Topological methods in combinatorics

Martin Balko

1st lecture

February 26th 2022

Basic info

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- Course webpage: *https://kam.mff.cuni.cz/topmet/*
  - basic info, topics covered, presentations, ...

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- Course webpage: *https://kam.mff.cuni.cz/topmet/*
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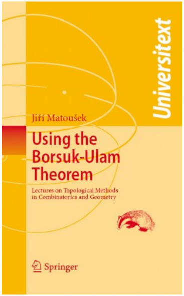
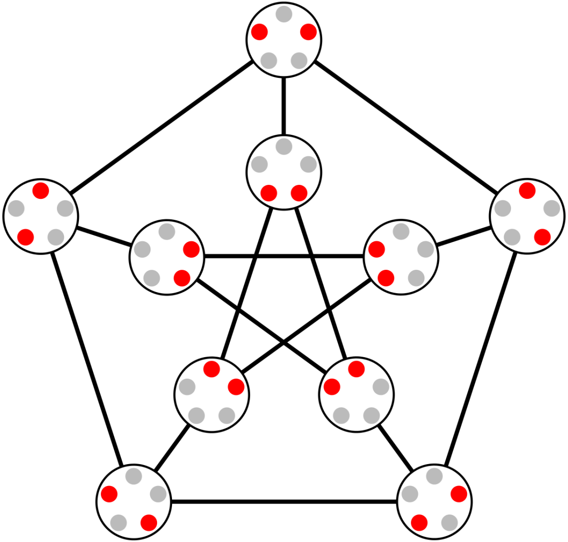
- Recommended literature:
  - J. Matoušek: Using the Borsuk–Ulam Theorem.



Source: https://link.springer.com/

Applications of topology



Source: https://en.wikipedia.org

Kneser's conjecture

Kneser's conjecture

- For all $n \geq 2k-1$, the chromatic number of the Kneser graph $KG_{n,k}$ is $n-2k+2$.

Figure: Martin Kneser (1928–2004) and Lászlo Lovász (born 1948).

Source: https://en.wikipedia.org and https://web.cs.elte.hu/ lovasz/

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Figure: Martin Kneser (1928–2004) and Lászlo Lovász (born 1948).

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- Solved by Lovász in 1978 using topological methods.

The Ham sandwich theorem

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- Given finite sets $A_1, \ldots, A_d$ of points in $\mathbb{R}^d$, there is a hyperplane H that contains at most $\lfloor |A_i|/2 \rfloor$ points from each set A_i in each open halfspace determined by H.

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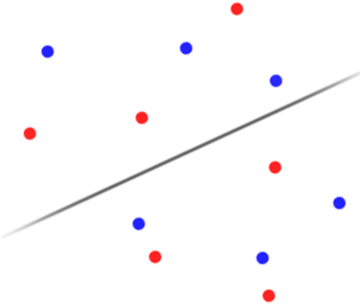

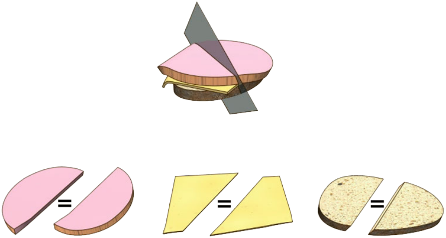

Sources: https://ejarzo.github.io and https://curiosamathematica.tumblr.com

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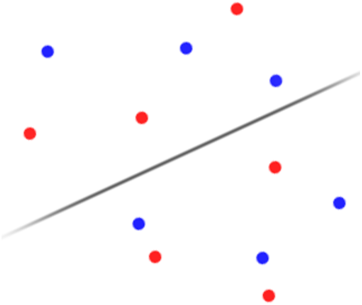

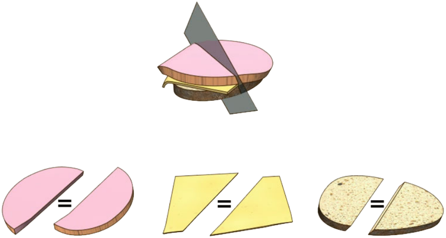

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- Again, proved by topological methods.

