On off-diagonal ordered Ramsey numbers of nested matchings

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September 4, 2021



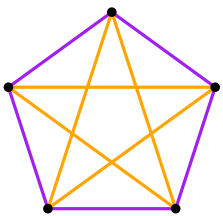
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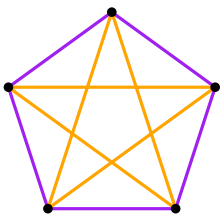
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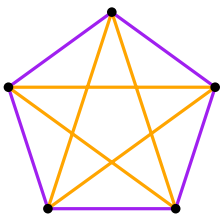
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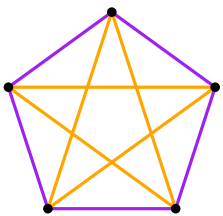
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- Ramsey numbers of bounded-degree graphs are at most linear.

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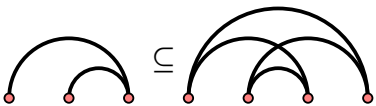
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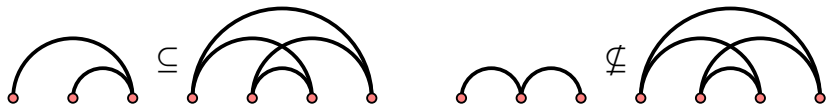
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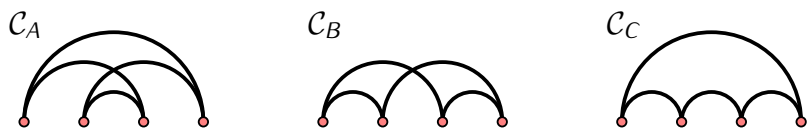
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- Theorem 1 is in sharp contrast with the case for unordered graphs.

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- The proof of Theorem 3 is based on nested matchings.

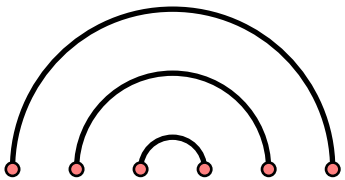
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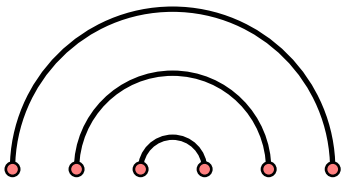
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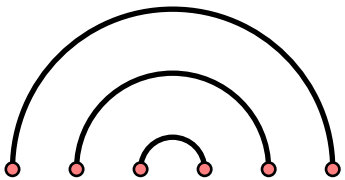
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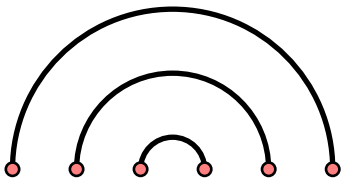
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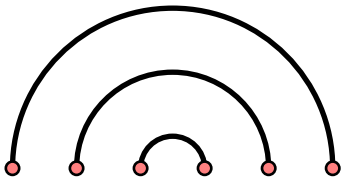
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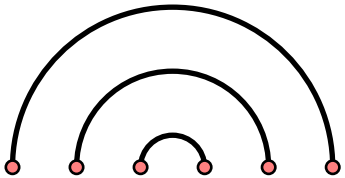
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Conjecture 1 (Rohatgi, 2018)

