_i, divided by n. Similarly, let $r(v)$, $b(v)$, and $g(v)$ be the numbers of all red/blue/green edges incident to v, divided by n.

Claim 12. *For every $v \in X_i$, we have $x_i - r_i(v), x_i - b_i(v), x_i - g_i(v) > 0.033$, where $i \in \{1,2,3,4\}$.*

Proof. Without loss of generality, let us assume $v \in X_1$ and $x_1 - r_1(v) \leq 0.033$. Denote $x_{\max} := \max\{x_2, x_3, x_4\}$. We bound the number of rainbow triangles containing v divided by n^2, i.e., $\frac{1}{2}\text{RBT}(v)$. For a rainbow triangle uvw, we distinguish several cases.

1. If $u, w \in X_1$, then the normalized number of rainbow triangles uvw can be upper bounded by $r_1(v)b_1(v) + r_1(v)g_1(v) + g_1(v)b_1(v)$. This is maximized when $g_1(v) = b_1(v) = \frac{1}{2}(x_1 - r_1(v))$, which gives the upper bound $(r_1(v) + \frac{x_1 - r_1(v)}{4})(x_1 - r_1(v))$ for triangles of this type.
2. If $u \in X_i$ and $w \in X_j$, where $1 < i < j \leq 4$, and all of uv, vw, uw are non-funky, then we obtain the upper bound $x_2 x_3 + x_2 x_4 + x_3 x_4 - d_f(v)(x_2 + x_3 + x_4 - x_{\max}) + \frac{1}{3}d_f(v)^2$ for triangles of this type, where the third term accounts for possible double counting in the second term.
3. If uw is a funky edge, then uvw might be rainbow and in this case we get the upper bound f for triangles of this type.
4. If $u \in X_0$ then w can be anywhere, which gives the bound x_0 for triangles of this type.
5. We can bound the number of rainbow triangles where both vu and vw are funky by $\frac{1}{3}d_f(v)^2$. The $\frac{1}{3}$ in the term comes from the fact that vu and vw must have different colors for the triangle to be rainbow.
6. If vu is funky and $w \in X_1$, then we get an upper bound of $d_f(v)r_1(v)$ for triangles of this type.
7. If vu is funky and v and w are in the same X_i (for $i \geq 2$), we get an upper bound of $d_f(v)x_{\max}$ for triangles of this type.

Note that it cannot happen that only vu is funky, $v \in X_i$, and $w \in X_j$, where $i, j \in \{2, 3, 4\}$ and $i \neq j$.

Counting all types together, we obtain

$$\frac{1}{2}\text{RBT}(v) \leq \left(r_1(v) + \frac{x_1 - r_1(v)}{4}\right)(x_1 - r_1(v)) + x_2 x_3 + x_2 x_4 + x_3 x_4$$
$$+ f + x_0 + d_f(v)(2x_{\max} + r_1(v) - x_2 - x_3 - x_4 + \frac{2}{3}d_f(v)) < 0.1991, \qquad (22)$$

14

which contradicts Corollary 9. The last inequality can be obtained by maximizing (22) in the following way.

If $x_1 - r_1(v) \leq 0.033$, then $r_1(v) \geq 0.244287 - 0.033$ and the partial derivative of the right hand side of (22) in direction $r_1(v)$ is $\frac{3}{4}x_1 - \frac{3}{2}r_1(v) + d_f(v)$, which is negative. Thus, to maximize the bound, we need to pick $r_1(v)$ minimal, and thus we may assume that $x_1 - r_1(v) = 0.033$.

Next, we get that the coefficient of $d_f(v)$ in (22) is

$$2x_{\max} + r_1(v) - x_2 - x_3 - x_4 + \tfrac{2}{3}d_f(v) = x_1 + 2x_{\max} - x_2 - x_3 - x_4 - 0.033 + \tfrac{2}{3}d_f(v)$$
$$\leq_{(21)} x_1 + 2x_{\max} - x_2 - x_3 - x_4 - 0.033 + \tfrac{2}{3}(x_2 + x_3 + x_4 - \tfrac{26}{9}x_1 + o(1))$$
$$= -\tfrac{25}{27}x_1 + 2x_{\max} - 0.033 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 + o(1) <_{(18)} 0,$$

so we may assume that $d_f(v) = 0$, and the right hand side of (22) becomes

$$((x_1 - 0.033) + \tfrac{0.033}{4})0.033 + x_2x_3 + x_2x_4 + x_3x_4 + f + x_0. \tag{23}$$

Now (23) is maximized when $x_2 = x_3 = x_4$ if we fix all the other variables. Note that this choice will not conflict with any other bounds. So we may assume that $x_2 = x_3 = x_4$.

This gives us
$$\tfrac{1}{2}\mathrm{RBT}(v) \leq (x_1 - 0.033 + \tfrac{0.033}{4})0.033 + 3x_2^2 + f + x_0,$$

while from (13):
$$6x_1x_2 - \tfrac{8}{3}x_2^2 - \tfrac{26}{9}x_1^2 + 0.027x_0 - 2f > 0.02760856.$$

The resulting program we want to solve is

$$(P) \begin{cases} \text{maximize} & (x_1 - 0.033 + \tfrac{0.033}{4})0.033 + 3x_2^2 + f + x_0 \\ \text{subject to} & 0.02760856 < 6x_1x_2 - \tfrac{8}{3}x_2^2 - \tfrac{26}{9}x_1^2 + 0.027x_0 - 2f, \\ & x_1 + 3x_2 + x_0 = 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & x_0 \geq 0, \\ & f \geq 0. \end{cases}$$

This program can be solved using Lagrange multipliers. We give the computation in Appendix B. The optimal solution is $x_1 \approx 0.246648$, $x_2 \approx 0.249389$, $f = 0$, and the value is less than 0.1991. $\qquad\square$

Let us call a vertex $v \in X_i$ *blue* if $x_i - b_i(v) \leq 0.075$, and similarly *red* or *green*, and finally *black* if it has none of the other colors. Note that each vertex has exactly one of the four colors.