Fair partitions of convex bodies

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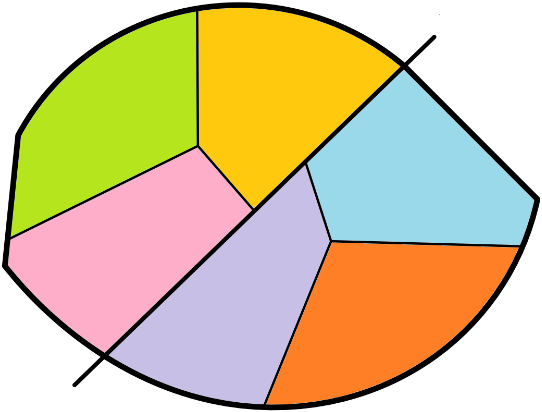
- Any convex body in the plane can be partitioned into m convex parts of equal areas and perimeters for any integer $m \geq 2$.



Source: Akopyan, Avvakumov, Karasev: Convex fair partitions into an arbitrary number of pieces (colored)

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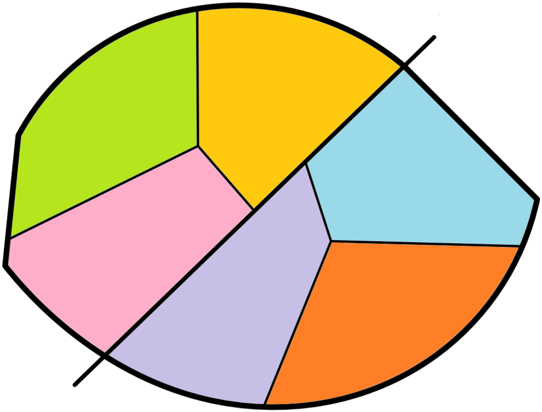

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- Proved by Akopyan, Avvakumov, and Karasev in 2021.

(Gentle) introduction to topology

Source: https://techblog.cisco.com/

Homeomorphisms

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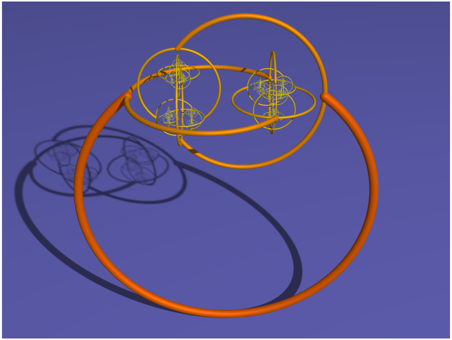
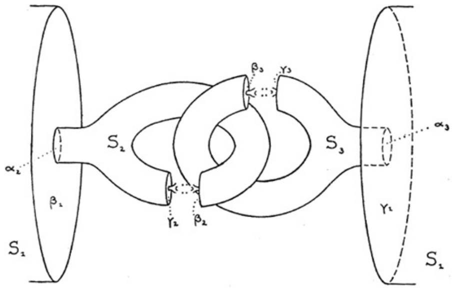
- The Alexander's horned sphere is homeomorphic to the 3-ball.