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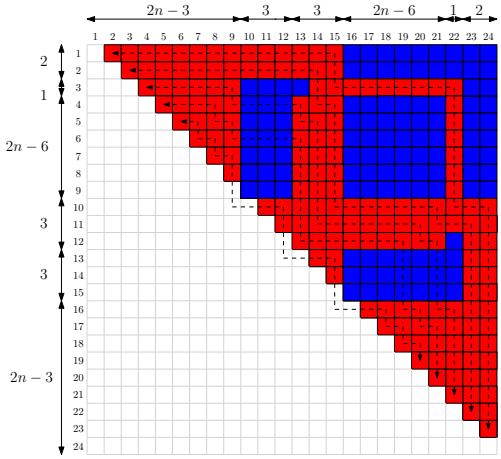
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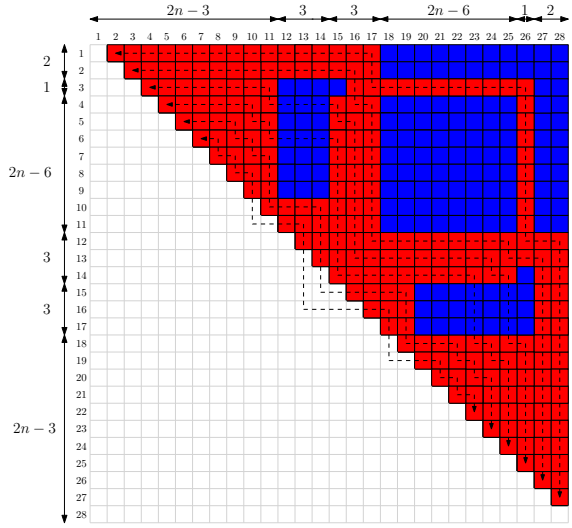
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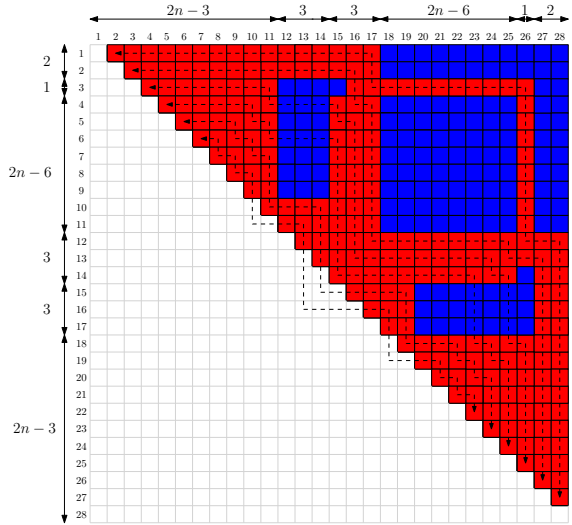
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- The idea was based on colorings found by computer.

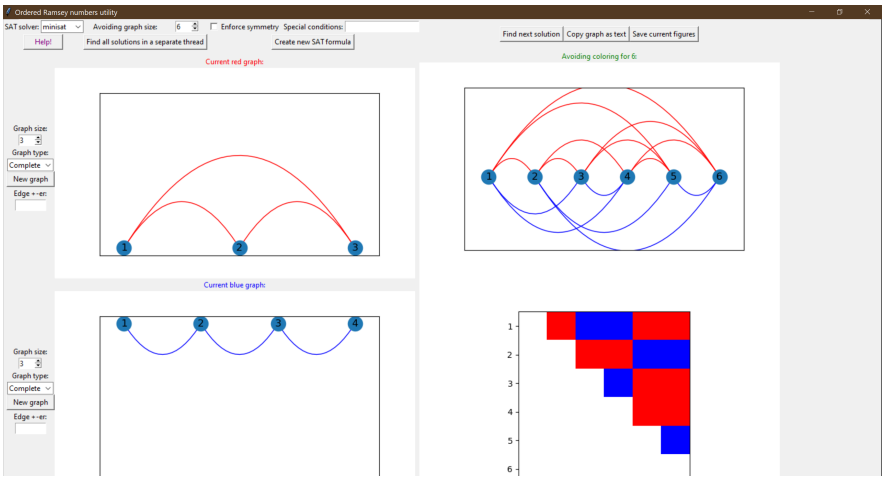
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Corollary 1 (B., Poljak, 2021+)

For every $k \geq 3$, the maximum chromatic number of k-queue graphs is at least $2k + 2$.

Ramsey goodness

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- What ordered graphs are good?

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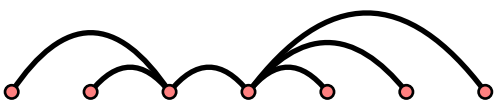
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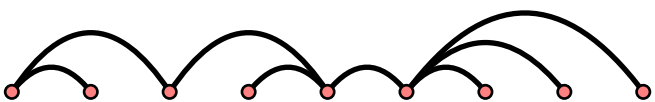
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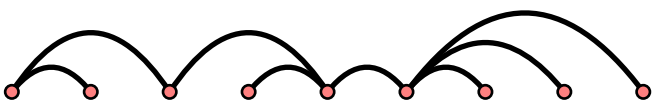
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Theorem 6 (B., Poljak, 2021+)

All monotone caterpillars are good.

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Thank you.