Ramsey numbers of ordered graphs

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Ramsey theory

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- Ramsey number $\mathrm{R}(G_1, \ldots, G_c)$ of $G_1, \ldots, G_c$ is the smallest such N.
- If all $G_1, \ldots, G_c$ are isomorphic to G, we write $\mathrm{R}(G; c)$ or $\mathrm{R}(G)$ if $c = 2$.

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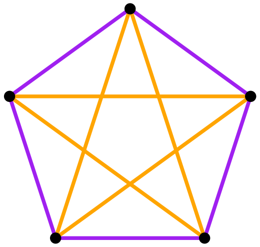
Example:



$\mathrm{R}(K_3) = \mathrm{R}(C_4) = 6$

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Observation

For ordered graphs $\mathcal{G}_1 = (G_1, \prec_1), \ldots, \mathcal{G}_c = (G_c \prec_c)$ we have

$$\mathrm{R}(G_1, \ldots, G_c) \leq \overline{\mathrm{R}}(\mathcal{G}_1, \ldots, \mathcal{G}_c) \leq \mathrm{R}(K_{|V(G_1)|}, \ldots, K_{|V(G_c)|}).$$

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Example:

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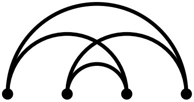
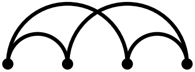


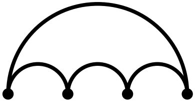
$\overline{\mathrm{R}}(\mathcal{C}_A) = 10$

$\mathcal{C}_B$



$\overline{\mathrm{R}}(\mathcal{C}_B) = 11$

$\mathcal{C}_C$



$\overline{\mathrm{R}}(\mathcal{C}_C) = 14$

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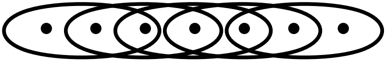

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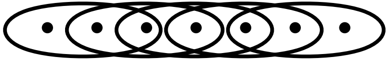

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- Fox, Pach, Sudakov, and Suk, 2011: $t_{k-1}(Cn^{c-1}) \leq \overline{\mathrm{R}}((P_n^k, \prec_{mon}); c) \leq t_{k-1}(C'n^{c-1}\log n)$.
- Moshkovitz and Shapira, 2012: $t_{k-1}(n^{c-1}/2\sqrt{c}) \leq \overline{\mathrm{R}}((P_n^k, \prec_{mon}); c) \leq t_{k-1}(2n^{c-1})$.
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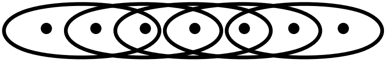

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  Every ordered path $\mathcal{P}_n$ satisfies $\overline{\mathrm{R}}(\mathcal{P}_n) \leq O(n^{\log n})$.
- Similar results discovered independently by Conlon, Fox, Lee, and Sudakov, 2014+.

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- Unordered case (Burr and Roberts, 1973):

$$\mathrm{R}(K_{1,n-1};c) = \begin{cases} c(n-2)+1 & \text{if } c \equiv n-1 \equiv 0 \pmod 2, \\ c(n-2)+2 & \text{otherwise.} \end{cases}$$

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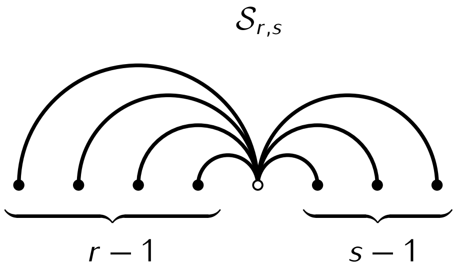

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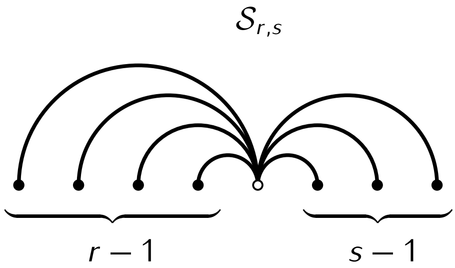

- The 2-colored ordered case was resolved by Choudum and Ponnusamy.

