On the Beer Index of Convexity and Its Variants

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Czech Republic

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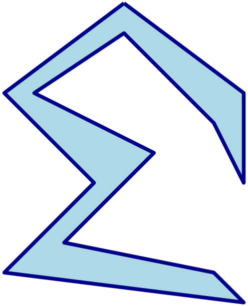

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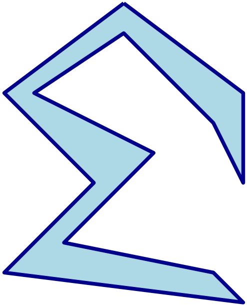

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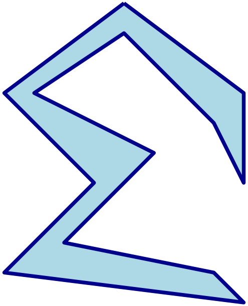

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- There are (at least) two known approaches.

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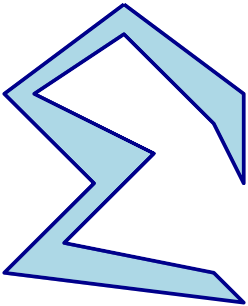

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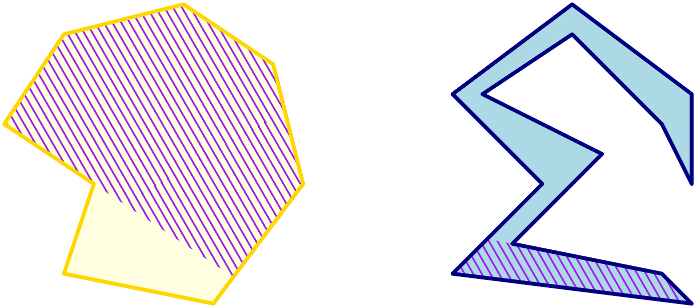

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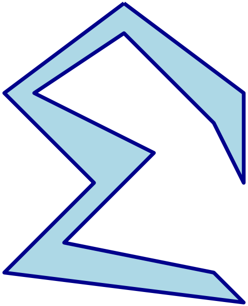

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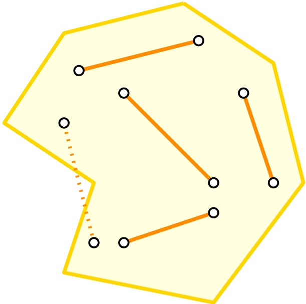

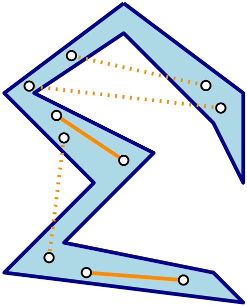

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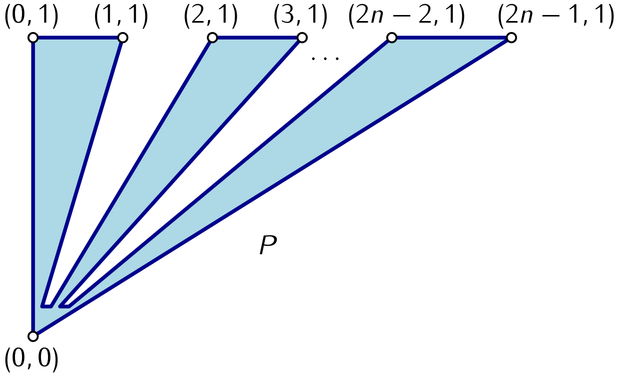

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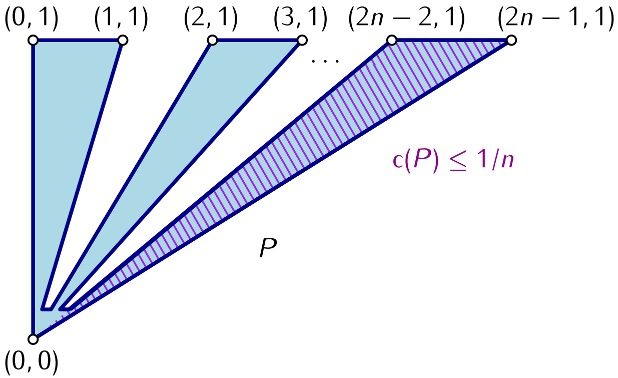

