On ordered Ramsey numbers of bounded-degree graphs

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Ramsey theory

Ramsey's theorem for graphs

For every graph $G$ there is an integer $N = N(G)$ such that every 2-coloring of the edges of $K_N$ contains a monochromatic copy of $G$.

Ramsey number $R(G)$ of $G$ is the smallest such $N$.

Example: $R(C_4) = 6$
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Ordered Ramsey numbers

An ordered graph $G$ is a pair $(G, \prec)$ where $G$ is a graph and $\prec$ is a total ordering of its vertices. $(H, \prec_1)$ is an ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.

The ordered Ramsey number $R(G)$ of an ordered graph $G$ is the least number $N$ such that every 2-coloring of edges of $K_N$ contains a monochromatic copy of $G$ as an ordered subgraph.
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![Diagram of ordered Ramsey number concept]
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Bounded-degree graphs

We consider graphs with the maximum degree bounded by a constant. There is a substantial difference between ordered and unordered case.

Theorem (Chvátal, Rödl, Szemerédi, Trotter, 1983)
Every graph $G$ on $n$ vertices with bounded maximum degree satisfies $R(G) \leq O(n)$.

Theorem (B., Cibulka, Král, Kynčl and Conlon, Fox, Lee, Sudakov, 2014)
There are arbitrarily large ordered matchings $M_n$ on $n$ vertices such that $R(M_n) \geq n\Omega(\log n \log \log n)$.

Conlon et al. showed that this holds for almost every ordered matching.
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There are arbitrarily large ordered matchings $\mathcal{M}_n$ on $n$ vertices such that

$$\overline{R}(\mathcal{M}_n) \geq n^{\Omega\left(\frac{\log n}{\log \log n}\right)}.$$
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Smallest ordered Ramsey numbers

There are \( n \)-vertex ordered matchings \( M \) with \( R(M) \) linear in \( n \).

Which orderings have asymptotically smallest ordered Ramsey numbers?

When can we attain linear ordered Ramsey numbers?

Problem (Conlon, Fox, Lee, Sudakov, 2014)

Do random 3-regular graphs have superlinear ordered Ramsey numbers for all orderings?
Smallest ordered Ramsey numbers

- There are $n$-vertex ordered matchings $\mathcal{M}$ with $\overline{R}(\mathcal{M})$ linear in $n$. 

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$\overline{R}(\mathcal{M}_1), \overline{R}(\mathcal{M}_2) \leq 2n - 2$
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Superlinear lower bound

We give a positive answer to the problem of Conlon et al. \( R(G) \) is the minimum of \( R(G) \) over all orderings \( G \) of \( G \).

**Theorem**

For every \( d \geq 3 \), almost every \( d \)-regular graph \( G \) on \( n \) vertices satisfies

\[
\min R(G) \geq \frac{n}{3} - \frac{1}{d} 4 \log n \log \log n.
\]

For \( 3 \)-regular graphs, we obtain \( \min R(G) \geq \frac{n}{7} 4 \log n \log \log n \).
Superlinear lower bound

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Superlinear lower bound

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For every \( d \geq 3 \), almost every \( d \)-regular graph \( G \) on \( n \) vertices satisfies

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\min\overline{R}(G) \geq \frac{n^{3/2-1/d}}{4 \log n \log \log n}.
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**Theorem**

For every $d \geq 3$, almost every $d$-regular graph $G$ on $n$ vertices satisfies

$$\min-\overline{R}(G) \geq \frac{n^{3/2 - 1/d}}{4 \log n \log \log n}.$$ 

For 3-regular graphs, we obtain $\min-\overline{R}(G) \geq \frac{n^{7/6}}{4 \log n \log \log n}$. 
Sketch of the proof

For $d \geq 3$, let $G$ be a $d$-regular graph on $n$ vertices.