Claim 13. *If $v \in X_1$ is black, then $d_f(v) = 0$.*

Proof. Let vw be a funky edge, and suppose that w is chosen such that $d_f(w)$ is minimized over all funky neighbors of v. Therefore, $d_f(v) \times d_f(w) \leq 2f$. By symmetry, we may assume that $w \in X_2$ and vw is red. As G has maximal rainbow triangle density, recoloring vw to green (making it not

15

funky) can only reduce the number of rainbow triangles. So let us bound the number of rainbow triangles containing vw before and after the recoloring.

Before: $\mathrm{RBT}(vw) \leq d_f(v) + d_f(w) + b_1(v) + b_2(w) + x_0$;

After: $\mathrm{RBT}(vw) \geq x_3 + x_4 - d_f(v, X_3 \cup X_4) - d_f(w, X_3 \cup X_4) \geq x_3 + x_4 - d_f(v) - d_f(w)$.

By the assumption that $\mathrm{RBT}(uw)$ does not increase when the color of uw is changed, we obtain that

$$-b_1(v) \leq b_2(w) - x_3 - x_4 + x_0 + 2d_f(v) + 2d_f(w). \tag{24}$$

By Claim 12, $b_2(w) \leq x_2 - 0.033$, which together with v being black gives

$$0.075 \leq x_1 - b_1(v) \leq x_1 + x_2 - 0.033 - x_3 - x_4 + x_0 + 2d_f(v) + 2d_f(w)$$
$$\leq 2(x_1 + x_2 + x_0) - 0.033 - 1 + 2d_f(v) + 2d_f(w).$$

Let us maximize the right hand side using (15), (17) and (21).

$$(P) \begin{cases} \text{maximize} & 2(x_1 + x_2 + x_0) - 0.033 - 1 + 2d_f(v) + 2d_f(w) \\ \text{subject to} & d_f(v) \times d_f(w) \leq 2f \leq 2 \times 0.000084609, \\ & d_f(v) \leq 1 - \frac{35}{9}x_1, \\ & d_f(w) \leq 1 - \frac{35}{9}x_2, \\ & 0.244287 \leq x_1 \leq 0.255713, \\ & 0.244287 \leq x_2 \leq 0.255713. \end{cases}$$

In order to simplify the computation and writeup, we omit the $o(1)$ term that is coming from constraints given by (21). The only change is that the objective functions in the following programs contain $+o(1)$.

To break the symmetry of (P) we assume that $x_1 \leq x_2$, making the bound on $d_f(w)$ lower than the bound on $d_f(v)$. This is allowed as all the relations of (P) are symmetric in x_1 and x_2. If x_0, x_1, x_2 are fixed, the maximum of (P) is attained when $d_f(v)$ is maximized, i.e., for $d_f(v) = 1 - \frac{35}{9}x_1$, and then $d_f(w)$ is maximized, i.e., for $d_f(w) = \min\{1 - \frac{35}{9}x_2, 2(0.000084609/d_f(v))\}$.

It follows from (20) that $x_2 + x_0 < 0.2563$, which gives the following relaxation (P_1) of (P) with only one variable:

$$(P_1) \begin{cases} \text{maximize} & 2(x_1 + 0.2563) - 0.033 - 1 + 2(1 - \frac{35}{9}x_1) + 4(0.000084609/(1 - \frac{35}{9}x_1)) \\ \text{subject to} & 0.244287 \leq x_1 \leq 0.255713. \end{cases}$$

Simplification of the objective function in (P_1) gives (P_1')

$$(P_1') \begin{cases} \text{maximize} & 1.4796 - \frac{52}{9}x_1 + 0.003045924/(9 - 35x_1) \\ \text{subject to} & 0.244287 \leq x_1 \leq 0.255713. \end{cases}$$

The maximum of P_1' is when $x_1 = 0.244287$ and gives $0.075 > x_1 - b_1(v)$ which contradicts $x_1 - b_1(v) \geq 0.075$.

$\square$

Claim 14. *If $v \in X_1 \cup \cdots \cup X_4$ is a vertex of color c that is not black, then v is not incident to any funky edges colored c or to funky edges whose non-funky color would be c. For example, a blue vertex $v \in X_1$ can be incident only to funky edges that are not blue and have the other endpoint in X_2 or X_3, in other words, $b_2(v) + b_3(v) + g_4(v) + r_4(v) = 0$.*

Proof. We assume without loss of generality that $v \in X_1$ is blue. Suppose for contradiction that there is a vertex w such that vw is funky and either $w \in X_4$ or if $w \in X_2 \cup X_3$ then uw is blue. Let us only look at the case that $w \in X_2$ and vw blue, the other cases are similar.

By similar arguments as in Claim 13 we count the number of rainbow triangles containing uw and the number after recoloring uw to green. We obtain

Before: $\mathrm{RBT}(vw) \leq d_f(v) + d_f(w) + r_1(v) + r_2(w) + x_0$;

After: $\mathrm{RBT}(vw) \geq x_3 + x_4 - d_f(v, X_3 \cup X_4) - d_f(w, X_3 \cup X_4) \geq x_3 + x_4 - d_f(v) - d_f(w)$.

