

Topological methods in combinatorics

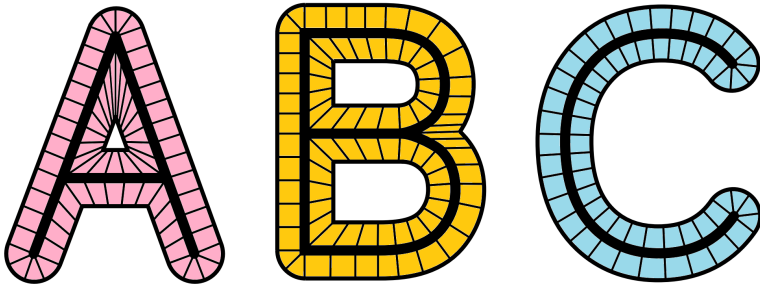
Martin Balko

2nd lecture

March 2nd 2026



Homotopy



Source: Hatcher: Algebraic topology (colored)

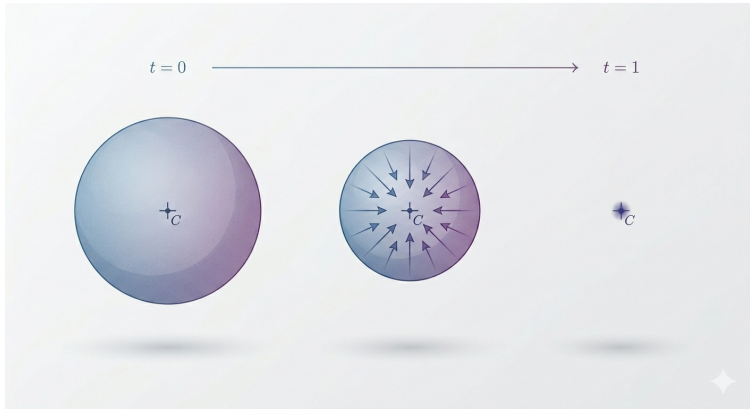
Homotopy: example

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- The ball B^3 is homotopy equivalent to the origin C .

Homotopy: example

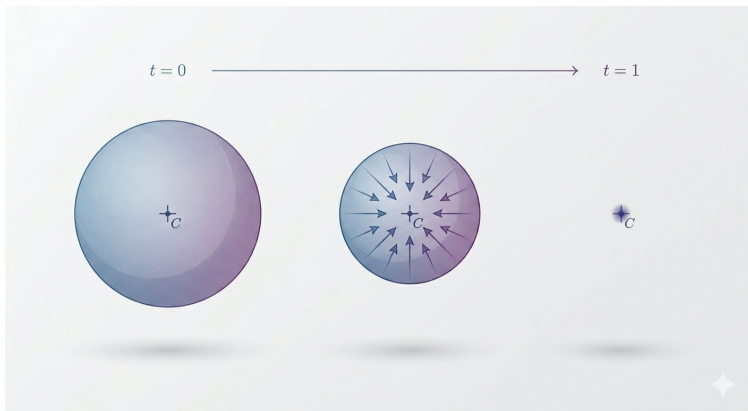
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Source: Gemini.

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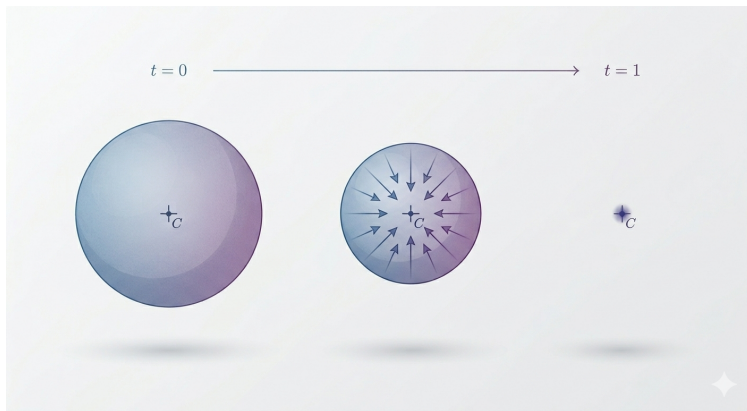


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Consider $f: B^3 \rightarrow \{C\}$ and $g: \{C\} \rightarrow B^3$ given by $f(x) = C$ and $g(C) = C$.

