Topological methods in combinatorics

Martin Balko

6th lecture

April 1st 2022

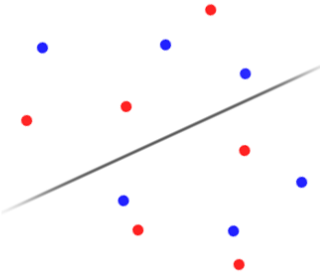
The Ham sandwich theorem and its applications

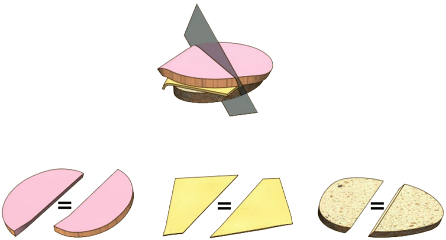
Source: https://www.seekpng.com/

The Ham sandwich theorem

The Ham sandwich theorem

- Given finite sets $A_1, \ldots, A_d$ of points in $\mathbb{R}^d$, there is a hyperplane H that contains at most $\lfloor |A_i|/2 \rfloor$ points from each set A_i in each open halfspace determined by H.

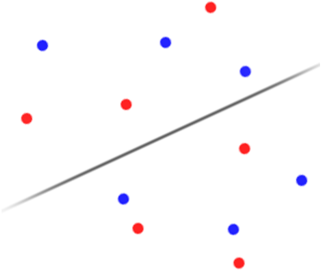


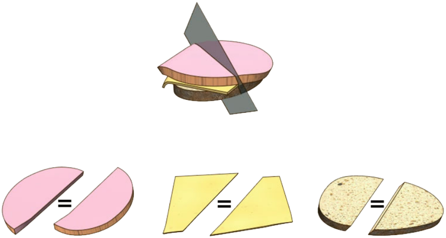


Sources: https://ejarzo.github.io and https://curiosamathematica.tumblr.com

The Ham sandwich theorem

- Given finite sets $A_1, \ldots, A_d$ of points in $\mathbb{R}^d$, there is a hyperplane H that contains at most $\lfloor |A_i|/2 \rfloor$ points from each set A_i in each open halfspace determined by H.





Sources: https://ejarzo.github.io and https://curiosamathematica.tumblr.com

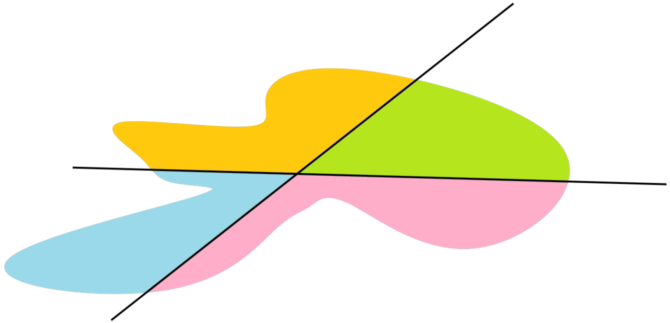
The Ham sandwich theorem for Borel measures

Let $\mu_1, \ldots, \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every hyperplane has measure 0 for each μ_i. Then there is a hyperplane H such that $\mu_i(H^+) = \mu_i(\mathbb{R}^d)/2$ for $i = 1, \ldots, d$.

Equipartitions by hyperplanes

Equipartitions by hyperplanes

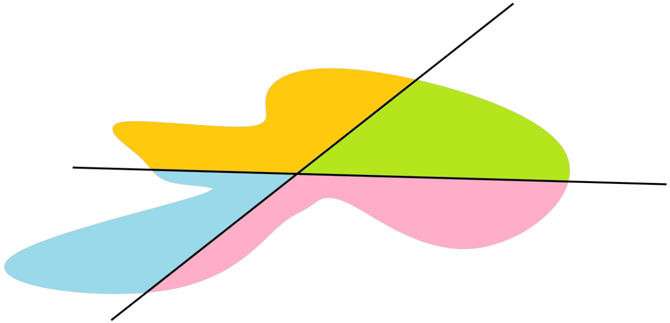
- The Ham Sandwich theorem for measures $\Rightarrow$ any mass distribution in the plane can be dissected into 4 equal parts by 2 lines (exercise).



