BORSUK-ULAM THEOREM:

- OUR MAIN TOOL FROM TOPOLOGY
- IT IS A FIXED POINT THEOREM
- MANY EQUIVALENT STATEMENTS
- RECALL - $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$
- **THEOREM 3.1** (THE BORSUK-ULAM THEOREM): FOR EVERY INTEGER $n \geq 0$, THE FOLLOWING 6 STATEMENTS ARE TRUE AND EQUIVALENT:

(BU1a) $\forall \overset{\text{continuous}}{f} : S^n \to \mathbb{R}^n \; \exists x \in S^n : f(x) = f(-x)$

(BU1b) $\forall$ ANTIPODAL $f : S^n \to \mathbb{R}^n \; \exists x \in S^n : f(x) = 0$

↳ $\forall x \in S^n : f(-x) = -f(x)$ + CONTINUOUS

(BU2a) $\nexists$ ANTIPODAL $f : S^n \to S^{n-1}$

(BU2b) $\nexists$ CONTINUOUS $f : B^n \to S^{n-1}$ THAT IS ANTIPODAL ON $\partial B^n = S^{n-1}$

LYUSTERNIK-SCHNIRELMAN ←

(LS-c) $\forall$ CLOSED COVERING $F_1, \dots, F_{n+1}$ OF S^n

$\exists i : F_i \cap (-F_i) \neq \emptyset$

↳ $= \{-x : x \in F_i\}$

(LS-o) $\forall$ OPEN COVERING $U_1, \dots, U_{n+1}$ OF S^n

$\exists i : U_i \cap (-U_i) \neq \emptyset$

- WITHOUT PROOF (COMPLICATED)
- WE PROVE THE EQUIVALENCES (10 IN TOTAL)
- PROOF PLAN:

$$\begin{tikzcd}
 & (BU1a) \arrow[dl, Rightarrow, "7"'] \arrow[dr, Rightarrow] & \\
(LS\text{-}c) \arrow[d, Rightarrow, shift right, "9"'] \arrow[drr, Rightarrow, "8"] & & (BU1b) \arrow[ul, Rightarrow, shift left, "2"] \arrow[d, Rightarrow, shift left, "3"] \\
(LS\text{-}o) \arrow[u, Rightarrow, shift right, "10"'] & & (BU2a) \arrow[u, Rightarrow, shift left, "4"] \arrow[dl, Rightarrow, shift left, "5"] \\
 & (BU2b) \arrow[ur, Rightarrow, shift left, "6"] &
\end{tikzcd}$$

PROOF OF THEOREM 3.1:

1) (BU1a) $\Rightarrow$ (BU1b):

- CLEAR: (BU1a) $\Rightarrow$ $\forall f: S^n \to \mathbb{R}^n \ \exists x \in S^n: f(x) = f(-x)$

  IF f IS ANTIPODAL THEN $f(x) = f(-x) = -f(x) \Rightarrow \underline{f(x) = 0}$

2) (BU1b) $\Rightarrow$ (BU1a):

- GIVEN $f: S^n \to \mathbb{R}^n$, DEFINE $g: S^n \to \mathbb{R}^n$ BY $g(x) = f(x) - f(-x)$
- THEN $g(-x) = f(-x) - f(x) = -g(x) \Rightarrow g$ IS ANTIPODAL
- (BU1b) $\Rightarrow \exists x \in S^n: g(x) = 0$

  $\Rightarrow 0 = g(x) = f(x) - f(-x) \Rightarrow \underline{f(-x) = -f(x)}$

3) (BU1b) $\Rightarrow$ (BU2a):

- SUPPOSE FOR CONTRADICTION THAT $f: S^n \to S^{n-1}$ IS ANTIPODAL
- THEN, f IS ANTIPODAL MAPPING FROM $S^n \to \mathbb{R}^n \supseteq S^{n-1}$
- SINCE $0 \notin S^{n-1}$, $f(x) \neq 0$ FOR $\forall x \in S^n$ $\to$ THIS CONTRADICTS (BU1b) ↯