Homotopy: example

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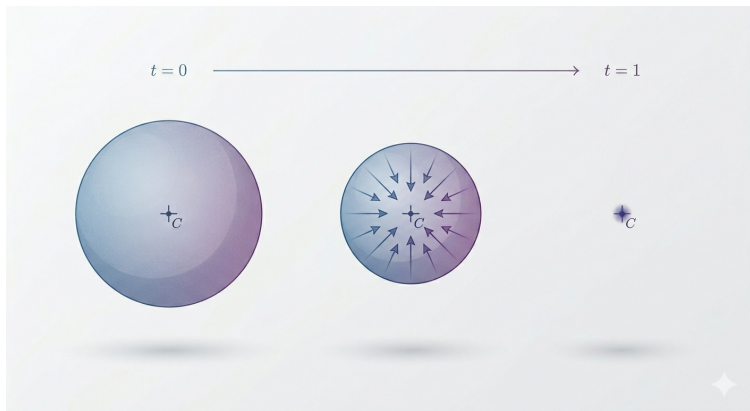


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Consider $f: B^3 \rightarrow \{C\}$ and $g: \{C\} \rightarrow B^3$ given by $f(x) = C$ and $g(C) = C$. Then, $f \circ g = id_{\{C\}}$

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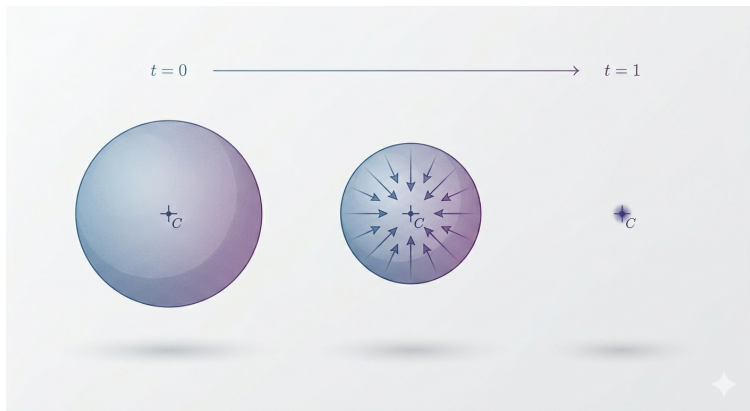


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Consider $f: B^3 \rightarrow \{C\}$ and $g: \{C\} \rightarrow B^3$ given by $f(x) = C$ and $g(C) = C$. Then, $f \circ g = id_{\{C\}}$ and $g \circ f \simeq id_{B^3}$ via $H: B^3 \times [0, 1] \rightarrow B^3$ given by $H(x, t) = tC + (1 - t)x$ for every $t \in [0, 1]$.

Homotopy: example

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- That is, B^3 is **contractible**.

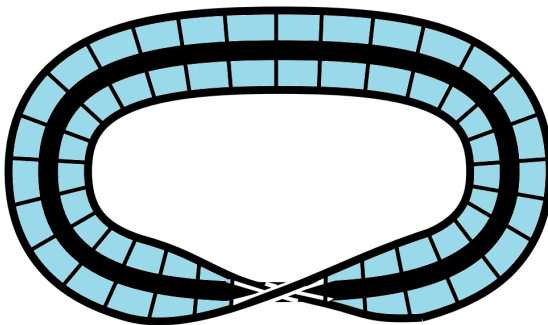
Homotopy: another example

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- The Möbius band M is homotopy equivalent to a circle S^1 .