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Theorem (Choudum and Ponnusamy, 2002)

For positive integers r_1, r_2, we have $\overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \mathcal{S}_{1,r_2}) = r_1 + r_2 - 2$ and for $r_2 \geq r_1 > 2$

$$\overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,1}) = \left\lfloor \frac{-1 + \sqrt{1 + 8(r_1 - 2)(r_2 - 2)}}{2} \right\rfloor + r_1 + r_2 - 2.$$

For arbitrary ordered stars we have

$$\overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,s_2}) = \overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,1}) + r_1 + s_2 - 3$$

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- For the multicolored case the ordered Ramsey numbers remain linear in the number of vertices.

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- Unordered case (Faudree and Schelp, 1974):

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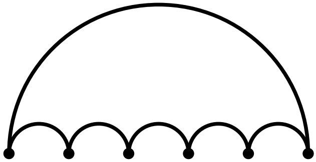

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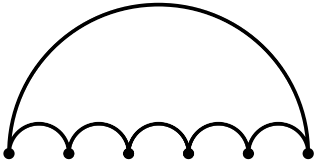

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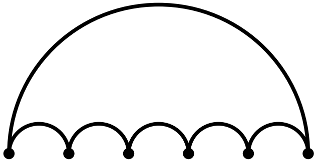

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- Settles a question of Károlyi et al. about geometric Ramsey numbers.

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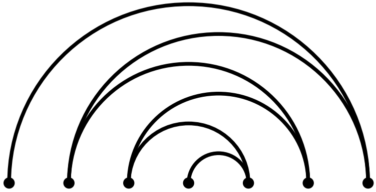

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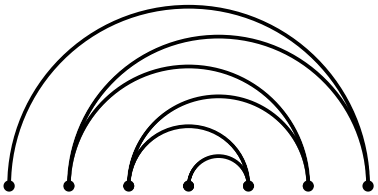

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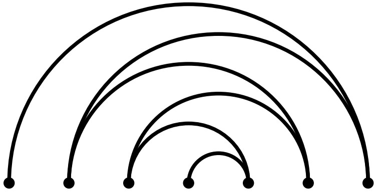

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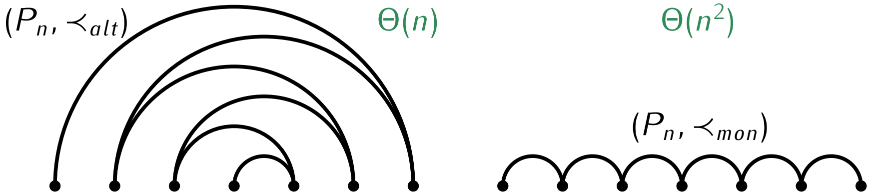

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- Conlon et al.: almost every ordered n-vertex matching $\mathcal{M}_n$ satisfies

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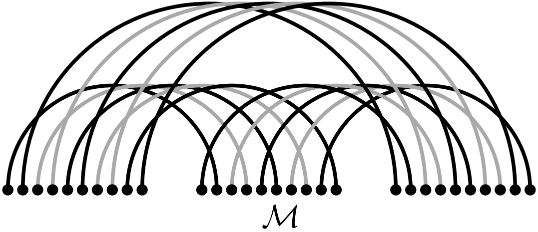

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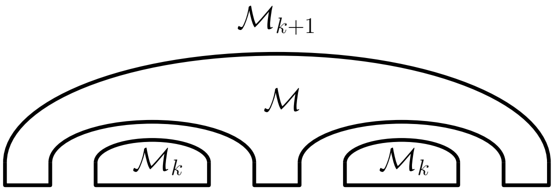

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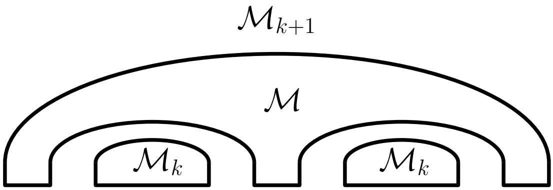

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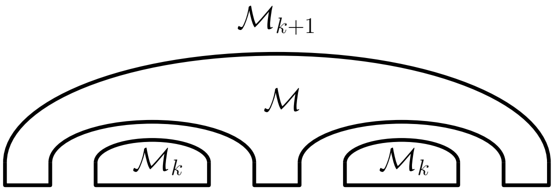

- The coloring is not constructive.

