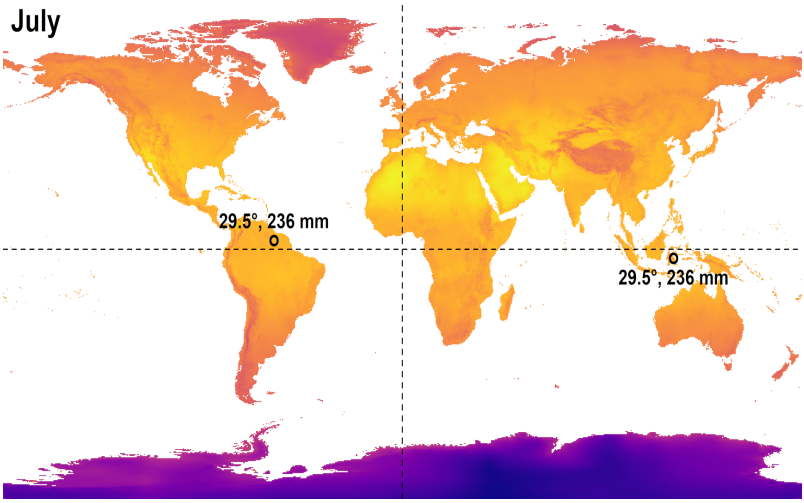
Topological methods in combinatorics

Martin Balko

3rd lecture

March 9th 2026

The Borsuk–Ulam Theorem



Source: https://scientificgems.wordpress.com/

The Borsuk–Ulam Theorem: history

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- The first proof was given by Karol Borsuk (1933), where the formulation of the problem was attributed to Stanislaw Ulam.

Figure: Karol Borsuk (1905–1982) a Stanislaw Ulam (1909–1984).

Sources: https://www.komputerswiat.pl/ and https://en.wikipedia.org

The Borsuk–Ulam Theorem: history

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- First mentioned by Lyusternik and Schnirelmann (1930).

Figure: Lazar Lyusternik (1899–1981) a Lev Schnirelmann (1909–1984).

Source: https://en.wikipedia.org

The Borsuk–Ulam Theorem: the original statement

Der Zweck dieser Arbeit ist, folgende drei Sätze zu beweisen:

Satz I [6]. *Jede antipodentreue Abbildung von S_n ist wesentlich.*

Satz II [7]. *Ist $f \in R^{n S_n}$ (d. h. bildet f die Sphäre S_n auf einen Teil von R^n ab), so gibt es einen derartigen Punkt $p \in S_n$, dass $f(p) = -f(p^*)$ ist.*

Satz III. *Sind $A_0, A_1, \ldots, A_n$ in sich kompakte Mengen von denen keine zwei antipodische Punkte der Sphäre S_n enthält, so enthält die Summe $\sum_{i=0}^{n} A_i$ die Sphäre S_n nicht.*

Source: Matoušek: Using the Borsuk–Ulam Theorem

The Borsuk–Ulam Theorem

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The Borsuk–Ulam Theorem

For every $n \geq 0$, the following statements are equivalent, and true:

(BU1a) For every $f : S^n \to \mathbb{R}^n$ there is $x \in S^n$ with $f(x) = f(-x)$.

(BU1b) For every antipodal $f : S^n \to \mathbb{R}^n$ there is $x \in S^n$ with $f(x) = 0$.

(BU2a) There is no antipodal $f : S^n \to S^{n-1}$.

(BU2b) There is no $f : B^n \to S^{n-1}$ that is antipodal on $\partial B^n = S^{n-1}$.

(LS-c) For any closed cover $F_1, \ldots, F_{n+1}$ of S^n, there is $i \in [n+1]$ and $x \in S^n$ with $x, -x \in F_i$.

(LS-o) For any open cover $U_1, \ldots, U_{n+1}$ of S^n, there is $i \in [n+1]$ and $x \in S^n$ with $x, -x \in U_i$.