Homotopy: another example

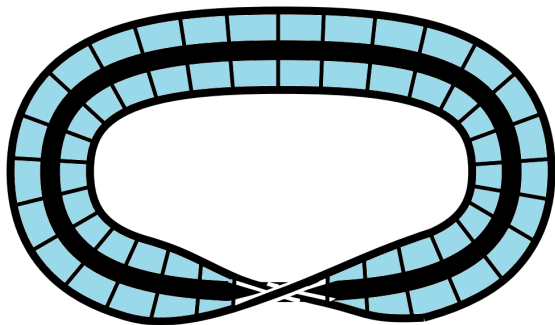
- The Möbius band M is **homotopy equivalent** to a circle S^1 . However, they are **not homeomorphic**.



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Homotopy: another example

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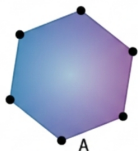
Source: Hatcher: Algebraic topology (colored)

- Formally, $M = [0, 1] \times [0, 1] / \sim$, where $(0, y) \sim (1, 1 - y)$, is homotopy equivalent to $S^1 = [0, 1] \times \{\frac{1}{2}\} / \sim$ via $H: M \times [0, 1] \rightarrow M$ given by $H((x, y), t) = (x, (1 - t)y + t\frac{1}{2})$ for every $t \in [0, 1]$.

Contractible spaces: examples

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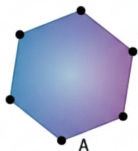
- Which of the following spaces are **contractible**?



Source: <https://en.wikipedia.org>

Contractible spaces: examples

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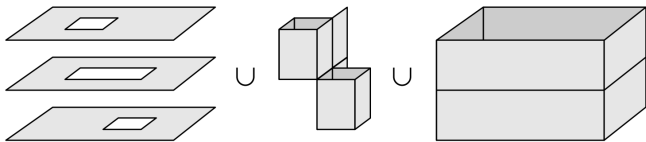
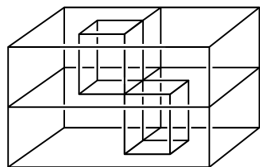


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- A, B, C** are, while **D, E, F** are not.

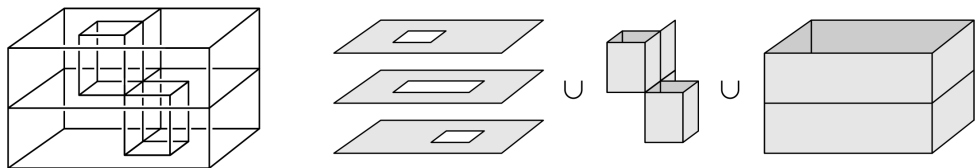
Contractible spaces: weird example

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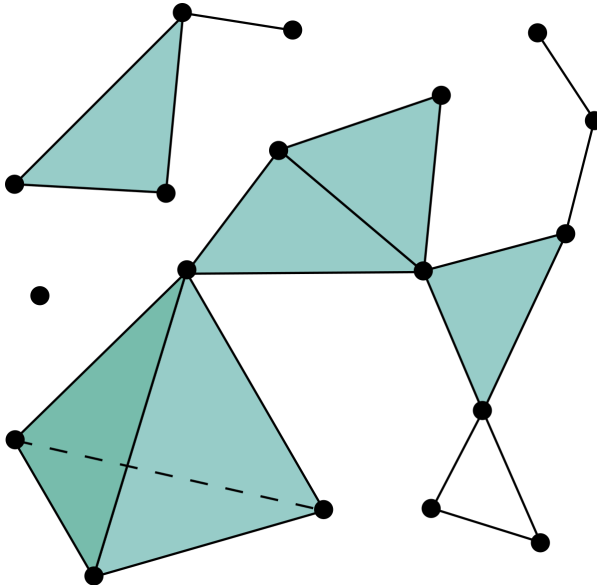
Contractible spaces: weird example



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- **Bing's house** is a contractible space (not so easy to see, right?).

