On ordered Ramsey numbers of bounded-degree graphs

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Ramsey theory

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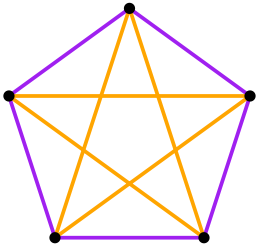
Example:



$\mathrm{R}(C_4) = 6$

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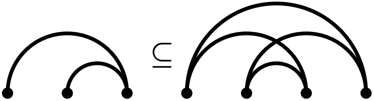

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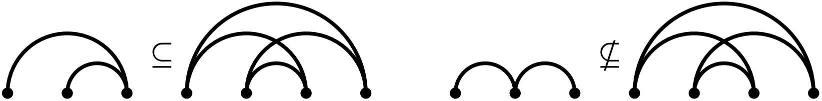

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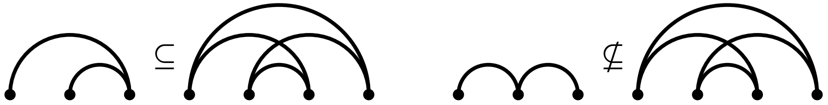

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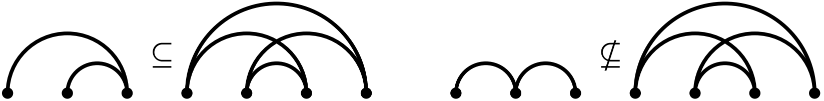

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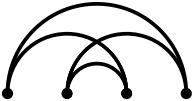
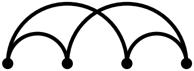


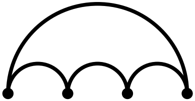
$\overline{\mathrm{R}}(\mathcal{C}_A) = 10$

$\mathcal{C}_B$



$\overline{\mathrm{R}}(\mathcal{C}_B) = 11$

$\mathcal{C}_C$



$\overline{\mathrm{R}}(\mathcal{C}_C) = 14$

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There are arbitrarily large ordered matchings $\mathcal{M}_n$ on n vertices such that

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- Conlon et al. showed that this holds for almost every ordered matching.

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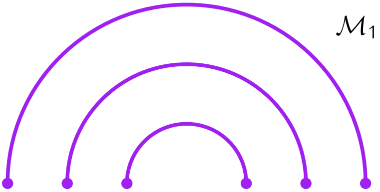

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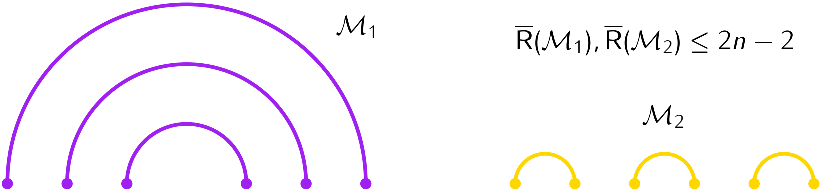

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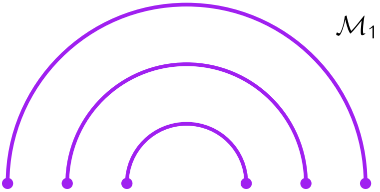

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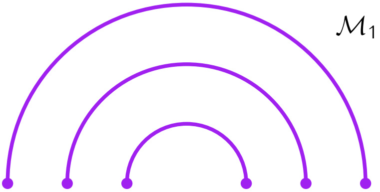

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Problem (Conlon, Fox, Lee, Sudakov, 2014)

Do random 3-regular graphs have superlinear ordered Ramsey numbers for all orderings?

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- For 3-regular graphs, we obtain min-$\overline{\mathrm{R}}(G) \geq \frac{n^{7/6}}{4 \log n \log \log n}$.