Source: Matoušek: Using the Borsuk–Ulam Theorem (colored)

Equipartitions by hyperplanes

- The Ham Sandwich theorem for measures $\Rightarrow$ any mass distribution in the plane can be dissected into 4 equal parts by 2 lines (exercise).

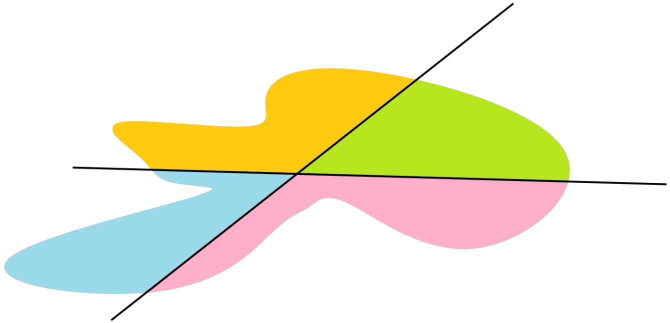


Source: Matoušek: Using the Borsuk–Ulam Theorem (colored)

- Any mass distribution in $\mathbb{R}^3$ can be partitioned into $2^3 = 8$ equal pieces by 3 planes (not easy).

Equipartitions by hyperplanes

- The Ham Sandwich theorem for measures $\Rightarrow$ any mass distribution in the plane can be dissected into 4 equal parts by 2 lines (exercise).

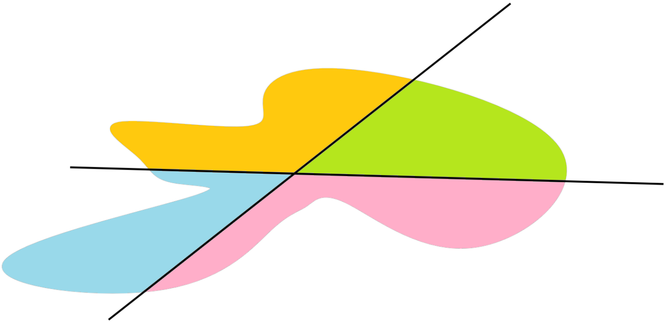


Source: Matoušek: Using the Borsuk–Ulam Theorem (colored)

- Any mass distribution in $\mathbb{R}^3$ can be partitioned into $2^3 = 8$ equal pieces by 3 planes (not easy).
- For $d \geq 5$, equipartition into 2^d equal parts by d hyperplanes fails.

Equipartitions by hyperplanes

- The Ham Sandwich theorem for measures $\Rightarrow$ any mass distribution in the plane can be dissected into 4 equal parts by 2 lines (exercise).



Source: Matoušek: Using the Borsuk–Ulam Theorem (colored)

- Any mass distribution in $\mathbb{R}^3$ can be partitioned into $2^3 = 8$ equal pieces by 3 planes (not easy).
- For $d \geq 5$, equipartition into 2^d equal parts by d hyperplanes fails.
- For $d = 4$, the problem is open.

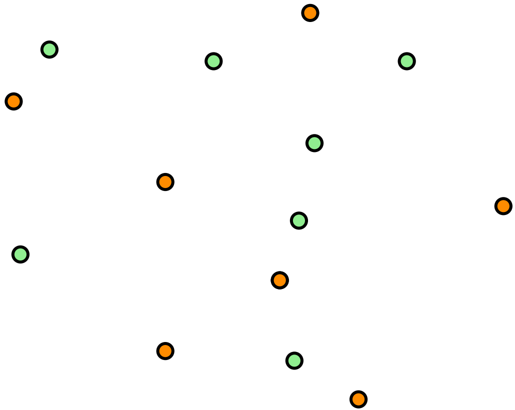
Multicolored partitions

Multicolored partitions

- For $d \geq 1$ and sets $A_1, \ldots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^{d} A_i$ in general position, the points from $\cup_{i=1}^{d} A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.

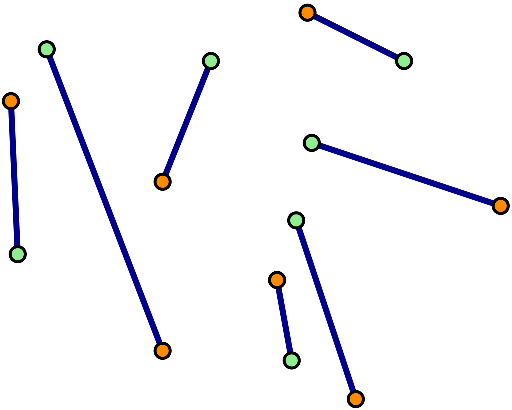
Multicolored partitions

- For $d \geq 1$ and sets $A_1, \ldots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^{d} A_i$ in general position, the points from $\cup_{i=1}^{d} A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.



