Ramsey numbers and monotone colorings

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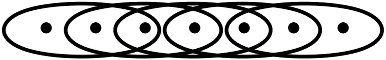

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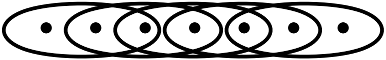
$\mathcal{P}_7^2$  $\mathcal{P}_7^3$ 

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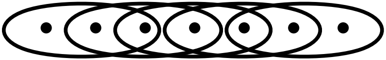

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- That is, what is the smallest $N \in \mathbb{N}$ such that every 2-coloring of the edges of $\mathcal{K}_N^r = \left([N], \binom{[N]}{r}\right)$ contains a monochromatic copy of $\mathcal{P}_n^r$.

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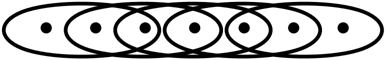
$\mathcal{P}_7^2$  $\mathcal{P}_7^3$ 

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- This question was raised by Fox, Pach, Sudakov, and Suk (2012) who proved

$$\overline{\mathrm{R}}(\mathcal{P}_n^r) \leq \mathrm{tow}_{r-1}(O(n \log n))$$

for $r \geq 3$, where $\mathrm{tow}_1(x) = x$ and $\mathrm{tow}_h(x) = 2^{\mathrm{tow}_{h-1}(x)}$ for $h \geq 2$, and asked whether

$$\overline{\mathrm{R}}(\mathcal{P}_n^r) \leq \mathrm{tow}_{r-1}(O(n)).$$

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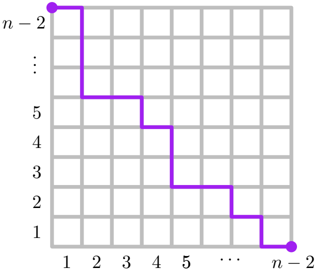

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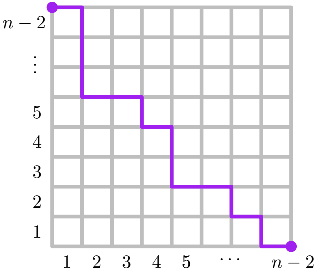

- In particular, $\overline{\mathrm{R}}(\mathcal{P}_n^3) = \binom{2n-4}{n-2} + 1$.

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- Note: not every 2-coloring of $E(K_N)$ can be obtained this way.

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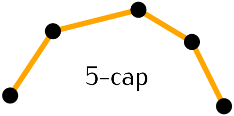

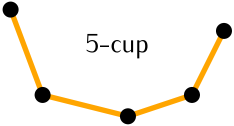

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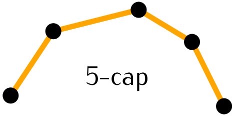

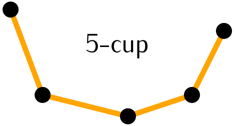

- The fact $\overline{\mathrm{R}}(\mathcal{P}^3_n) = \binom{2n-4}{n-2} + 1$ yields new proof of the Cap-Cup Theorem.
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- Then red monotone paths are cups and blue ones are caps.

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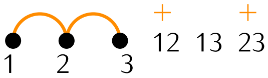

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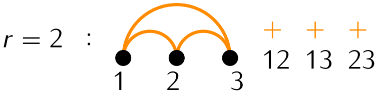

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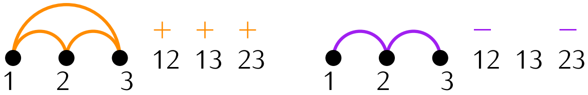

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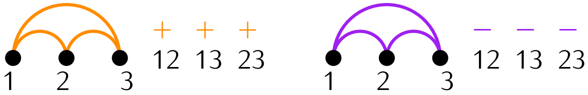

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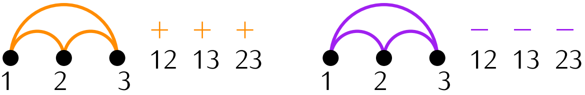