Simplicial complexes

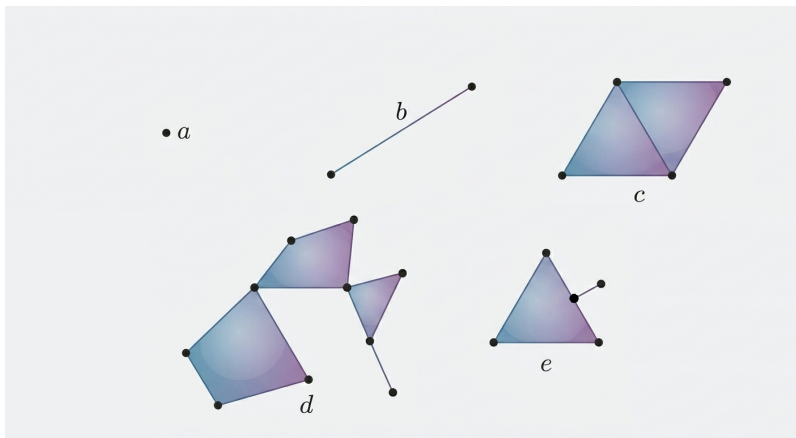


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Simplicial complexes: examples

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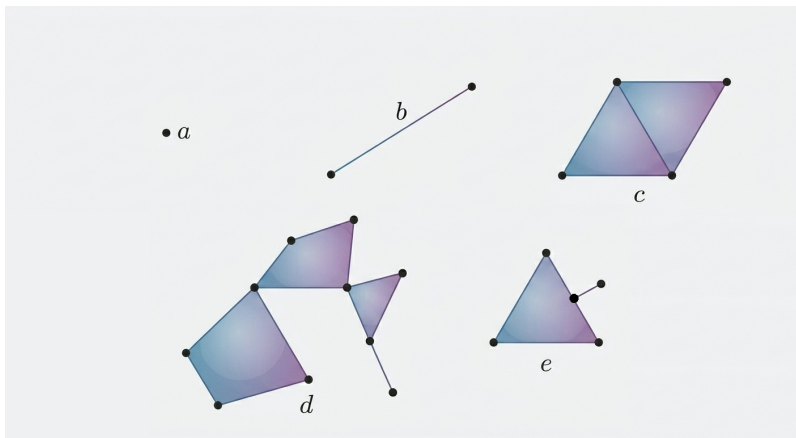
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Simplicial complexes: examples

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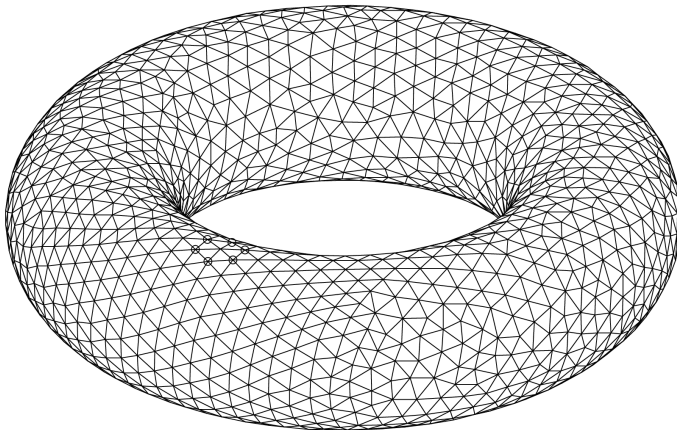


Source: Gemini

- A**, **B**, **C**, are, while **D** and **E** are not.