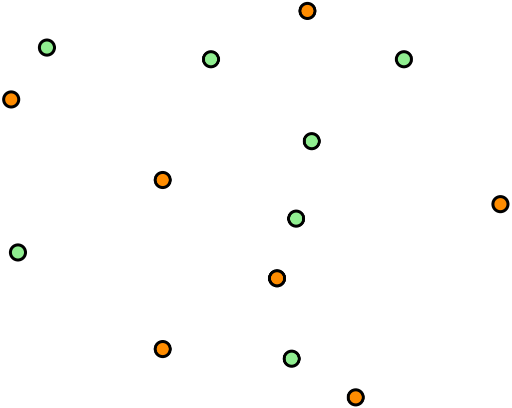
Multicolored partitions

- For $d \geq 1$ and sets $A_1, \ldots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^{d} A_i$ in general position, the points from $\cup_{i=1}^{d} A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.



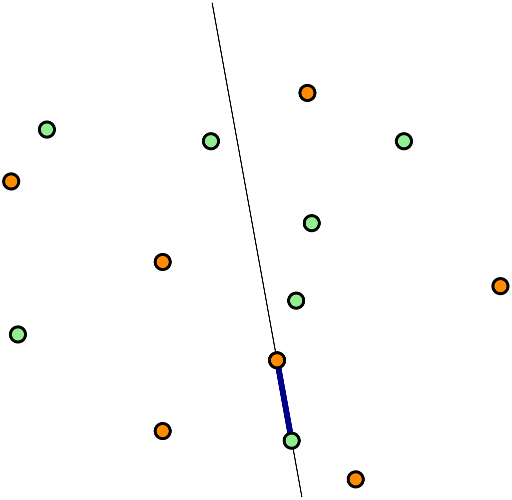
Multicolored partitions

- For $d \geq 1$ and sets $A_1, \dots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^{d} A_i$ in general position, the points from $\cup_{i=1}^{d} A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.



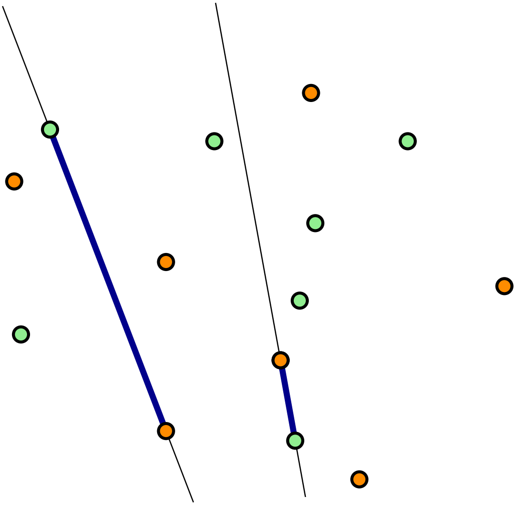
Multicolored partitions

- For $d \geq 1$ and sets $A_1, \ldots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^{d} A_i$ in general position, the points from $\cup_{i=1}^{d} A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.



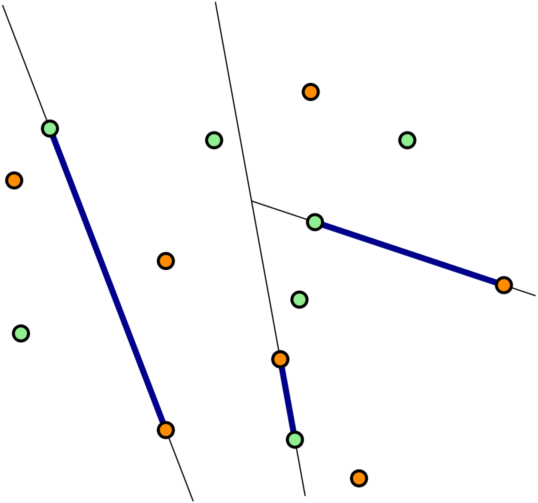
Multicolored partitions

- For $d \geq 1$ and sets $A_1, \ldots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^{d} A_i$ in general position, the points from $\cup_{i=1}^{d} A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.