Source: https://en.wikipedia.org and https://mathworld.wolfram.com/

Homeomorphisms: Poincaré conjecture

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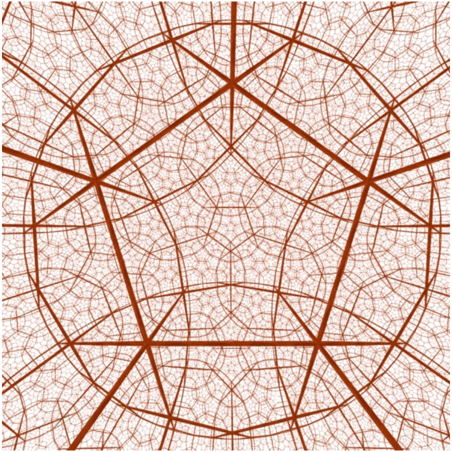
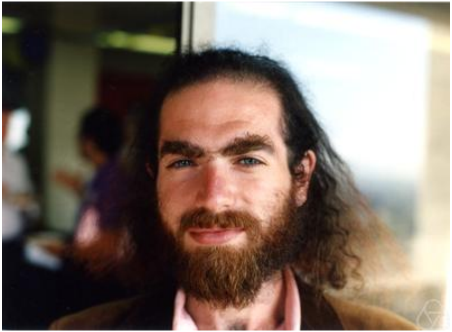
- Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.





Source: https://www.claymath.org/ and https://en.wikipedia.org

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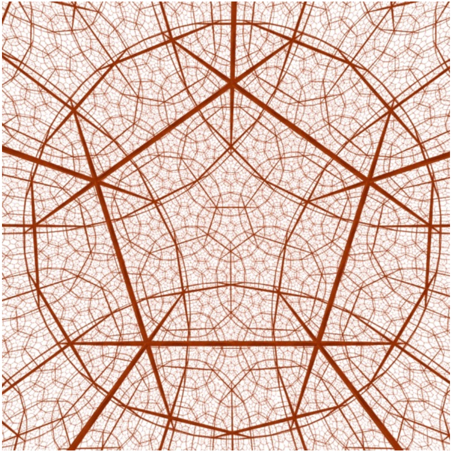

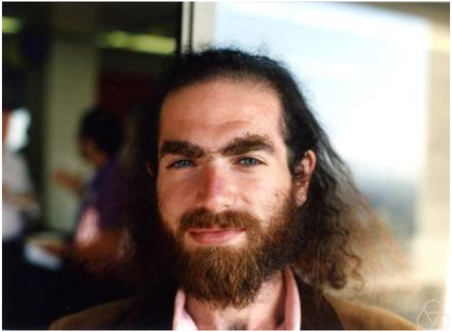

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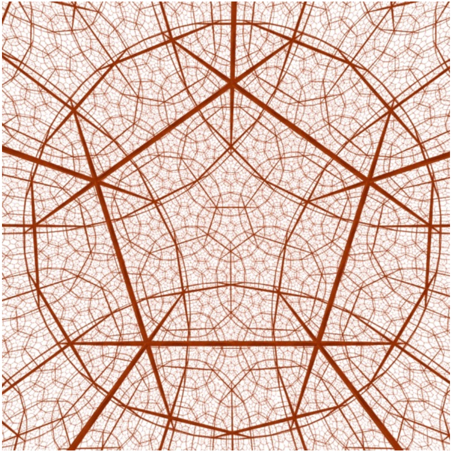

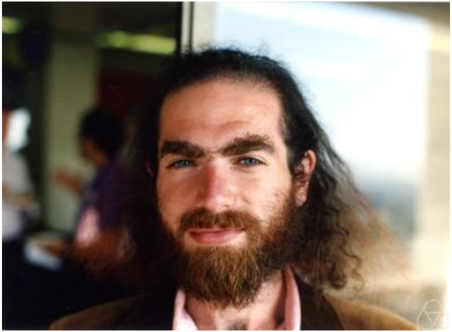

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- The only solved Millennium Prize Problem.
- Solved by Perelman in 2003.