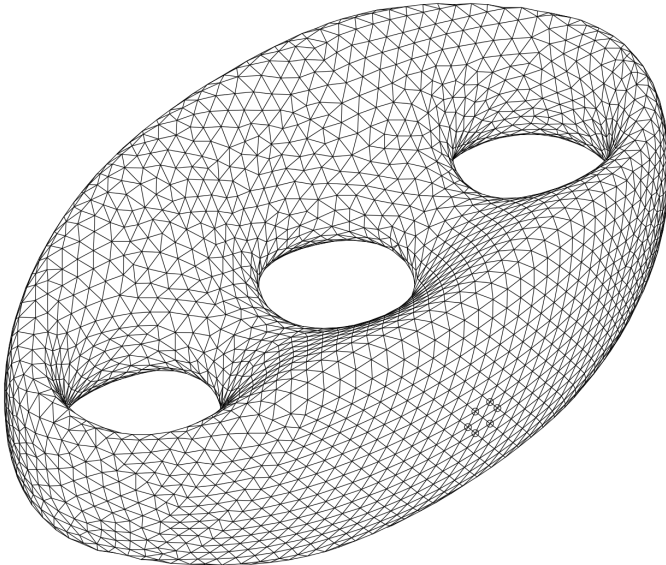
Triangulation: examples

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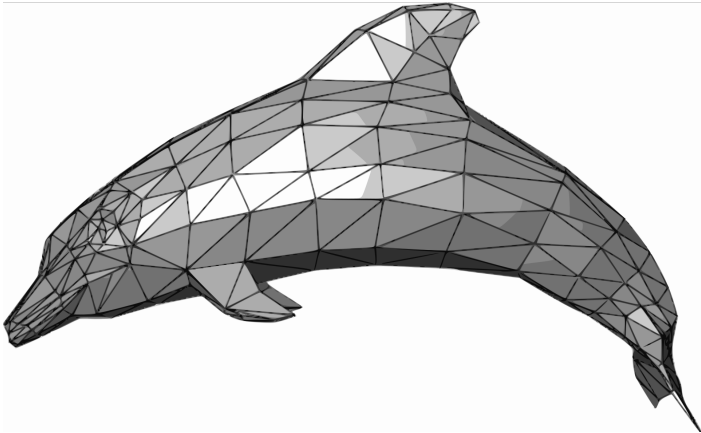
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Triangulation: examples



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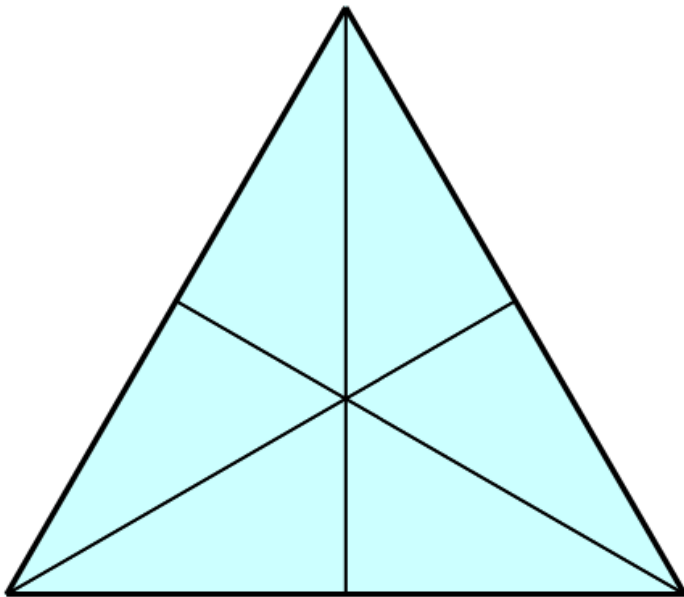
Triangulation: examples