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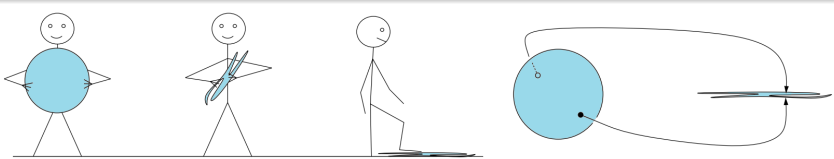

Source: Matoušek: Using the Borsuk–Ulam Theorem (colored)

The Borsuk–Ulam Theorem: another intepretation

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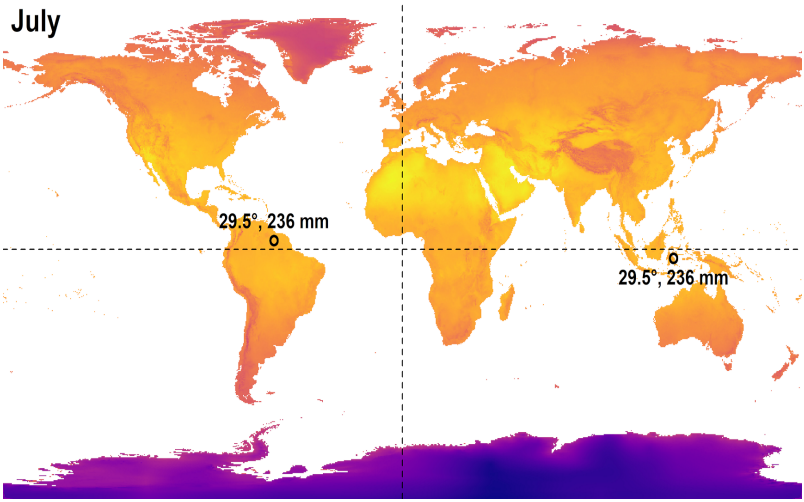

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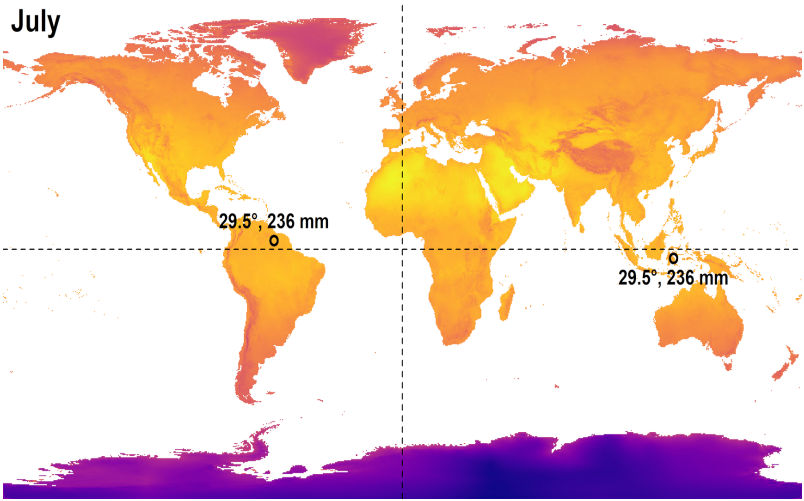

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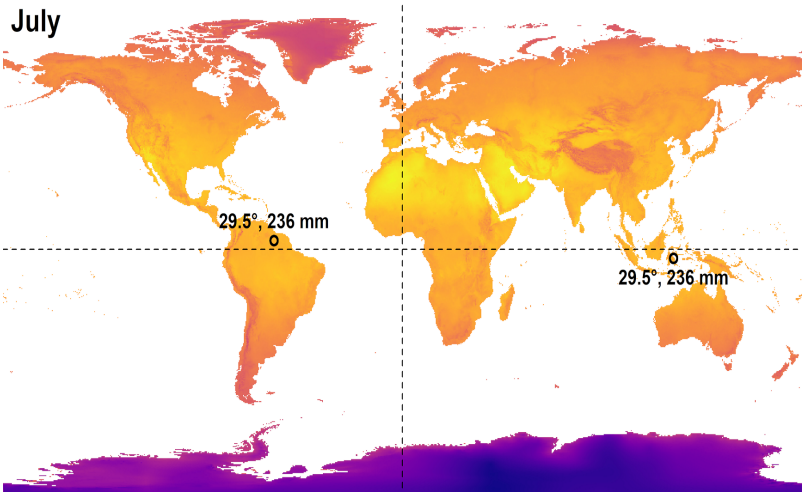

