# Topological methods in combinatorics

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4th lecture

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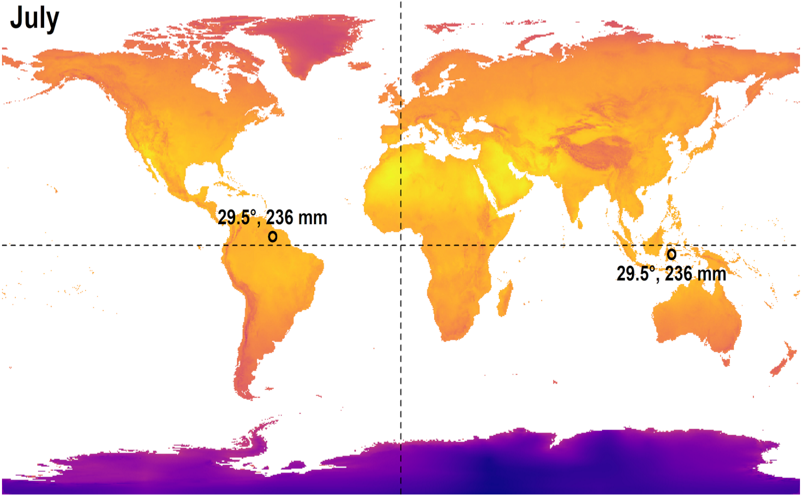
# The Borsuk–Ulam theorem




Source: https://scientificgems.wordpress.com/

# The Borsuk–Ulam theorem: history

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- First mentioned by Lyusternik and Shnirel'man (1930). The first proof was given by Karol Borsuk (1933), where the formulation of the problem was attributed to Stanislaw Ulam.

Figure: Karol Borsuk (1905–1982) a Stanislaw Ulam (1909–1984).

Sources: https://www.komputerswiat.pl/ and https://en.wikipedia.org

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Figure: Karol Borsuk (1905–1982) a Stanislaw Ulam (1909–1984).

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- Implies the Brouwer's fixed point theorem.

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(BU1a) For every $f\colon S^n \to \mathbb{R}^n$ there is $x \in S^n$ with $f(x) = f(-x)$.

(BU1b) For every antipodal $f\colon S^n \to \mathbb{R}^n$ there is $x \in S^n$ with $f(x) = 0$.

(BU2a) There is no antipodal $f\colon S^n \to S^{n-1}$.

(BU2b) There is no $f\colon B^n \to S^{n-1}$ that is antipodal on $\partial B^n = S^{n-1}$.

(LS-c) For any closed cover $F_1, \ldots, F_{n+1}$ of $S^n$, there is $i \in [n+1]$ and $x \in S^n$ with $x, -x \in F_i$.

(LS-o) For any open cover $U_1, \ldots, U_{n+1}$ of $S^n$, there is $i \in [n+1]$ and $x \in S^n$ with $x, -x \in U_i$.

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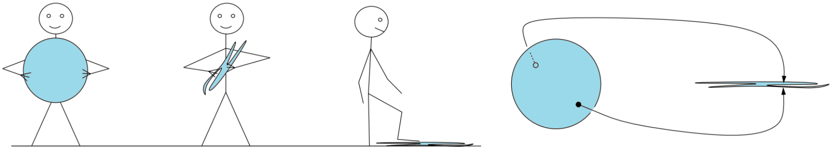
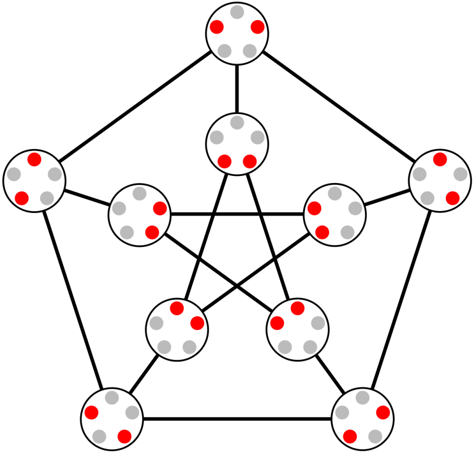


Source: Matoušek: Using the Borsuk–Ulam Theorem (colored)