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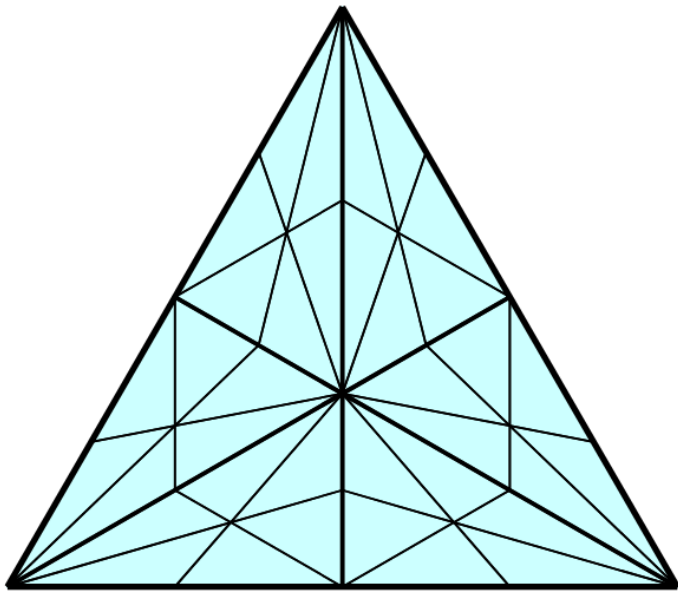
Barycentric subdivision: example

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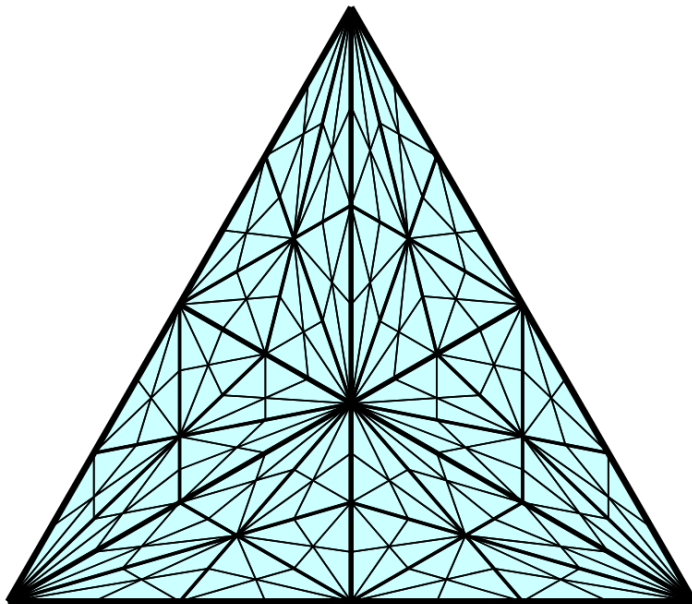
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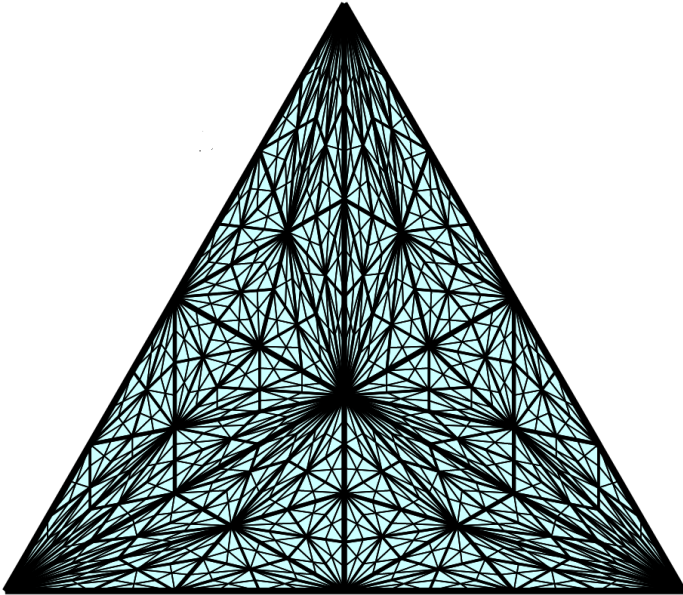
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Thank you for your attention.