Connectivity: example

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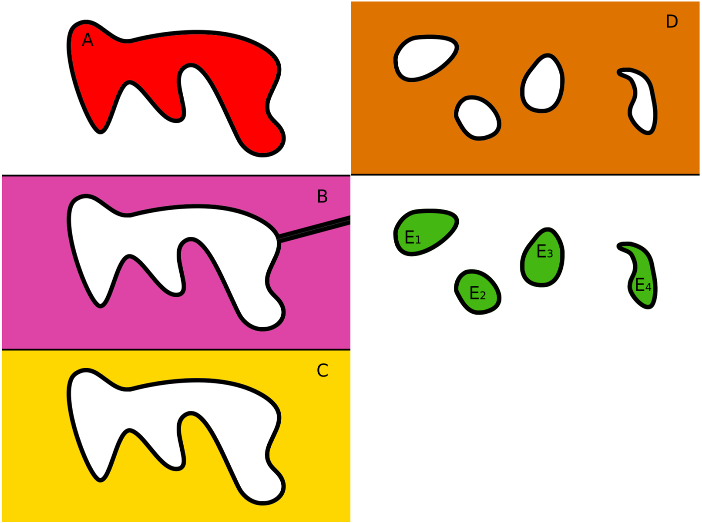
- Which of the following spaces are (path-)connected?



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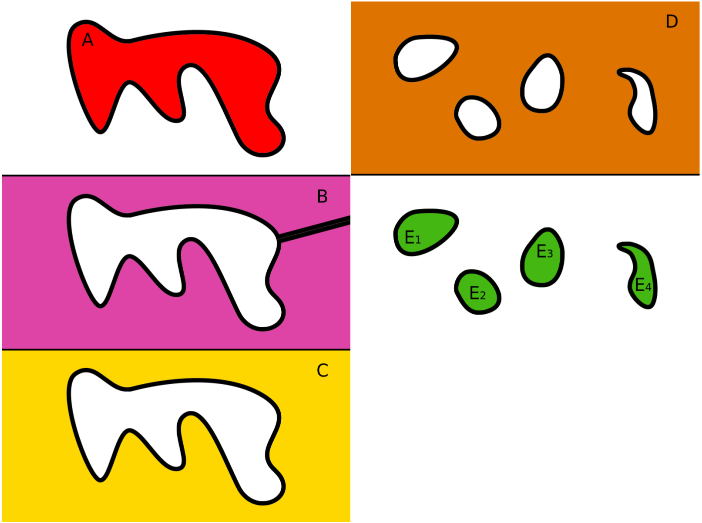

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- A, B, C, D are all path-connected and E is not even connected.

Connectivity: weird example

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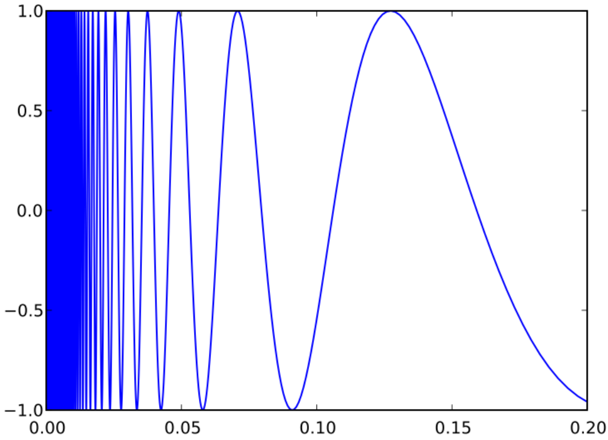
- The topologist's sine curve $\{(x, \sin\frac{1}{x})\colon x \in [0,1)\} \cup \{(0,0)\}$ is connected, but it is not path connected.



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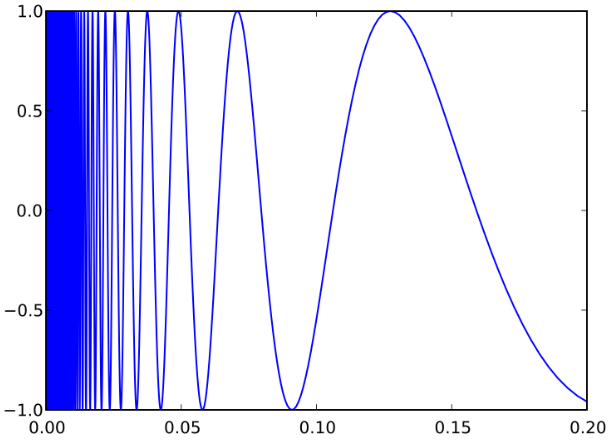

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- There is no way to link the function to the origin so as to make a path.