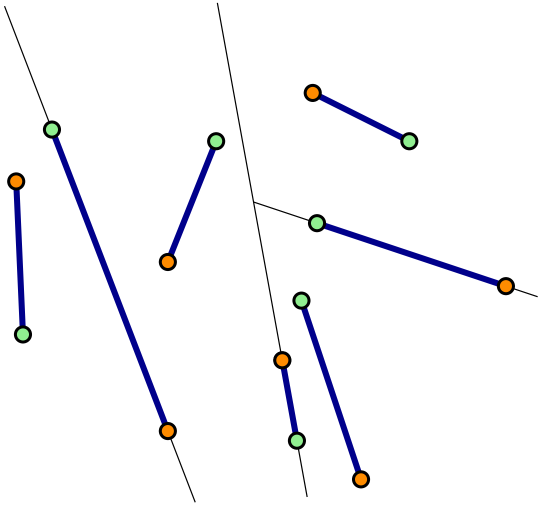
Multicolored partitions

- For $d \geq 1$ and sets $A_1, \ldots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^d A_i$ in general position, the points from $\cup_{i=1}^d A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.



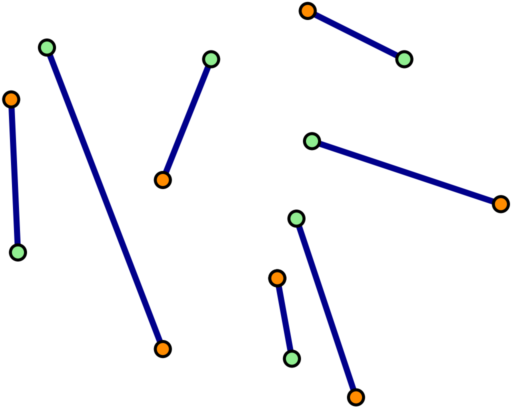
Multicolored partitions

- For $d \geq 1$ and sets $A_1, \ldots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^{d} A_i$ in general position, the points from $\cup_{i=1}^{d} A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.



Multicolored partitions

- For $d \geq 1$ and sets $A_1, \ldots, A_d$, each containing n points from $\mathbb{R}^d$ with $\cup_{i=1}^d A_i$ in general position, the points from $\cup_{i=1}^d A_i$ can be partitioned into rainbow d-tuples with disjoint convex hulls.



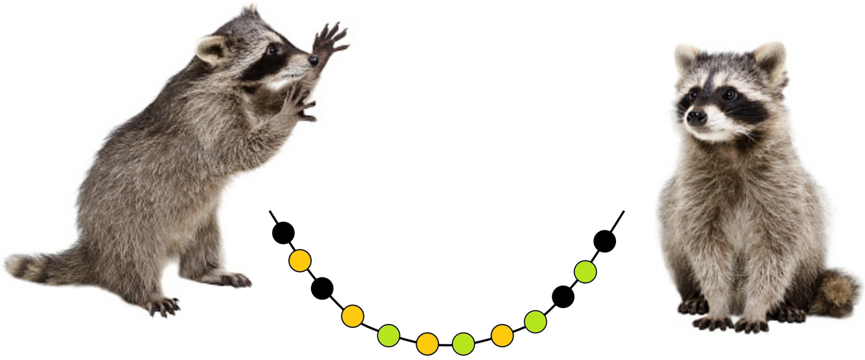
Division of a necklace

Division of a necklace

- Two thieves have stolen a necklace with precious stones (even number of each kind) and they want to divide the stones of each kind evenly by as few cuts as possible.