Since switching vw to green may not increase the number of RBT, we get an analogue of (24)

$$-r_1(v) \leq d_f(v) + d_f(w) + r_2(w) + x_0 - (x_3 + x_4 - d_f(v) - d_f(w)). \tag{25}$$

Since v is blue, $r_1(v) \leq 0.075$. With (21) and by adding $x_1 + r_1(v)$ to both sides of (25) we get

$$
\begin{aligned}
x_1(v) \leq_{(25)} \ & x_1 + r_1(v) + x_2 - x_3 - x_4 + x_0 + 2d_f(v) + 2d_f(w) \\
\leq_{(21)} \ & r_1(v) + 4 - \tfrac{61}{9}(x_1 + x_2) - x_3 - x_4 + x_0 \\
= \ & r_1(v) + 4 - \tfrac{52}{9}(x_1 + x_2) - (x_0 + x_1 + x_2 + x_3 + x_4) + 2x_0 \\
\leq \ & r_1(v) + 3 - \tfrac{52}{9}0.493403 + 2 \cdot 0.0059605 \\
< \ & 0.162 + r_1(v) \leq 0.237,
\end{aligned}
$$

which contradicts (15). □

For every $v \in V(G)$ we define $d_{mono}(v) := max\{r(v), g(v), b(v)\}$.

Claim 15. *The number of vertices v with $d_f(v) > 0$ is less than $0.00937n$. This implies that $d_f(v) < 0.00937$ for all vertices in $V \setminus X_0$.*

Proof. Using (8) and the definition of d_{mono} we get

$$0.33343492 > \tfrac{1}{3}\mathrm{TCT} + \mathrm{MONOT} = \frac{1}{n}\sum_{v \in V}(r(v)^2 + g(v)^2 + b(v)^2) - o(1)$$

$$\geq \frac{1}{n}\sum_{v \in V}(d_{mono}(v)^2 + \tfrac{1}{2}(1 - d_{mono}(v))^2) - o(1) \geq \frac{1}{3} - o(1),$$

and hence

$$0.333435 > \frac{1}{n}\sum_{v \in V}(d_{mono}(v)^2 + \tfrac{1}{2}(1 - d_{mono}(v))^2). \tag{26}$$

By Claim 13, any v with $d_f(v) > 0$ is not black. Without loss of generality we assume $v \in X_1$ is blue, hence $r_4(v) = g_4(v) = 0$ by Claim 14. Then we have

$$d_{mono}(v) \geq b(v) \geq x_1 - 0.075 + x_4 >_{(16)} 0.4184.$$

So
$$d_{mono}(v)^2 + \tfrac{1}{2}(1 - d_{mono}(v))^2 > 0.344188.$$

By this and (26), we conclude that the number of vertices v with $d_f(v) > 0$ can be at most

$$\frac{0.333435 - \tfrac{1}{3}}{0.344188 - \tfrac{1}{3}} n < 0.009367 < 0.00937n.$$

<div style="text-align: right;">□</div>

Claim 16. *For all $v \in X_1 \cup X_2 \cup X_3 \cup X_4$ we have $d_f(v) = 0$.*

Proof. Suppose that vw is funky, say $v \in X_1$, $w \in X_2$, and vw is red. Then, using (24) and the bounds for $d_f(v)$ from Claim 15,

$$x_1 - b_1(v) + x_2 - b_2(w) \le x_1 + x_2 - x_3 - x_4 + x_0 + 2d_f(v) + 2d_f(w)$$
$$\le_{(16)} 0.506597 - 0.493403 + 0.0059605 + 4 \times 0.00937 = 0.0566345,$$

contradicting Claim 12, which implies that $x_1 - b_1(v) + x_2 - b_2(w) \ge 0.066$. □

Next, we want to show that $X_0 = \emptyset$. For this, suppose that there exists $x \in X_0$. We will add x to one of the X_i such that $d_f(x)$ is minimal. By symmetry, we may assume that x is added to X_1. Note that adding a single vertex to X_1 changes the density bounds we used above by at most $o(1)$.

Claim 17. *For every $x \in X_0$, if x was part of X_1 then $d_f(x) \ge 0.0099$.*

Proof. Let xw be a funky edge, where $w \in X_2$. Since G is extremal, making xw not funky cannot increase the number of rainbow triangles which gives a relation analogous to (24).

$$\text{Before: } \mathrm{RBT}(xw) \le d_f(x) + b_1(x) + b_2(w) + x_0;$$
$$\text{After: } \mathrm{RBT}(xw) \ge x_3 + x_4 - d_f(x).$$

By the assumption that $\mathrm{RBT}(xw)$ does not increase when the color of xw is changed, we obtain that

$$-b_1(x) - b_2(w) \le -x_3 - x_4 + x_0 + 2d_f(x). \tag{27}$$

We also use the trivial bounds $b_1(x) \le x_1$ and $b_2(w) \le x_2 - 0.033$. Then

$$-x_1 - (x_2 - 0.033) \le -b_1(x) - b_2(w) \le_{(27)} -x_3 - x_4 + x_0 + 2d_f(x),$$

$$2d_f(x) \ge x_3 + x_4 + 0.033 - (x_0 + x_1 + x_2) = x_3 + x_4 + 0.033 - (1 - x_3 - x_4)$$
$$= 2x_3 + 2x_4 - 0.967 >_{(16)} 0.019802 > 2 \times 0.0099.$$

<div style="text-align: right;">□</div>

Using yet a different way of bounding $d_f(x)$ and combining it with Claim 17 we get the following improved bound on $d_f(x)$.