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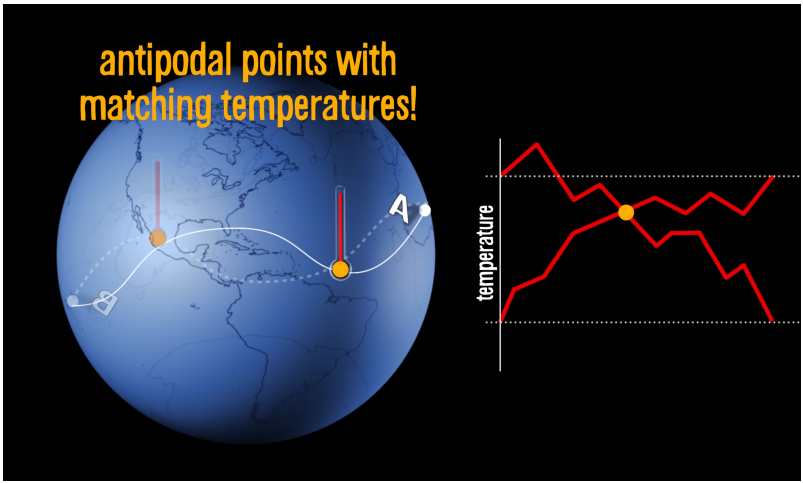

Source: Fixed poits (Vsauce)

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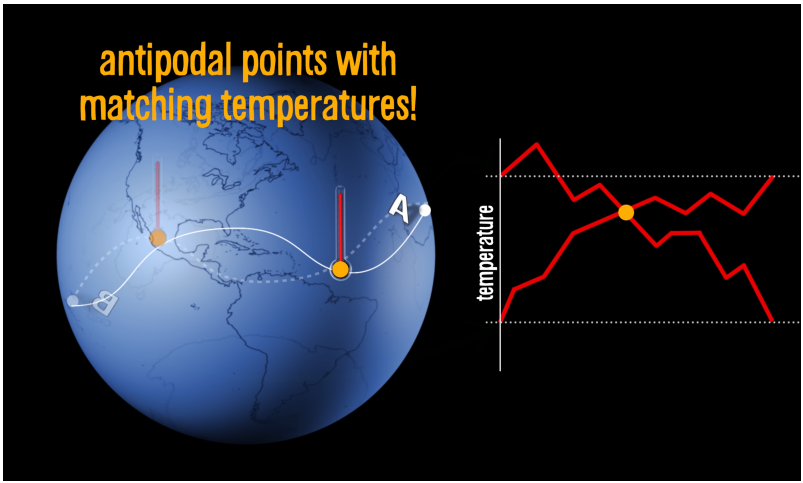

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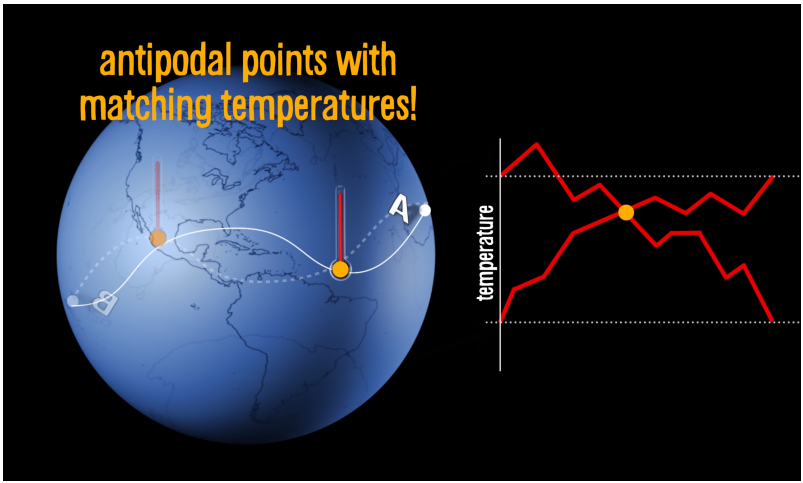

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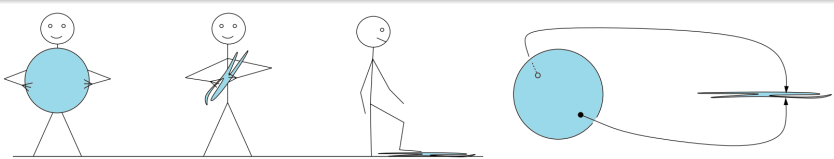

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For each $n \in \mathbb{N}$ and every continuous $f : B^n \to B^n$, there exists a fixed point $x \in B^n$ for f, that is, $f(x) = x$.

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Figure: L. E. J. Brouwer (1881–1966).

Source: https://arxiv.org/pdf/1612.06820.pdf

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- A simple corollary of the The Borsuk–Ulam Theorem.


(BU1a)
(BU1b)
(BU2a)
(BU2b)
(LS-c)
(LS-o)

Thank you for your attention.