4) (BU2a) $\Rightarrow$ (BU1b):

- SUPPOSE FOR CONTRADICTION THAT $\exists$ ANTIPODAL $f: S^n \to \mathbb{R}^n$ WITH $f(x) \neq 0$ $\forall x \in S^n$
- DEFINE $g: S^n \to S^{n-1}$ BY $g(x) = \frac{f(x)}{\underbrace{\|f(x)\|}_{\neq 0}}$
- THEN $g: S^n \to S^{n-1}$ IS ANTIPODAL $\to$ THIS CONTRADICTS (BU2a) ↯

  BOOK

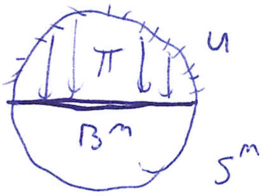
5) (BU2a) $\Rightarrow$ (BU2b):

- SUPPOSE FOR CONTRADICTION THAT $g: B^n \to S^{n-1}$ IS ANTIPODAL ON S^{n-1}
- DEFINE $f(x) = g(\pi(x))$ FOR $\forall x \in U$ = UPPER HEMISPHERE OF S^n

  $f(-x) = -g(\pi(x))$ WHERE $\pi: U \to B^n$

  $\pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$

  ↳ HOMEOMORPHISM



CONSISTENT AS g IS ANTIPODAL ON $S^{n-1} = \partial B^n$

AND $\pi(-x) = -\pi(x)$ FOR $x \in \partial B^n = S^{n-1}$

$\Rightarrow f$ IS CONTINUOUS ANTIPODAL FROM $S^n \to S^{n-1}$

↓ EXERCISE

$\to$ THIS CONTRADICTS (BU2a) ↯

6) (BU2b) ⇒ (BU2a):

- SUPPOSE FOR CONTRADICTION THAT $f: S^n \to S^{n-1}$ IS ANTIPODAL
- THEN, $g: B^n \to S^{n-1}$ GIVEN BY $g(x) = f(\pi^{-1}(x))$ IS ANTIPODAL ON S^{n-1} → THIS CONTRADICTS (BU2b) ↯

  $\pi^{-1}(x) = -\pi^{-1}(-x)$ FOR $x \in \partial B^n = S^{n-1}$

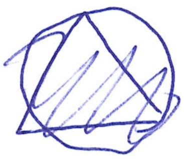
7) (BU1a) ⇒ (LS-c):

- LET $F_1,\dots,F_{n+1}$ BE A CLOSED COVER OF S^n
- DEFINE $f: S^n \to \mathbb{R}^n$ BY $f(x) = (\mathrm{dist}(x,F_1),\dots,\mathrm{dist}(x,F_n))$
- (BU1a) ⇒ $\exists\, x \in S^n: f(x) = f(-x) =: y$
- IF $\exists\, i \in [n]: y_i = 0 \Rightarrow f(x), f(-x) \in F_i$
- OTHERWISE $f(x), f(-x) \in F_{n+1}$

8) (LS-c) ⇒ (BU2a):

- AUXILIARY RESULT: ∃ CLOSED COVER $F_1,\dots,F_{n+1}$ OF S^{n-1} WITH $F_i \cap (-F_i) = \emptyset$ FOR $\forall\, i \in [n+1]$

  ↑ 1 DIMENSION LESS



(F_i = ~~...~~ PROJECTION OF i-TH FACET OF Δ_n)

- SUPPOSE FOR CONTRADICTION THAT ∃ ANTIPODAL $f: S^n \to S^{n-1}$
- THEN $f^{-1}(F_1),\dots,f^{-1}(F_{n+1})$ CONTRADICTS (LS-c) ↯

  $x, -x \in f^{-1}(F_i) \Rightarrow f(x), \underbrace{f(-x)}_{= -f(x)} \in F_i$

9) (LS-c) ⇒ (LS-o):