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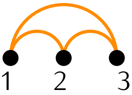

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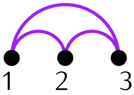

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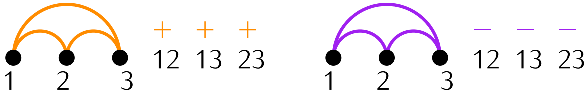

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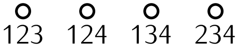

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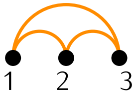

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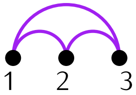

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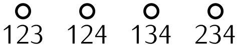

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- However, such colorings for higher uniformities are not!

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What is the growth rate of $\overline{\mathrm{R}}_{trans}(\mathcal{P}_n^r)$ for $r > 4$?

- We settle this problem even for more restrictive colorings.

Monotone colorings

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- Since r-monotone colorings admit geometric interpretations, we obtain estimates for geometric Ramsey-type statements.

Sketch of the construction

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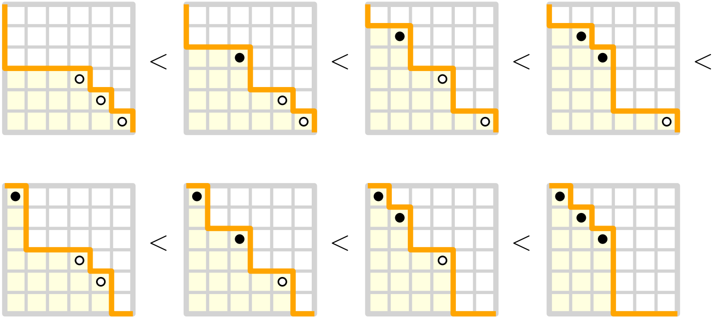

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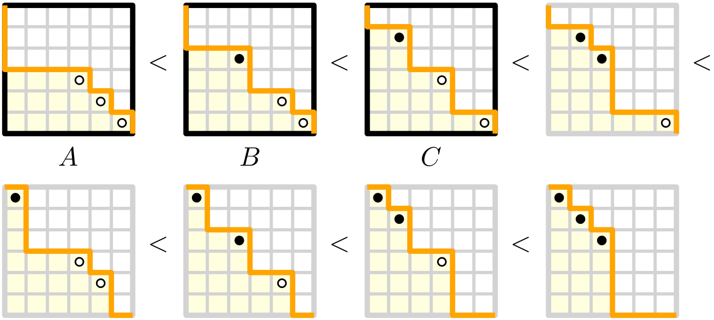

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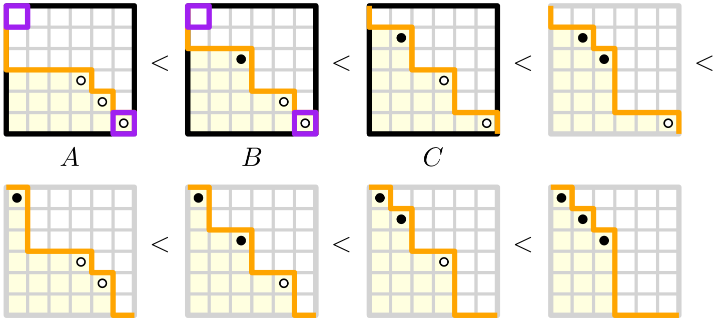

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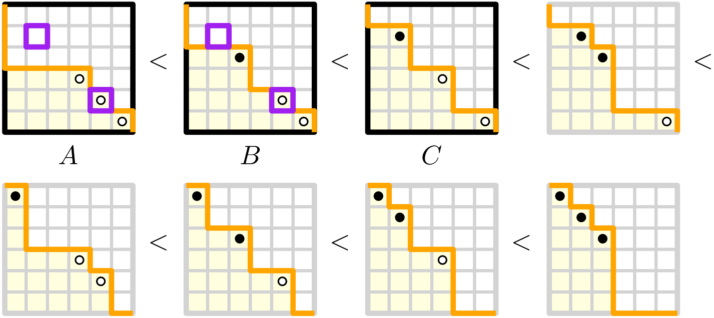