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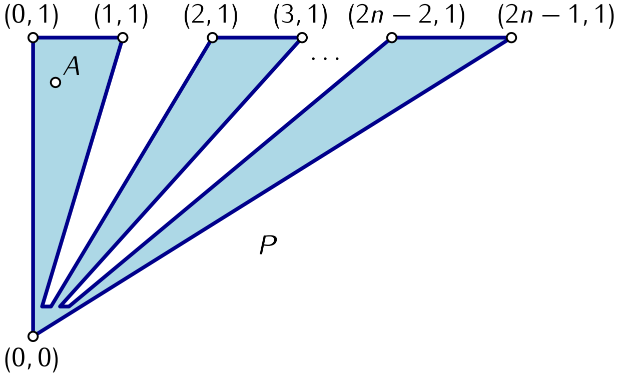

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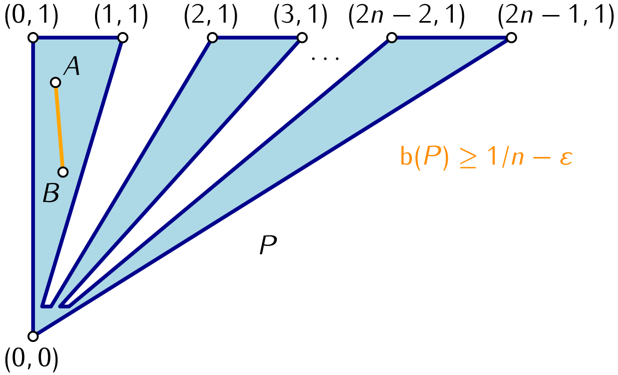

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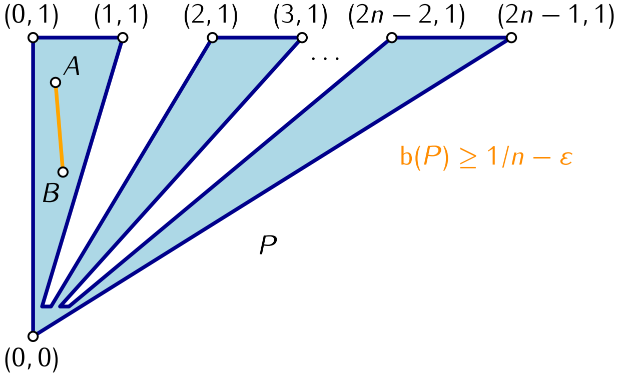

- Thus $\mathrm{b}(P)$ is not bounded from above by a sublinear function of $\mathrm{c}(P)$.

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Conjecture (Cabello et al., 2014)

There is $\alpha > 0$ so that for every simple polygon P we have $\mathrm{b}(P) \leq \alpha\,\mathrm{c}(P)$.

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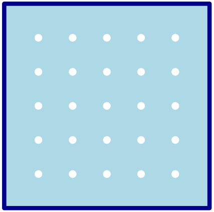

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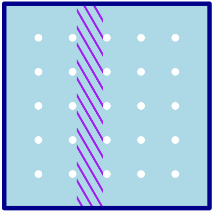

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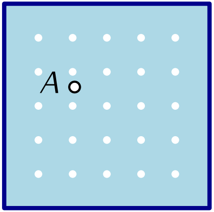

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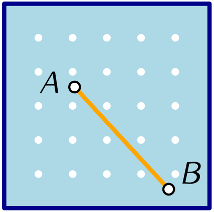

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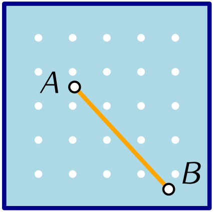

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- In fact, $S := [0,1]^2 \setminus \mathbb{Q}^2$ gives $\mathrm{c}(S) = 0$ and $\mathrm{b}(S) = 1$.