- LET $U_1,\dots,U_{n+1}$ BE OPEN COVER OF S^n
- $\forall\, x \in S^n$: SET V_x := OPEN NEIGHBORHOOD OF x S.T. $\mathrm{cl}\,V_x \subseteq U_i$ FOR SOME i
- S^n IS COMPACT ⇒ ∃ FINITE SUBCOVER $V_1,\dots,V_m$ OF $(V_x)_{x \in S^n}$
- SET $F_i = \bigcup_{j:\, V_j = V_x \text{ WITH } \mathrm{cl}\,V_x \subseteq U_i} \mathrm{cl}\,V_j$ ⇒ $F_1,\dots,F_{n+1}$ IS CLOSED COVER OF S^n WITH $F_i \subseteq U_i$ FOR $\forall\, i \in [n+1]$
- (LS-c) ⇒ $\exists\, i: F_i \cap (-F_i) \neq \emptyset \Rightarrow U_i \cap (-U_i) \neq \emptyset$

10) $(LS\text{-}a) \Rightarrow (LS\text{-}c)$: → EASIER ARGUMENT THAN IN THE BOOK

- SUPPOSE FOR CONTRADICTION THAT $\exists F_1,\dots,F_{n+1}$ CLOSED COVER OF S^n WITH $F_i \cap (-F_i) = \emptyset \ \forall i \in [n+1]$
- $\varepsilon = \min_{i=1,\dots,n+1} \text{DIST}(F_i, -F_i) \Rightarrow \varepsilon > 0$
- $\forall i \in [n+1]$, LET $U_i = \{x \in S^n : \text{DIST}(x, F_i) \leq \frac{\varepsilon}{4}\}$ ($F_i \cap (-F_i) = \emptyset \Rightarrow U_i \cap (-U_i) = \emptyset$)

$\Rightarrow U_1,\dots,U_{n+1}$ = OPEN COVER OF S^n WITH $U_i \cap (-U_i) = \emptyset \ \forall i \in [n+1]$

~~(...)~~ → THIS CONTRADICTS $(LS\text{-}c)$ ↯

☒

- FIRST APPLICATION → THE BROUWER FIXED POINT THEOREM
- THEOREM 3.2 (BROUWER'S FIXED POINT THEOREM):

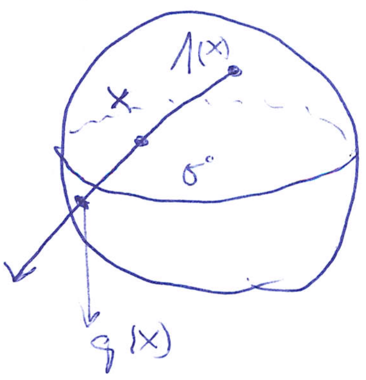
  $\forall$ CONTINUOUS $f : B^n \to B^n \ \exists\, x \in B^n : f(x) = x$

  ($\Rightarrow \exists$ NASH EQUILIBRIA IN ANY FINITE NORMAL-FORM GAME)

  ($\Rightarrow$  → INTERSECTION)

- PROOF OF THEOREM 3.2:
  - LET $f : B^n \to B^n$ BE CONTINUOUS
  - SUPPOSE FOR CONTRADICTION: $f(x) \neq x \ \forall x \in B^n$
  - DEFINE $g : B^n \to S^{n-1}$ BY $g(x)$ = INTERSECTION OF $S^{n-1} = \partial B^n$ WITH THE RAY $\overrightarrow{f(x)\,x}$

B^n



($\Rightarrow g$ IS RETRACTION OF B^n TO S^{n-1}, AS $g \restriction S^{n-1} = id_{S^{n-1}}$)

- SINCE $g \restriction S^{n-1} = id_{S^{n-1}}$, WE HAVE $g(-x) = -g(x)$

$\Rightarrow g$ IS ANTIPODAL ON S^{n-1} → THIS CONTRADICTS (BU2b) ↯ ☒