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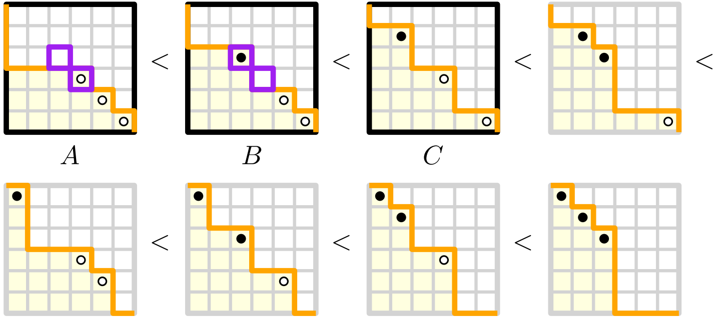

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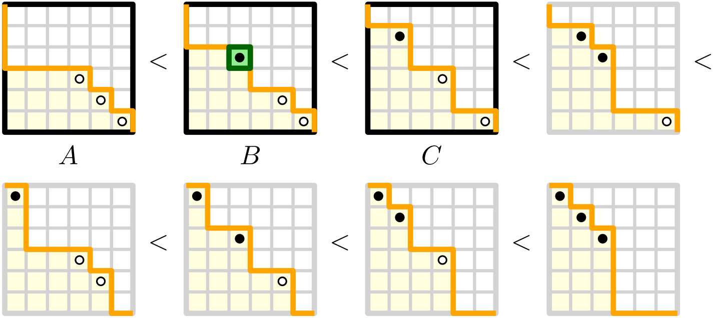

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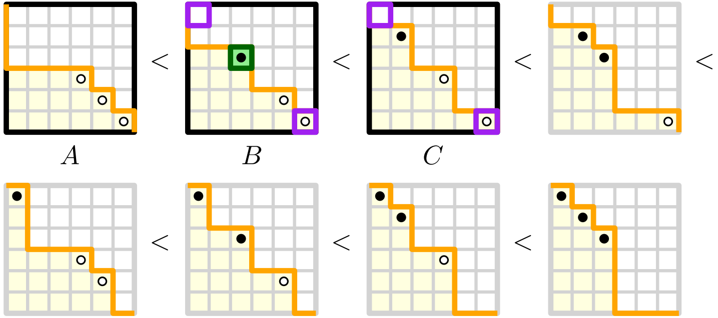

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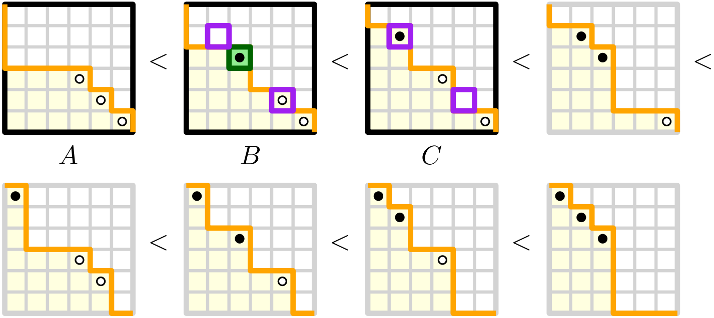