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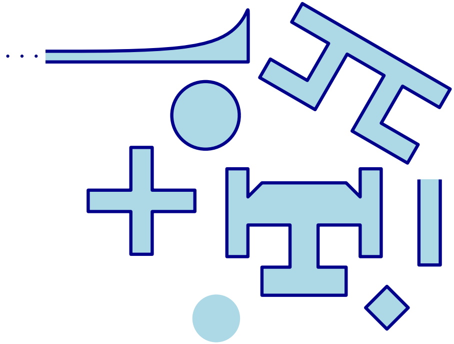

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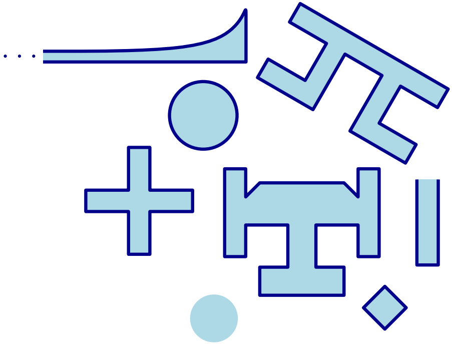

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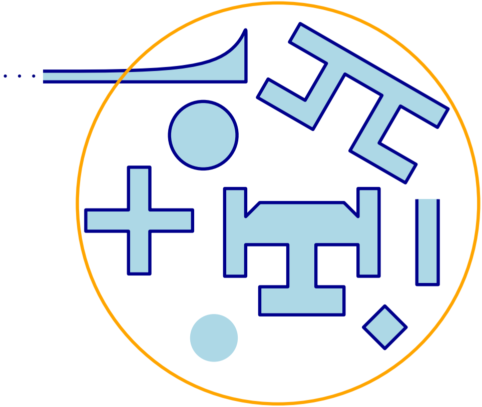

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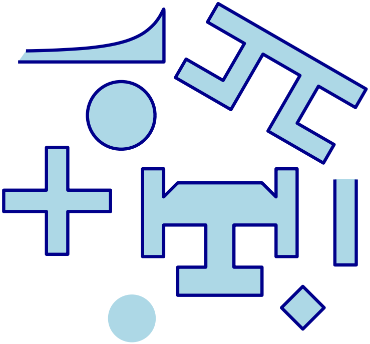

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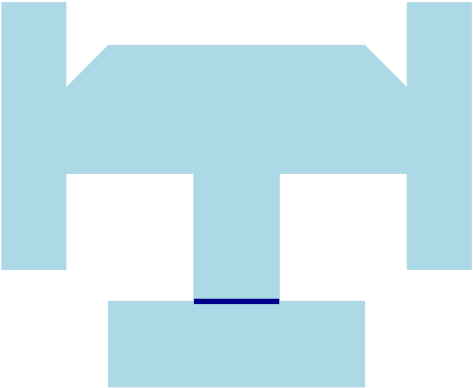

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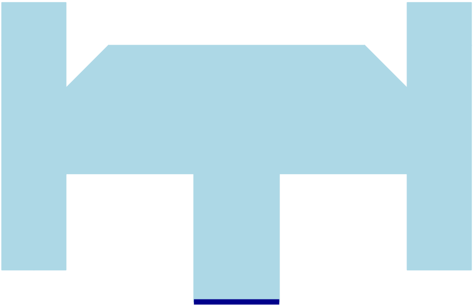

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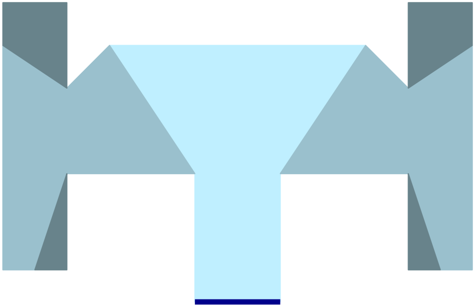

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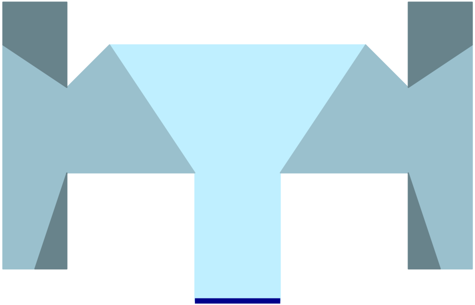

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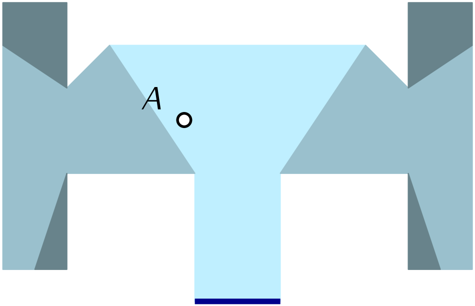