Division of a necklace

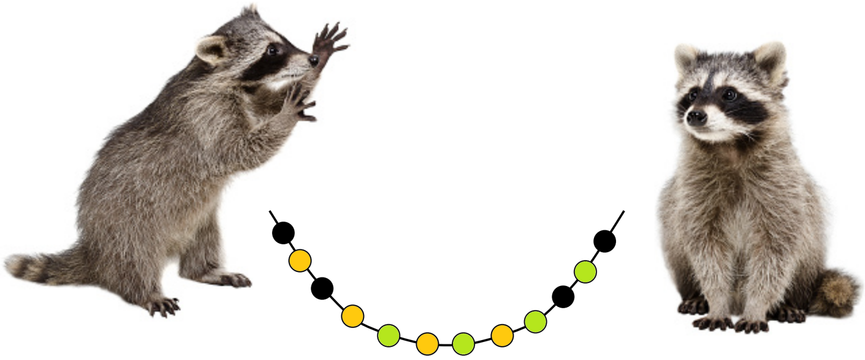
- Two thieves have stolen a necklace with precious stones (even number of each kind) and they want to divide the stones of each kind evenly by as few cuts as possible.



Source: Matoušek: Using the Borsuk–Ulam Theorem (colored) and https://media.istockphoto.com/

Division of a necklace

- Two thieves have stolen a necklace with precious stones (even number of each kind) and they want to divide the stones of each kind evenly by as few cuts as possible.

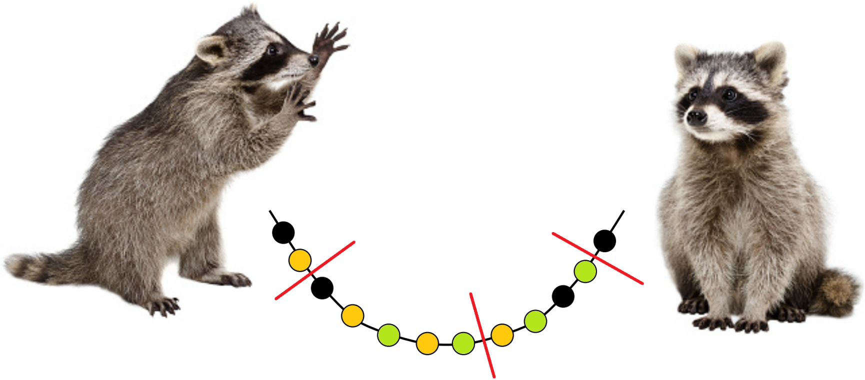


Source: Matoušek: Using the Borsuk–Ulam Theorem (colored) and https://media.istockphoto.com/

- Every open necklace with d kinds of stones can be divided between two thieves using no more than d cuts.

Division of a necklace

- Two thieves have stolen a necklace with precious stones (even number of each kind) and they want to divide the stones of each kind evenly by as few cuts as possible.

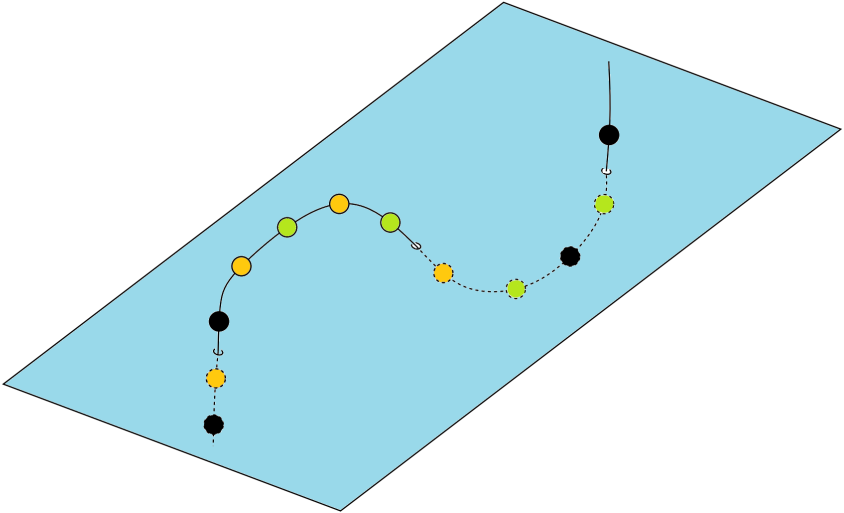


Sources: Matoušek: Using the Borsuk–Ulam Theorem (colored) and https://media.istockphoto.com/

- Every open necklace with d kinds of stones can be divided between two thieves using no more than d cuts.

Division of a necklace by the Ham sandwich theorem

Division of a necklace by the Ham sandwich theorem



Source: Matoušek: Using the Borsuk–Ulam Theorem (colored)


bye-bye
ICANHASCHEEZBURGER.COM

Thank you for your attention.