Claim 18. *For every $x \in X_0$, if x was part of X_1, then $d_f(x) > 0.12866$.*

<div style="text-align: center;">18</div>

Proof. Suppose for a contradiction that $d_f(x) < 0.12866$. First we derive lower bounds on d_{mono} of vertices in funky edges containing x. Suppose that xw is funky, say $w \in X_2$ and xw is red. By arguments very similar to the proof of Claim 13, we have

$$\text{Before:} \quad \text{RBT}(xw) \leq b_1(x) + b_2(w) + g_3(x) + x_0;$$
$$\text{After:} \quad \text{RBT}(xw) \geq x_3 + x_4 - (d_f(x) - r_2(x)).$$

We conclude that

$$b_2(w) \geq x_3 + x_4 - x_0 - b_1(x) - d_f(x) - g_3(x) + r_2(x).$$

Next, we give a lower bound on $d_{mono}(w)$:

$$d_{mono}(w) \geq b(w) = b_2(w) + x_3 \geq 2x_3 + x_4 - x_0 - b_1(x) - d_f(x) - g_3(x) + r_2(x)$$
$$>_{(19)} 0.484987 + x_1 - b_1(x) - d_f(x) - g_3(x) + r_2(x)$$
$$\geq 0.484987 - d_f(x) - g_3(x) + r_2(x).$$

Similar bounds hold for all other funky edges incident to x. We give only a conclusion here:

$$d_{mono}(w) \geq \begin{cases} 0.484987 - d_f(x) - g_3(x) + r_2(x) & \text{if } w \in X_2 \text{ and } xw \text{ is red;} \\ 0.484987 - d_f(x) - g_4(x) + b_2(x) & \text{if } w \in X_2 \text{ and } xw \text{ is blue;} \\ 0.484987 - d_f(x) - r_2(x) + g_3(x) & \text{if } w \in X_3 \text{ and } xw \text{ is green;} \\ 0.484987 - d_f(x) - r_4(x) + b_3(x) & \text{if } w \in X_3 \text{ and } xw \text{ is blue;} \\ 0.484987 - d_f(x) - b_3(x) + r_4(x) & \text{if } w \in X_4 \text{ and } xw \text{ is red;} \\ 0.484987 - d_f(x) - b_2(x) + g_4(x) & \text{if } w \in X_4 \text{ and } xw \text{ is green.} \end{cases} \quad (28)$$

Observe that the bound when $w \in X_2$ and xw is red contains the same variables as if $w \in X_3$ and xw is green. The same is true also for $w \in X_2$ with blue xw and $w \in X_4$ with green xw and also for the last pair. In order to fit the following computation on one page, we write it only for the first pair. For the other two pairs, we use analogous operations. It follows from (26), (28) and

$d_f(x) = r_2(x) + g_3(x) + b_2(x) + g_4(x) + b_3(x) + r_4(x)$ that

$$0.333435 >_{(26)} \frac{1}{n} \sum_{v \in V(G)} (d_{mono}(v)^2 + \tfrac{1}{2}(1 - d_{mono}(v))^2)$$

$$\geq \tfrac{1}{3}(1 - d_f(x)) + \tfrac{1}{2}d_f(x)$$
$$+ r_2(x)[\tfrac{3}{2}(0.484987 - d_f(x) - g_3(x) + r_2(x))^2 - (0.484987 - d_f(x) - g_3(x) + r_2(x))]$$
$$+ g_3(x)[\tfrac{3}{2}(0.484987 - d_f(x) + g_3(x) - r_2(x))^2 - (0.484987 - d_f(x) + g_3(x) - r_2(x))]$$
$$+ b_2(x)(...) + g_4(x)(...) + b_3(x)(...) + r_4(x)(...)$$
$$= \tfrac{1}{3}(1 - d_f(x)) + \tfrac{1}{2}d_f(x)$$
$$+ d_f(x)(\tfrac{3}{2}(0.484987 - d_f(x))^2 - 0.484987 + d_f(x))$$
$$+ r_2(x)[3(0.484987 - d_f(x))(r_2(x) - g_3(x)) + \tfrac{3}{2}(r_2(x) - g_3(x))^2 - (r_2(x) - g_3(x))]$$
$$+ g_3(x)[3(0.484987 - d_f(x))(g_3(x) - r_2(x)) + \tfrac{3}{2}(g_3(x) - r_2(x))^2 - (g_3(x) - r_2(x))] + \cdots$$
$$= \tfrac{1}{3}(1 - d_f(x)) + \tfrac{1}{2}d_f(x) + d_f(x)(\tfrac{3}{2}(0.484987 - d_f(x))^2 - 0.484987 + d_f(x))$$
$$+ r_2(x)[3(0.484987 - d_f(x)) - 1)(r_2(x) - g_3(x)) + \tfrac{3}{2}(r_2(x) - g_3(x))^2]$$
$$+ g_3(x)[3(0.484987 - d_f(x)) - 1)(g_3(x) - r_2(x)) + \tfrac{3}{2}(g_3(x) - r_2(x))^2] + \cdots$$
$$= \tfrac{1}{3}(1 - d_f(x)) + \tfrac{1}{2}d_f(x) + d_f(x)(\tfrac{3}{2}(0.484987 - d_f(x))^2 - 0.484987 + d_f(x))$$
$$+ (3(0.484987 - d_f(x)) - 1)(r_2(x) - g_3(x))^2 + \tfrac{3}{2}(r_2(x) - g_3(x))^2(r_2(x) + g_3(x)) + \cdots .$$

If $d_f(x) < 0.12866$, then $3(0.484987 - d_f(x)) - 1 > 0$. Hence,

$$(3(0.484987 - d_f(x)) - 1)(r_2(x) - g_3(x))^2 + \tfrac{3}{2}(r_2(x) - g_3(x))^2(r_2(x) + g_3(x)) \geq 0,$$

and we can obtain the following lower bound:

$$0.333435 \geq \tfrac{1}{3}(1 - d_f(x)) + \tfrac{1}{2}d_f(x) + d_f(x)(\tfrac{3}{2}(0.484987 - d_f(x))^2 - 0.484987 + d_f(x))$$
$$= \tfrac{3}{2}d_f(x)^3 + (1 - 3 \times 0.484987)d_f(x)^2 + (\tfrac{1}{6} + \tfrac{3}{2} \times 0.484987^2 - 0.484987)d_f(x) + \tfrac{1}{3},$$

so

$$0 \geq \tfrac{3}{2}d_f(x)^3 - 0.454961d_f(x)^2 + 0.03449825d_f(x) - 0.000102.$$