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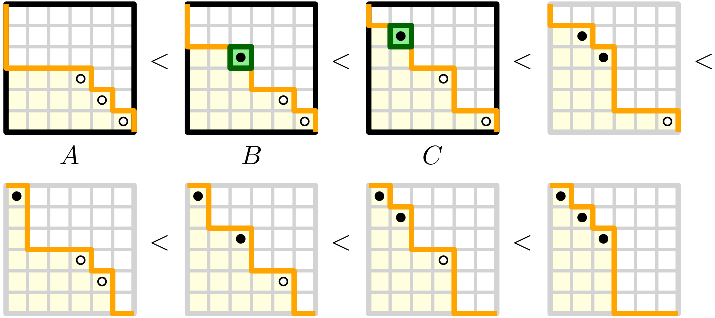

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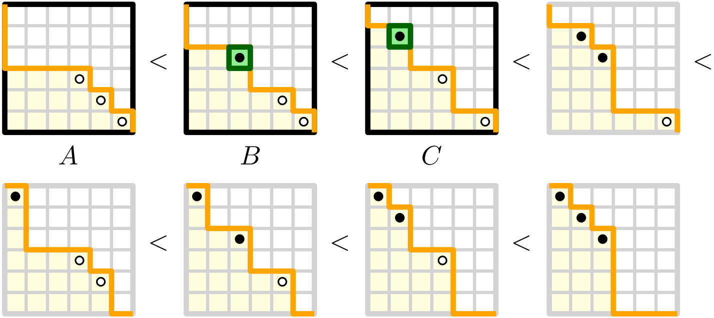

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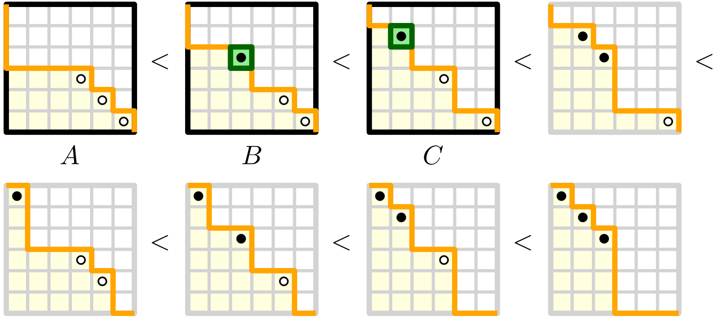

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- For $r = 4$, these eight vertices form the "new diagonal".

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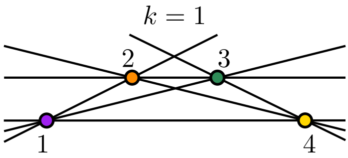

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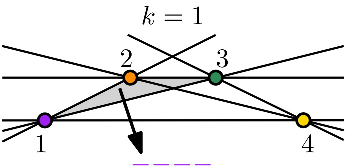

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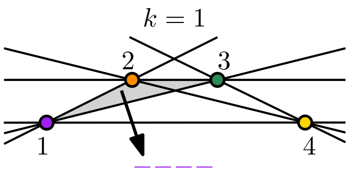

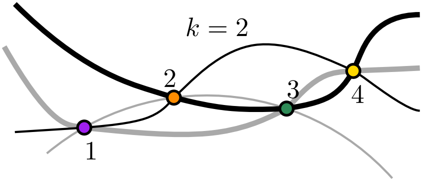

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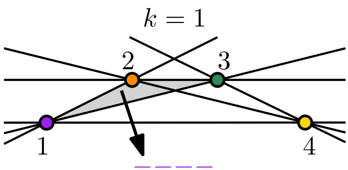

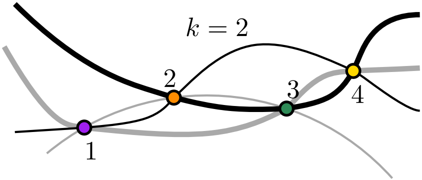

Theorem 7 (Miyata, 2017)

For $k, n \in \mathbb{N}$, there is a bijection between sign functions of simple k-pseudoconfigurations of n points and $(k+2)$-monotone colorings of $\mathcal{K}_n^{k+2}$.

Geometric interpretations: higher-order Erdős–Szekeres theorems

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Thank you.