Corollary

There is arbitrarily large n-vertex graph G with two orderings $\mathcal{G}'$ and $\mathcal{G}'$ such that $\overline{\mathrm{R}}(\mathcal{G})$ is super-polynomial in n and $\overline{\mathrm{R}}(\mathcal{G}')$ is linear in n.

Small ordered Ramsey numbers I

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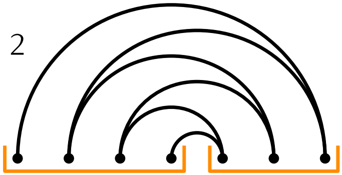

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Every ordered matching $(M, \prec)$ with $\chi_{\prec}(M) = 2$ and n vertices satisfies

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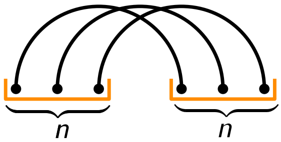

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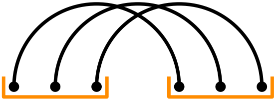

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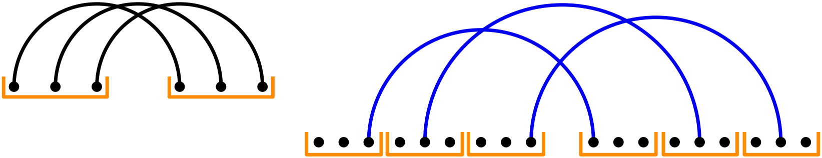

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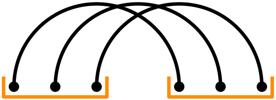

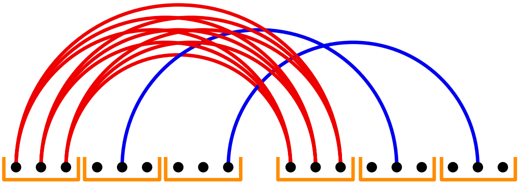

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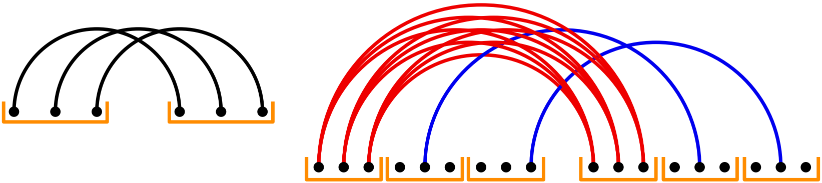

Theorem

For arbitrary k and p every k-degenerate ordered graph $(G, \prec)$ with n vertices and $\chi_{\prec}(G) = p$ satisfies

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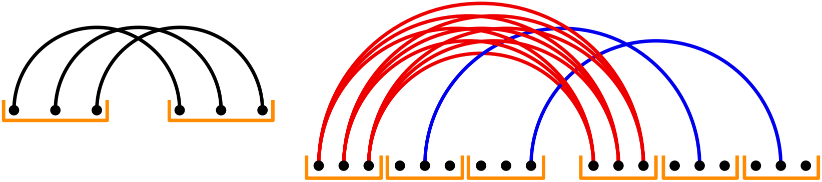

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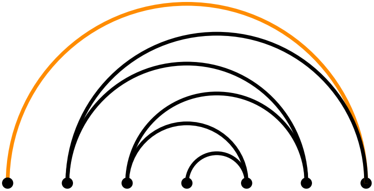

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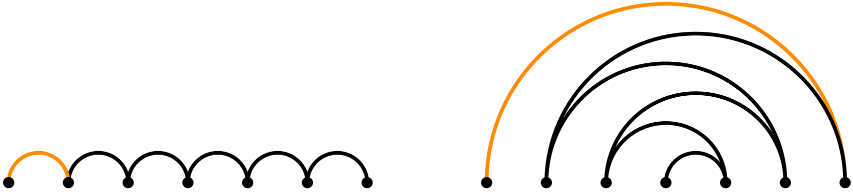

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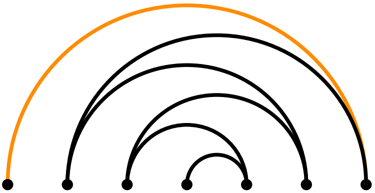

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- Solves a problem of Conlon et al.

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Thank you.