All $d_f(x)$ that satisfy the last inequality are in $(-\infty, 0.0031) \cup (0.12866, 0.1716)$. Claim 17 implies that $d_f(x)$ is not in $(-\infty, 0.0031)$, hence $d_f(x) > 0.12866$, which is a contradiction to the assumption $d_f(x) < 0.12866$. $\qquad \square$

Claim 19. *The set X_0 is empty.*

Proof. We will show that $\mathrm{RBT}(x) < 0.397$ for any $x \in X_0$, contradicting Corollary 9. For the ease of notation, we will write r_i for $r_i(x)$ etc.

$$\tfrac{1}{2}\mathrm{RBT}(x) \leq \quad \tfrac{1}{2}x_0^2 + x_0(1 - x_0)$$
$$+ r_1g_1 + r_1b_1 + g_1b_1 + r_2g_2 + r_2b_2 + g_2b_2 + r_3g_3 + r_3b_3 + g_3b_3 + r_4g_4 + r_4b_4 + g_4b_4$$
$$+ r_1(b_2 + g_4) + b_2g_4 + g_1(b_3 + r_4) + b_3r_4 + b_1(r_2 + g_3) + r_2g_3 + g_2(r_3 + b_4) + r_3b_4$$
$$\leq_{(*)} \tfrac{1}{2}x_0^2 + x_0(1 - x_0) + 0.1945(1 - x_0)^2 <_{(14)} 0.1982,$$

where $(*)$ comes from a massive computation described in Appendix C. This contradiction proves the claim. $\qquad \square$

Claim 20. *For n large enough, we have $|X_i| - |X_j| \leq 1$.*

Proof. By symmetry, for a contradiction we assume $|X_1| - |X_2| \geq 2$. Then we move a vertex from X_1 to X_2 and show that doing so increases the number of rainbow triangles. Recall that $\mathrm{RBT}(v)$ denotes the rooted density of RBT at v. Denote

$$F_{\mathrm{avg}}(m) = \frac{1}{m} \sum_{v \in V(G_m)} \mathrm{RBT}(v) \binom{m-1}{2} = 3\frac{F(m)}{m},$$

where G_m is an extremal graph on m vertices. Let

$$l = \lim_{m \to \infty} \frac{F_{\mathrm{avg}}(m)}{\binom{m-1}{2}}.$$

The limit exists since $F_{\mathrm{avg}}(m)/\binom{m-1}{2} = F(m)/\binom{m}{3}$ is non-increasing and lower bounded by 0.4. Corollary 9 implies that $0.40005 \geq l \geq 0.4$. Let $a_i = |X_i| = nx_i$ for $i \in \{1, 2, 3, 4\}$. We delete v from X_1, where $\mathrm{RBT}(v)$ is minimized over vertices in X_1, and add a duplicate w' of $w \in X_2$, where $\mathrm{RBT}(w)$ is maximized over vertices in X_2. We color ww' arbitrarily.

$$\text{Before: } \mathrm{RBT}(v)\binom{n-1}{2} \leq F_{\mathrm{avg}}(a_1) + a_2a_3 + a_2a_4 + a_3a_4,$$

$$\text{After: } \mathrm{RBT}(w')\binom{n-1}{2} \geq F_{\mathrm{avg}}(a_2) + (a_1 - 1)a_3 + (a_1 - 1)a_4 + a_3a_4.$$

Since G is extremal, $\mathrm{RBT}(v) \geq \mathrm{RBT}(w')$. Now we estimate $F_{\mathrm{avg}}(a_1) - F_{\mathrm{avg}}(a_2)$. Since $F_{\mathrm{avg}}(m)/\binom{m-1}{2}$ is non-increasing and its limit is l, for n large enough we have

$$F_{\mathrm{avg}}(a_1) = a_1^2 \cdot l/2 + \varepsilon_1 a_1^2, \qquad F_{\mathrm{avg}}(a_2) = a_2^2 \cdot l/2 + \varepsilon_2 a_2^2$$

and $\varepsilon_1 \leq \varepsilon_2 \leq 0.01$. Then we have $F_{\mathrm{avg}}(a_1) - F_{\mathrm{avg}}(a_2) \leq (l/2 + 0.01)(a_1^2 - a_2^2)$ and obtain

$$0 \leq (\mathrm{RBT}(v) - \mathrm{RBT}(w'))\binom{n-1}{2}$$

$$\leq F_{\mathrm{avg}}(a_1) + a_2a_3 + a_2a_4 + a_3a_4 - F_{\mathrm{avg}}(a_2) - (a_1 - 1)a_3 - (a_1 - 1)a_4 - a_3a_4$$

$$\leq (0.5l + 0.01)(a_1^2 - a_2^2) - (a_1 - 1 - a_2)(a_3 + a_4) < 0.22(a_1 - a_2)(a_1 + a_2) - 0.5(a_1 - a_2)(a_3 + a_4)$$

$$\leq (a_1 - a_2)(0.22(a_1 + a_2) - 0.5(a_3 + a_4)) < 0,$$

which is a contradiction. □

Claim 20 gives a proof of Theorem 4.