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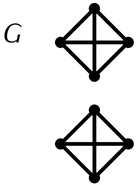

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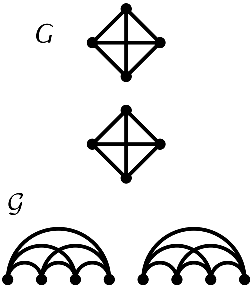

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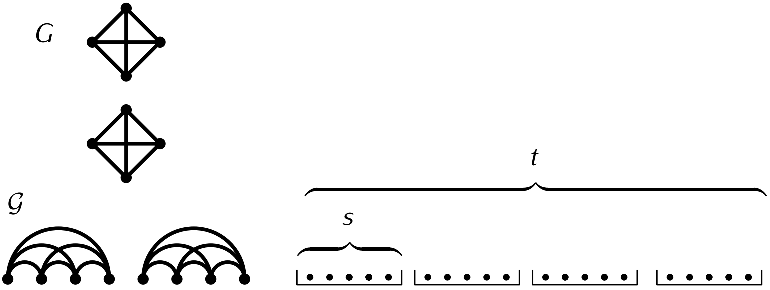

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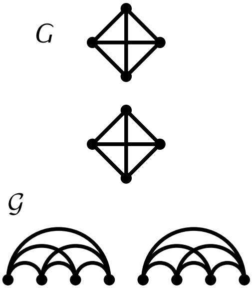

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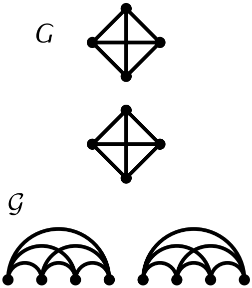

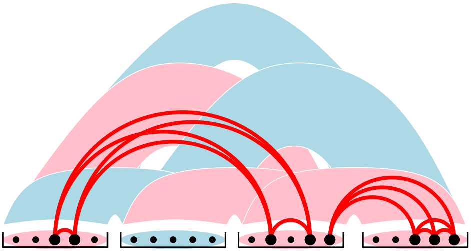

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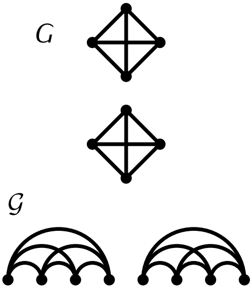

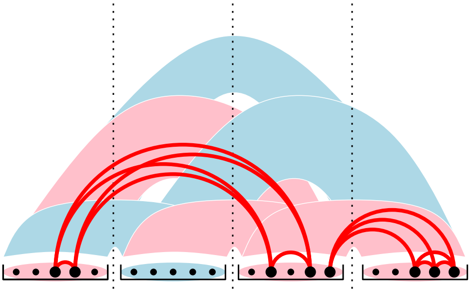

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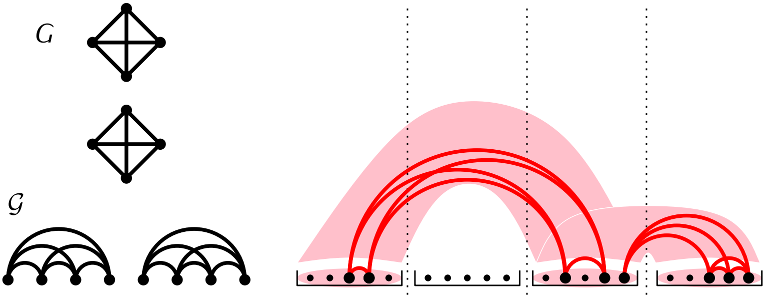

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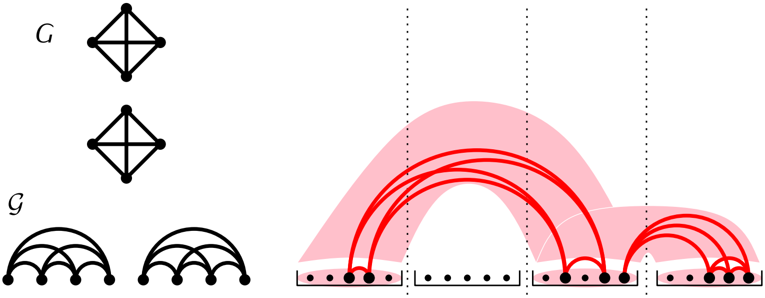

- We use an estimate by Bender and Canfield and by Wormald for the number of d-regular graphs on n vertices.