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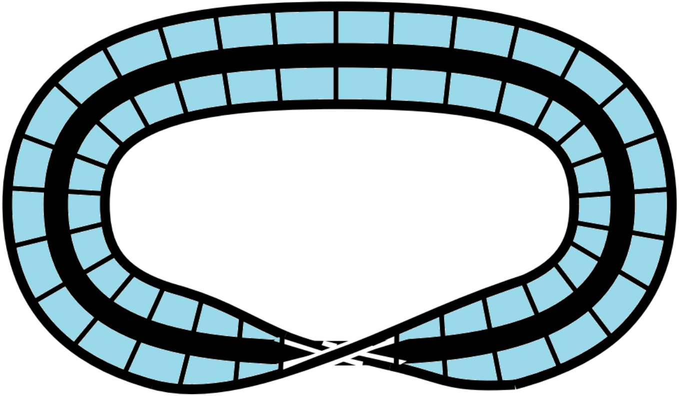

Source: Hatcher: Algebraic topology (colored)

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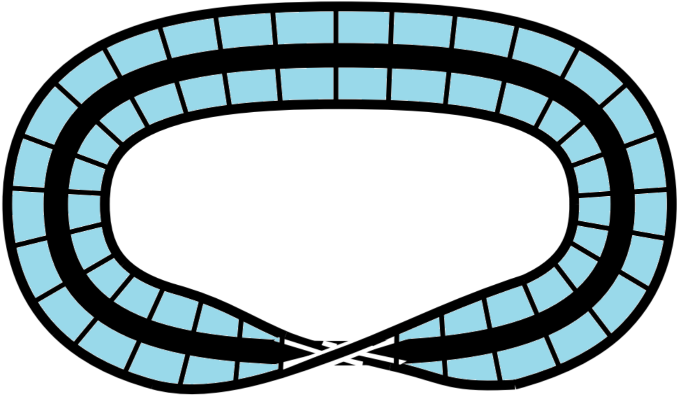

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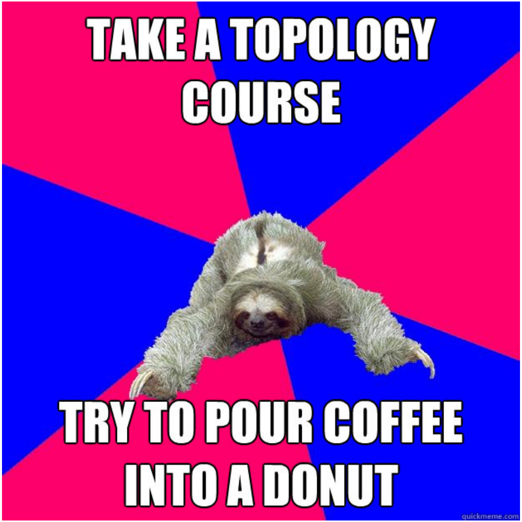
- However, they are not homeomorphic.



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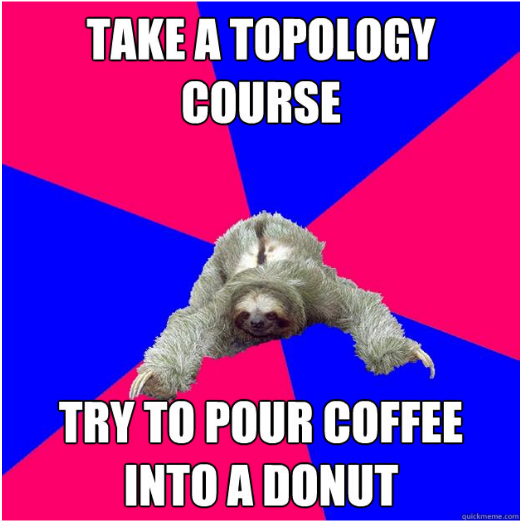

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Thank you for your attention.