Proof of Theorem 4. Let G be an extremal graph on n vertices, where n is sufficiently large, such that Claim 20 holds. Denote $a = |X_1|$, $b = |X_2|$, $c = |X_3|$ and $d = |X_4|$. By Claim 20, a, b, c, d are as equal as possible. Moreover, by Claims 16 and 19, rainbow triangles are either entirely in one X_i for $1 \leq i \leq 4$, or intersect three of the X_i's. It then follows from the extremality of G that

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd,$$

which completes the proof of the recurrence. Notice that X_1, X_2, X_3, and X_4 satisfy the claimed blow-up property by Claim 16. □

21

6 Extremal graphs

Now that we know the limit object, we look at the extremal graphs on n vertices. Using a standard blow-up argument, Theorem 6 implies that any 3-edge-colored graph G contains at most $(n^3 - n)/15$ rainbow triangles.

Corollary 21. *Every 3-edge-colored graph on n vertices contains at most $(n^3 - n)/15$ rainbow triangles.*

Proof. Suppose there exists a 3-edge-colored graph G on k vertices with $r = (k^3 - k)/15 + \ell$ rainbow triangles for some $\ell > 0$. Without loss of generality, G is a 3-edge-coloring of K_n. Let $G_0 := G$ and G_{i+1}, for $i \in \mathbb{N}$, will be obtained by blowing up every vertex of G by a factor k^i and placing G_i inside every blob. It follows that $v(G_i) = k^{i+1}$ and $F(G_i) = k^{3i} \cdot r + k \cdot F(G_{i-1})$. Recall that $F(G_i)$ denotes the number of rainbow triangles in G_i. Expanding the recurrence, it follows that

$$F(G_i) = \sum_{j=0}^{i} k^{3j} \cdot k^{i-j} \cdot r = \frac{k^{3i}\left(k^3 - k + 15\ell\right)}{15} \cdot \sum_{t=0}^{i} \frac{1}{k^{2t}}.$$

Therefore,

$$\lim_{i \to \infty} \frac{F(G_i)}{\binom{v(G_i)}{3}} = \frac{k^2}{k^2 - 1} \cdot \frac{6 \cdot \left(k^{3i+3} - k^{3i+1} + 15\ell \cdot k^{3i}\right)}{15 \cdot k^{3(i+1)}} = \frac{2}{5} \cdot \left(1 + \frac{15\ell}{k^3 - k}\right).$$

However, any convergent subsequence of $(G_i)_{i \in \mathbb{N}}$ converges to a homomorphism with the density of rainbow triangles equal to $\frac{2}{5} \cdot \left(1 + \frac{15\ell}{k^3 - k}\right) > \frac{2}{5}$, which contradicts Theorem 6. $\qquad\square$

The iterated blow-up of RB1111 shows that for n being a power of 4, the bound $(n^3 - n)/15$ is best possible. In this case, we show that the iterated blow-up of RB1111 is actually a unique extremal construction.

Proof of Theorem 5. Denote by R^ℓ the $(\ell - 1)$-times iterated blow-up of RB1111, so R^ℓ has 4^ℓ vertices. Theorem 5 is easily seen to be true for $k = 1$, so suppose for a contradiction that there is a graph G on $n = 4^k$ vertices with $F(G) = F(n) = (n^3 - n)/15$ that is not isomorphic to R^k for a minimal $k \geq 2$.

If G has the structure described in Theorem 4, then G is isomorphic to R^k by the minimality of k, a contradiction. Therefore, $V(G)$ cannot be partitioned into four parts X_1, X_2, X_3, X_4 with $|X_i| = 4^{k-1}$ as described in Theorem 4.

Fix an integer ℓ such that $4^\ell > n_0$, where n_0 is taken from the statement of Theorem 4. Let $\overline{G}$ be the graph obtained by blowing up every vertex of G by a factor of 4^ℓ, and inserting R^ℓ in every part. It follows that $\overline{G}$ has $4^{k+\ell}$ vertices, and

$$F(\overline{G}) = n \cdot F\left(R^\ell\right) + F(G) \cdot 4^{3\ell} = \frac{n \cdot 4^{3\ell} - n \cdot 4^\ell + n^3 \cdot 4^{3\ell} - n \cdot 4^{3\ell}}{15} = \frac{4^{3(k+\ell)} - 4^{k+\ell}}{15}.$$

So $\overline{G}$ must be extremal. However, Theorem 4 implies that $\overline{G}$ can be partitioned into four parts $\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4$ with $|\overline{X}_i| = 4^{k+\ell-1}$ as described in Theorem 4. Since any two vertices from $V(\overline{G})$ that arise from blowing up the same vertex of G need to be in the same part, the partition $\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4$ provides also a partition of the vertices of G. But this is a partition of G into four parts of the same size as described in Theorem 4, a contradiction. $\qquad\square$

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References

[1] R. Baber. Turán densities of hypercubes. Submitted.

[2] R. Baber and J. Talbot. Hypergraphs do jump. *Combin. Probab. Comput.*, 20(2):161–171, 2011.

[3] R. Baber and J. Talbot. A Solution to the 2/3 Conjecture. *SIAM J. Discrete Math.*, 28(2):756–766, 2014.

[4] J. Balogh, M. Delcourt, B. Lidický, and C. Palmer. Rainbow copies of C_4 in edge-colored hypercubes. Submitted.

[5] J. Balogh, P. Hu, B. Lidický, and H. Liu. Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube. *European Journal of Combinatorics*, 35:75–85, 2014.

[6] J. Balogh, P. Hu, B. Lidický, O. Pikhurko, B. Udvari, and J. Volec. Minimum number of monotone subsequences of length 4 in permutations. *To appear in Combinatorics, Probability and Computing.*

[7] J. Cummings, D. Kráľ, F. Pfender, K. Sperfeld, A. Treglown, and M. Young. Monochromatic triangles in three-coloured graphs. *J. Combin. Theory Ser. B*, 103(4):489–503, 2013.

[8] P. Erdős and A. Hajnal. On Ramsey like theorems. Problems and results. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pages 123–140. Inst. Math. Appl., Southend, 1972.