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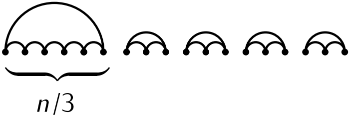

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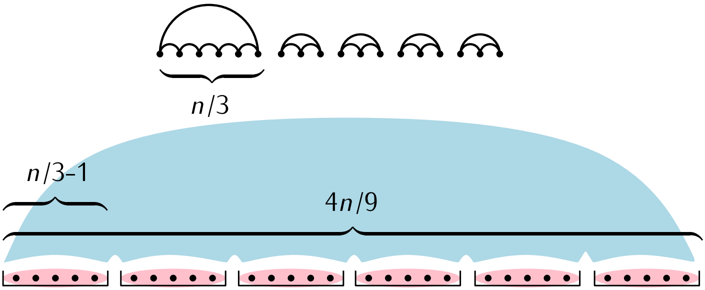

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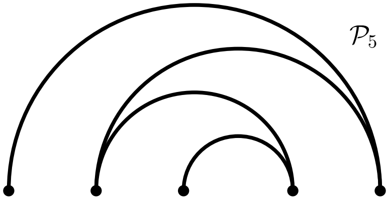

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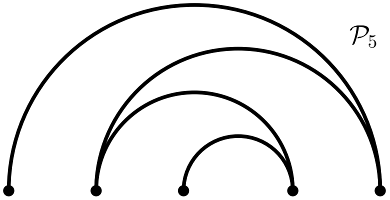

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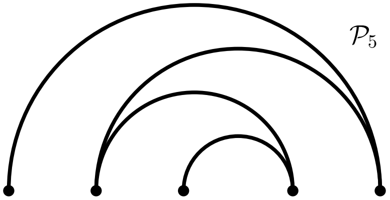

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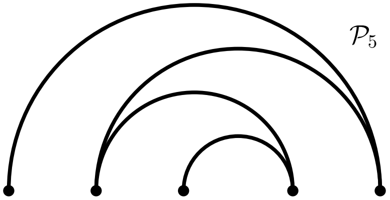

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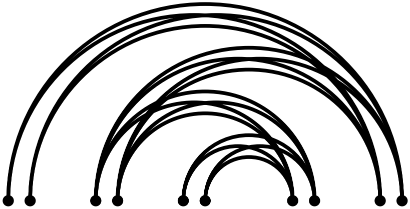

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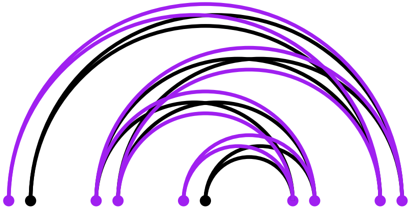

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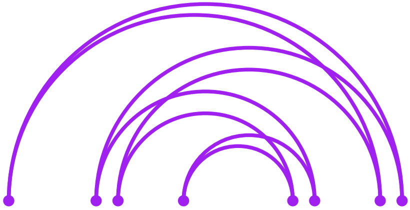

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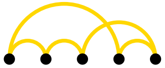

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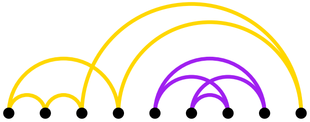

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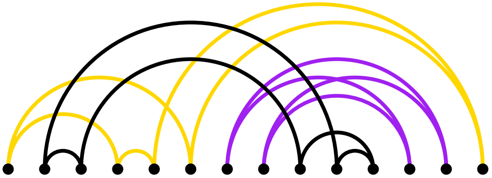

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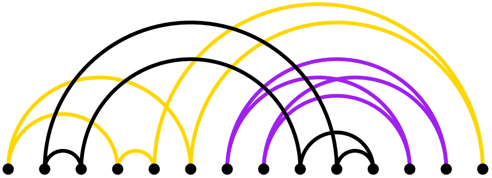

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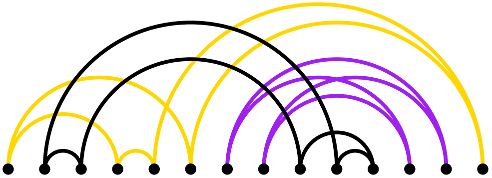

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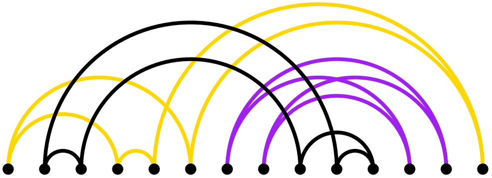

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Thank you.