# Kneser's conjecture



Source: Hatcher: Algebraic topology (colored)

# Kneser's conjecture

**Aufgabe 360:** $k$ und $n$ seien zwei natürliche Zahlen, $k \leqq n$; $N$ sei eine Menge mit $n$ Elementen, $N_k$ die Menge derjenigen Teilmengen von $N$, die genau $k$ Elemente enthalten; $f$ sei eine Abbildung von $N_k$ auf eine Menge $M$, mit der Eigenschaft, daß $f(K_1) \neq f(K_2)$ ist falls der Durchschnitt $K_1 \frown K_2$ leer ist; $m(k, n, f)$ sei die Anzahl der Elemente von $M$ und $m(k, n) = \operatorname*{Min}_f m(k, n, f)$. Man beweise: Bei festem $k$ gibt es Zahlen $m_0 = m_0(k)$ und $n_0 = n_0(k)$ derart, daß $m(k, n) = n - m_0$ ist für $n \geqq n_0$; dabei ist $m_0(k) \geqq 2k - 2$ und $n_0(k) \geqq 2k - 1$; in beiden Ungleichungen ist vermutlich das Gleichheitszeichen richtig.

Heidelberg. MARTIN KNESER.

Source: Matoušek: Using the Borsuk–Ulam Theorem

# Kneser's conjecture

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- For all $n \geq 2k - 1$, the chromatic number of the Kneser graph $KG_{n,k}$ is $n - 2k + 2$.

Figure: Martin Kneser (1928–2004) a Lászlo lovász (born 1948).

Source: https://en.wikipedia.org and https://web.cs.elte.hu/ lovasz/

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Figure: Martin Kneser (1928–2004) a Lászlo lovász (born 1948).

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- Solved by Lovász in 1978 using topological methods.

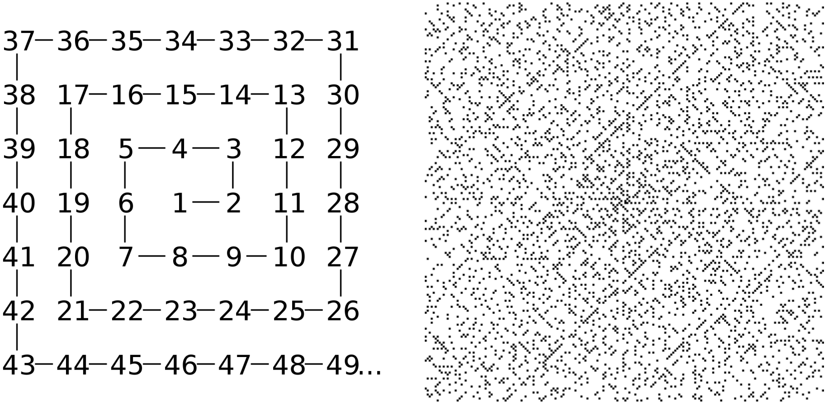
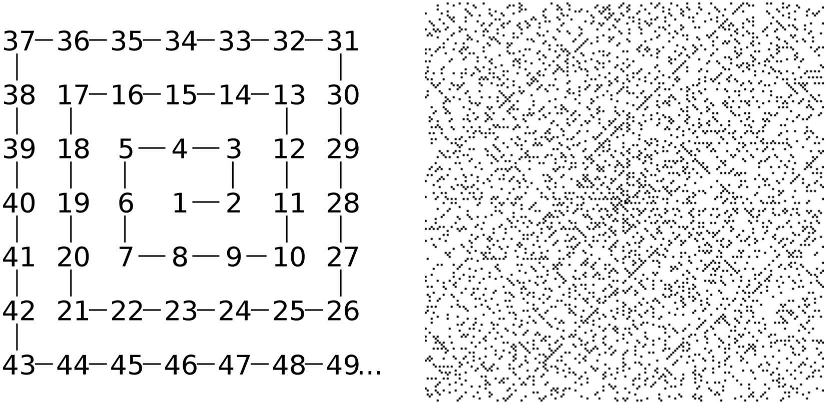



Figure: Ulam's spiral.

Source: https://en.wikipedia.org

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# Thank you for your attention.