[9] V. Falgas-Ravry and E. R. Vaughan. Turán h-densities for 3-graphs. *Electron. J. Combin.*, 19:R40, 2012.

[10] E. Gawrilow and M. Joswig. Polymake: a framework for analyzing convex polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.

[11] R. Glebov, D. Kráľ, and J. Volec. A problem of Erdős and Sós on 3-graphs. Submitted.

[12] A. Grzesik. On the maximum number of five-cycles in a triangle-free graph. *J. Combin. Theory Ser. B*, 102(5):1061–1066, 2012.

[13] H. Hatami, J. Hladký, D. Kráľ, S. Norine, and A. Razborov. On the number of pentagons in triangle-free graphs. *J. Combin. Theory Ser. A*, 120(3):722–732, 2013.

[14] J. Hladký, D. Kráľ, and S. Norine. Counting flags in triangle-free digraphs. In *European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2009)*, volume 34 of *Electron. Notes Discrete Math.*, pages 621–625. Elsevier Sci. B. V., Amsterdam, 2009.

[15] H. Huang. On the maximum induced density of directed stars and related problems. *To appear in SIAM J. on Discrete Math.*

[16] D. Kráľ, L. Mach, and J.-S. Sereni. A new lower bound based on Gromov's method of selecting heavily covered points. *Discrete Comput. Geom.*, 48(2):487–498, 2012.

[17] J. Nešetřil and V. Rödl, editors. *Mathematics of Ramsey theory*, volume 5 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 1990.

[18] V. Nikiforov. The number of cliques in graphs of given order and size. *Trans. Amer. Math. Soc.*, 363(3):1599–1618, 2011.

[19] O. Pikhurko and E. R. Vaughan. Minimum number of k-cliques in graphs with bounded independence number. *To appear in Combinatorics, Probability and Computing.*

[20] A. A. Razborov. Flag algebras. *Journal of Symbolic Logic*, 72(4):1239–1282, 2007.

[21] A. A. Razborov. On the minimal density of triangles in graphs. *Combin. Probab. Comput.*, 17(4):603–618, 2008.

[22] A. A. Razborov. Flag Algebras: An Interim Report. In *The Mathematics of Paul Erdős II*, pages 207–232. Springer, 2013.

[23] A. A. Razborov. On the Caccetta-Häggkvist conjecture with forbidden subgraphs. *J. Graph Theory*, 74(2):236–248, 2013.

[24] C. Reiher. The clique density theorem, 2012.

[25] K. Sperfeld. The inducibility of small oriented graphs. Submitted.

A Giving bounds on the x_i

Here we show how to prove (14) – (20). Suppose we want to derive the upper bound from (15). It means solving the following program:

$$(P) \begin{cases} \text{maximize} & x_1 \\ \text{subject to} & 2\sum_{1 \le i < j \le 4} x_i x_j - 2f - \frac{26}{9}\sum_{1 \le i \le 4} x_i^2 + \frac{27}{1000} x_0 > 0.02760856, \\ & x_1 + x_2 + x_3 + x_4 + x_0 = 1, \\ & x_i \ge 0 \text{ for } i \in \{0, \ldots, 4\}, \\ & f \ge 0. \end{cases}$$

As a quick check, it can be written to a heuristic online solver APMonitor. We provide the source of the program in file `APM.xi.txt`. However, this method may get stuck in local optima, so it does not provide a proof of global maximization.

A rigorous way is to use the method of Lagrange Multipliers. Since we need to solve several of the programs, we implemented the method in Sage. We provide a commented code in file `solve-xi.py`.

B   The computation in Claim 12

Recall that we want to solve the following program

$$(P) \begin{cases} \text{maximize} & (x_1 - 0.033 + \frac{0.033}{4})0.033 + 3x_2^2 + f + x_0 \\ \text{subject to} & 0.02760856 < 6x_1x_2 - \frac{8}{3}x_2^2 - \frac{26}{9}x_1^2 + 0.027x_0 - 2f, \\ & x_1 + 3x_2 + x_0 = 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & x_0 \geq 0, \\ & f \geq 0. \end{cases}$$

We give a solution using Lagrange multipliers. We also implemented a script in Sage performing the computation. The script is in file `solve-claim12.py`.

First observe that if $x_1 = 0$ or $x_2 = 0$, then the program is not feasible. Hence $x_1 > 0$ and $x_2 > 0$. We are left with inequalities $x_0 \geq 0$ and $f \geq 0$, which may be tight. Moreover, we always use $x_1 + 3x_2 + x_0 = 1$ for substitution. To solve this, we divide the analysis in four cases, and use Lagrange multipliers again:

Case 1: If $f = 0$ and $x_0 = 0$, this comes down to solving

$$(P) \begin{cases} \text{maximize} & 0.033x_1 + \frac{1}{3}(1 - x_1)^2 - \frac{3}{4}(0.033)^2 \\ \text{subject to} & 0.02760856 < 2x_1(1 - x_1) - \frac{8}{27}(1 - x_1)^2 - \frac{26}{9}x_1^2. \end{cases}$$

The constraint can be simplified to $0.02760856 < -\frac{2}{27}(4 - 35x_1 + 70x_1^2)$. This quadratic program in one variable has the optimal solution $x_1 \approx 0.24424$, and so $\frac{1}{2}\text{RBT}^v < 0.1985$.