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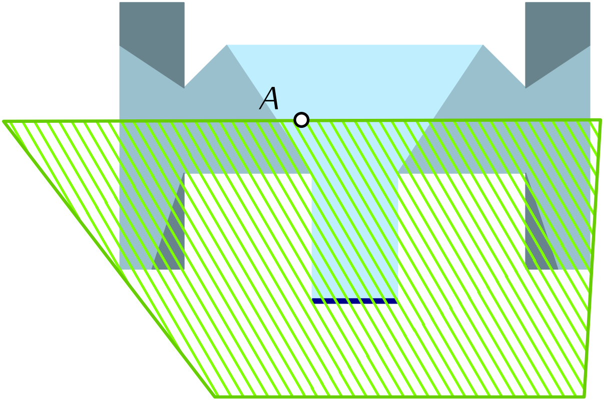

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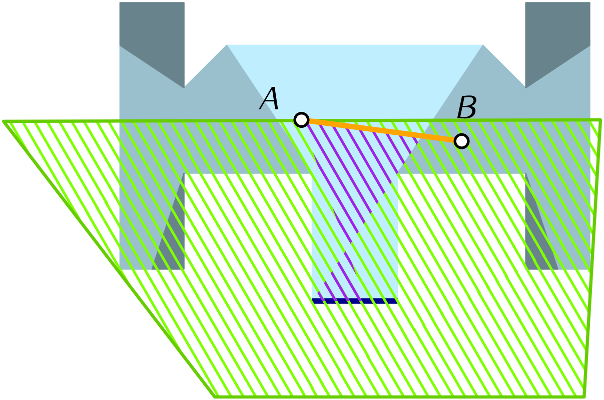

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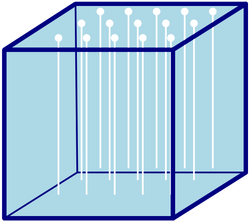

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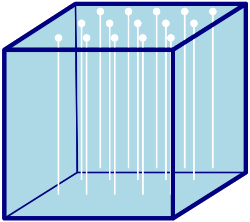
$\mathrm{c}(P) \sim 1/n$  $\mathrm{b}(P) = 1$

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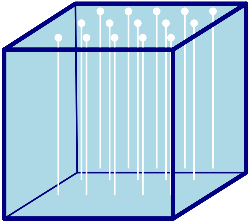
$c(P) \sim 1/n$  $b(P) = 1$

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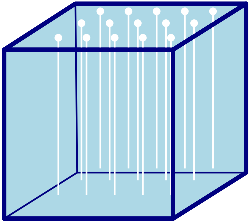

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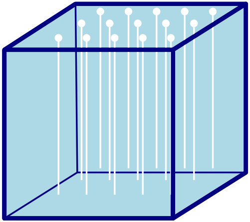

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$$b_k(S) := \frac{\lambda_{(k+1)d}(\{(A_1, \ldots, A_{k+1}) \in S^{k+1} \colon \operatorname{Conv}\{A_1, \ldots, A_{k+1}\} \subseteq S\})}{\lambda_d(S)^{k+1}}.$$

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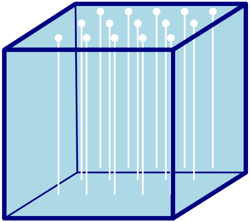
$c(P) \sim 1/n$  $b(P) = 1$

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- Note that $b_k(S) \in [0, 1]$ and $b_1(S) = b(S)$.

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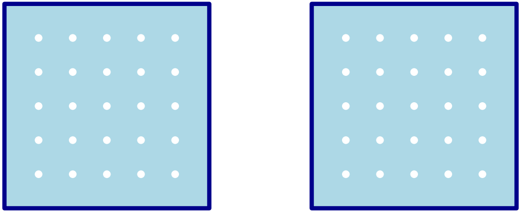

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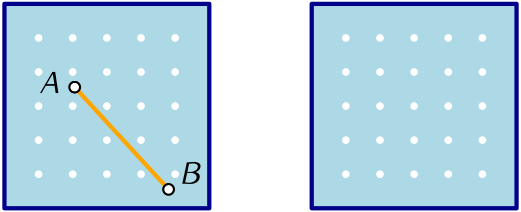

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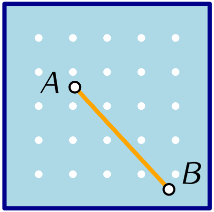

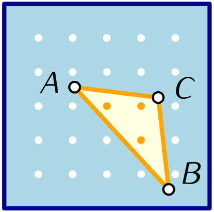

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