Key lemma: Almost every such $G$ satisfies the following: for every partition of $V(G)$ into few sets $X_1, \ldots, X_t$, each of size at most $s$, there are many pairs $(X_i, X_j)$ with an edge between them. Here, $t = \frac{n}{2 \log n \log \log n}$ and $s = \frac{n}{2} - \frac{1}{2d}$. We use an estimate by Bender and Canfield and by Wormald for the number of $d$-regular graphs on $n$ vertices.
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![Graph $G$](image)
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![Diagram of a $d$-regular graph](image)
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![Diagram of graphs](image-url)
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We use an estimate by Bender and Canfield and by Wormald for the number of $d$-regular graphs on $n$ vertices.
2-regular graphs?
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- Each \( n \)-vertex 1-regular graph has an ordering \( \mathcal{M} \) with \( \bar{R}(\mathcal{M}) \) linear in \( n \).
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- Each $n$-vertex 1-regular graph has an ordering $\mathcal{M}$ with $\overline{R}(\mathcal{M})$ linear in $n$.
- No longer true for $d$-regular graphs with $d \geq 3$. 
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- How about 2-regular graphs?
2-regular graphs

Theorem

Every graph $G$ with $n$ vertices and with maximum degree at most two satisfies $\min R(G) \leq O(n)$.

First, for every $n$, we find an ordering $C_n$ of $C_n$ with $R(C_n)$ linear in $n$.

Second, we find an ordering of a disjoint union $G$ of these ordered cycles with linear $R(G)$. Placing cycles sequentially does not work.
Theorem

Every graph $G$ with $n$ vertices and with maximum degree at most two satisfies

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- Placing cycles sequentially does not work.
Sketch of the proof

Linear ordering of cycles are based on alternating paths $P_n$. B., Cibulka, Král, Kyncl showed $R(P_n) \leq O(n)$.

Lemma

For every $\varepsilon > 0$ and every $n \in \mathbb{N}$, every ordered graph on $N \geq n/\varepsilon$ vertices with at least $\varepsilon N^2$ edges contains $P_n$ as an ordered subgraph.

We use this result to "blow-up" $P_n$ and obtain linear orderings of cycles.
Sketch of the proof I

- Linear ordering of cycles are based on alternating paths $\mathcal{P}_n$. 

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Orderings of disjoint union of cycles are constructed as follows. For bipartite 2-regular graphs we obtain a stronger Turán-type result.

Theorem

For each ε > 0, there is C(ε) such that, for every n ∈ \mathbb{N}, every bipartite graph G on n vertices with maximum degree 2 admits an ordering G of G that is contained in every ordered graph with N ≥ C(ε)n vertices and with at least εN^2 edges.

No longer true if G contains an odd cycle.
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Theorem

For each $\varepsilon > 0$, there is $C(\varepsilon)$ such that, for every $n \in \mathbb{N}$, every bipartite graph $G$ on $n$ vertices with maximum degree 2 admits an ordering of $G$ that is contained in every ordered graph with $N \geq C(\varepsilon)n$ vertices and with at least $\varepsilon N^2$ edges.

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- No longer true if $\mathcal{G}$ contains an odd cycle.
Final remarks

A result of Conlon, Fox, Lee, Sudakov gives a polynomial upper bound.

Corollary

Every graph $G$ with $n$ vertices and with maximum degree $d$ satisfies

$$\min R(G) \leq O(n(d+1)\lceil \log(d+1) \rceil + 1).$$

The upper and lower bounds for $\min R(G)$ are far apart.

Problem

Improve the upper and lower bounds on $\min R(G)$ for 3-regular graphs $G$.

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**Corollary**

Every graph $G$ with $n$ vertices and with maximum degree $d$ satisfies

$$\min-\overline{R}(G) \leq O(n^{(d+1)[\log(d+1)]+1}).$$
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Every graph $G$ with $n$ vertices and with maximum degree $d$ satisfies

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Improve the upper and lower bounds on $\min\overline{\chi}(G)$ for 3-regular graphs $G$.

Thank you.