Case 2: If $f = 0$ and $x_0 > 0$, it comes down to solving

$$(P) \begin{cases} \text{maximize} & 0.033x_1 + 3x_2^2 - x_1 - 3x_2 + 1 - \frac{3}{4}(0.033)^2 \\ \text{subject to} & 0.02760856 < 6x_1x_2 - \frac{8}{3}x_2^2 - \frac{26}{9}x_1^2 + 0.027(1 - x_1 - 3x_2), \\ & 0.24 \leq x_1 \leq 0.26, \\ & 0.24 \leq x_2 \leq 0.26. \end{cases}$$

Taking gradients, we get

$$\begin{pmatrix} -0.967 \\ -3 + 6x_2 \end{pmatrix} = \lambda \begin{pmatrix} -\frac{52}{9}x_1 + 6x_2 - 0.027 \\ 6x_1 - \frac{16}{3}x_2 - 0.081 \end{pmatrix},$$

which gives $x_1 \approx 0.24662$, $x_2 \approx 0.24936$, and $\frac{1}{2}\text{RBT}^v < 0.19991$ as the only feasible solution.

Case 3: If $f > 0$ and $x_0 = 0$, it comes down to solving

$$(P) \begin{cases} \text{maximize} & 0.033x_1 + \frac{1}{3}(1 - x_1)^2 + f - \frac{3}{4}(0.033)^2 \\ \text{subject to} & 0.02760856 < 2x_1(1 - x_1) - \frac{8}{27}(1 - x_1)^2 - \frac{26}{9}x_1^2 - 2f. \end{cases}$$

The constraint can be simplified to $0.02760856 < -\frac{2}{27}(4 - 35x_1 + 70x_1^2) - 2f$. Taking gradients, we get

$$\begin{pmatrix} 0.033 - \frac{2}{3} + 2x_1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} \frac{70}{27} - \frac{280}{27}x_1 \\ -2 \end{pmatrix},$$

whose solution $x_1 \approx 0.20803$ together with the constraint implies $f < 0$, a contradiction.

Case 4: If $f > 0$ and $x_0 > 0$, it comes down to solving

$$(P) \begin{cases} \text{maximize} & 0.033x_1 + 3x_2^2 + 1 - x_1 - 3x_2 + f \\ \text{subject to} & 0.02760856 < 6x_1 x_2 - \frac{8}{3}x_2^2 - \frac{26}{9}x_1^2 + 0.027(1 - x_1 - 3x_2) - 2f. \end{cases}$$

Taking gradients, we get

$$\begin{pmatrix} 0.967 \\ -3 + 6x_2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} -\frac{52}{9}x_1 + 6x_2 - 0.027 \\ 6x_1 - \frac{16}{3}x_2 - 0.081 \\ -2 \end{pmatrix}.$$

Similarly to the previous case, we again have $f < 0$, a contradiction.

C   The computation in Claim 19

The term we want to maximize does not include anything from X_0, so we can assume that $x_0 = 0$. Since $r_1 + g_1 + b_1 = x_1$, we can use bounds involving $x_1, \ldots, x_4$. First, we use $x_1 + x_2 + x_3 + x_4 = 1$. Then we use the lower bounds for (15) on all x_i. We also use the four bounds implied by Claim 18 (since there are four options where to put x). Finally, we add the bounds $r_i, g_i, b_i \geq 0$. So we solve the following program:

$$(P) = \begin{cases} \text{maximize} & r_1 g_1 + r_1 b_1 + g_1 b_1 + r_2 g_2 + r_2 b_2 + g_2 b_2 \\ & +r_3 g_3 + r_3 b_3 + g_3 b_3 + r_4 g_4 + r_4 b_4 + g_4 b_4 \\ & +r_1(b_2 + g_4) + b_2 g_4 + g_1(b_3 + r_4) + b_3 r_4 \\ & +b_1(r_2 + g_3) + r_2 g_3 + g_2(r_3 + b_4) + r_3 b_4 \\ \text{subject to} & \sum_{i=1}^{4} r_i + g_i + b_i = 1, \\ & r_i + g_i + b_i \geq 0.244287 \text{ for } i \in \{1, 2, 3, 4\}, \\ & r_2 + b_2 + g_3 + b_3 + r_4 + g_4 \geq 0.12866, \\ & r_1 + b_1 + r_3 + g_3 + g_4 + b_4 \geq 0.12866, \\ & g_1 + b_1 + r_2 + g_2 + r_4 + b_4 \geq 0.12866, \\ & r_1 + g_1 + g_2 + b_2 + r_3 + b_3 \geq 0.12866, \\ & r_i, g_i, b_i \geq 0 \text{ for } i \in \{1, 2, 3, 4\}. \end{cases}$$

The optimal solution to the program has value less than 0.1945 and it is achieved at $r_1 \approx 0.03854, g_1 \approx 0.16720, b_1 \approx 0.03854, r_2 = 0, g_2 \approx 0.24670, b_2 = 0, r_3 \approx 0.19243, g_3 = 0, b_3 \approx 0.06658, r_4 \approx 0.06208, g_4 = 0, b_4 \approx 0.18792$.

For each of the bounds, we consider the two cases that the bound is active (i.e. tight) or inactive, giving us a total of 2^{20} cases. In each of the cases, we have to solve a system of linear equations with up to 12 variables, and check the solution for feasibility. Obviously, this is done by a computer using rational arithmetic. We wrote a program in Sage which performs the computation. We reduce the number of programs to solve by eliminating the cases where some sets of constraints cannot be tight at the same time. For example, it is not possible that $r_1 = g_1 = b_1 = 0$ at the same time. Note that feasible solutions with dimension greater than zero will occur again as lower dimensional solutions in cases with more active bounds, so we only have to analyze discrete solutions. We could

use symmetries, and we could analyze the feasibility polytope closer to only check the faces which actually appear (the program Polymake [10] can yield this output), reducing the number of cases to check to a few thousand. But we decided to use this brute-force analysis, as this makes it easier to check the code, and the running time is still very reasonable.

The code performing the computation as well as the outputs can be downloaded at `http://www.math.uiuc.edu/~jobal/cikk/rbt`.