Induced Ramsey-type results and binary predicates for point sets

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Introduction

Let $P$ and $Q$ be finite sets of points in $\mathbb{R}^2$ in general position.

Let $(X)_p$ be the set of all ordered $p$-tuples of distinct elements from $X$.

We use $\Delta_P : (P)_3 \to \{-, +\}$ to denote the function that assigns an orientation to every triple from $(P)_3$.

The sets $P$ and $Q$ have the same order type if there is a bijection $f : P \to Q$ such that every $T \in (P)_3$ has the same orientation as $f(T)$.
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Ramsey point sets

For $k, p \in \mathbb{N}$, a point set $Q$ is $(k, p)$-Ramsey if there is a point set $P$ such that for every $k$-coloring of $(P, p)$ there is a subset of $P$ that has monochromatic $p$-tuples and has the same order type as $Q$.

Which point sets are $(k, p)$-Ramsey?

Known results (Neˇsetˇril and Valtr, 1994–98):

For $k \in \mathbb{N}$, all point sets are $(k, 1)$-Ramsey.

If $k, p \geq 2$, then not all point sets are $(k, p)$-Ramsey.

For $k \in \mathbb{N}$, the non-convex 4-tuple is $(k, 2)$-Ramsey.
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Ordered Ramsey point sets

We introduce a new family of \((k,2)\)-Ramsey point sets. To do so, we first introduce an ordered variant of \((k,p)\)-Ramsey sets. Point sets \(P = \{p_1, \ldots, p_n\}\) and \(Q = \{q_1, \ldots, q_n\}\) ordered by increasing \(x\)-coordinate have the same signature, if \(\Delta P(p_i, p_j, p_k) = \Delta Q(q_i, q_j, q_k)\) for all \(1 \leq i < j < k \leq n\).

Distinguishing point sets by signatures is finer than by order types. A point set \(Q\) is ordered \((k,p)\)-Ramsey if there is a point set \(P\) such that for every \(k\)-coloring of \((P)p\) there is a subset of \(P\) that has monochromatic \(p\)-tuples and has the same signature as \(Q\). If a point set is ordered \((k,p)\)-Ramsey, then it is \((k,p)\)-Ramsey.
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![Diagram showing same order type, distinct signatures.]

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\[
P = \{p_1, p_2, p_3, p_4, p_5\}
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- A point set \(Q\) is ordered \((k, p)\)-Ramsey if there is a point set \(P\) such that for every \(k\)-coloring of \(\binom{P}{p}\) there is a subset of \(P\) that has monochromatic \(p\)-tuples and has the same signature as \(Q\).
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- If a point set is ordered \((k, p)\)-Ramsey, then it is \((k, p)\)-Ramsey.
Decomposable sets are ordered Ramsey

A point set $P$ is decomposable if $|P| = 1$ or if $P$ admits the following partition into non-empty decomposable sets $P_1$ and $P_2$:

Theorem 1
For every $k \in \mathbb{N}$, every decomposable set is ordered $(k, 2)$-Ramsey.

For each $k \in \mathbb{N}$, all point sets are ordered $(k, 1)$-Ramsey.

For $k \geq 2$ and $p \geq 3$, $(k, p)$-Ramsey sets are exactly sets in convex position and ordered $(k, p)$-Ramsey sets are exactly caps and cups.

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- For each \( k \in \mathbb{N} \), all point sets are ordered \((k, 1)\)-Ramsey.
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![Partition of a point set into non-empty decomposable sets](image)

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Point-set predicates

Let $P$ be the set of all finite point sets in the plane in general position. For $t \in \mathbb{N}$ and a finite set $Z$, a $t$-ary point-set predicate with codomain $Z$ is a collection $\Gamma = \{ \Gamma_P : P \in P \}$, where $\Gamma_P : (P)^t \to Z$.

Example: ternary predicate $\Delta = \{ \Delta_P : P \in P \}$ with codomain $\{-, +\}$.

We say that $\Gamma$ encodes the order types if whenever there is a bijection $f : P \to Q$ such that $\Gamma_P(p_1, \ldots, p_t) = \Gamma_Q(f(p_1), \ldots, f(p_t))$ for every $(p_1, \ldots, p_t) \in (P)^t$, then $P$ and $Q$ have the same order type via $f$.

For $n \in \mathbb{N}$, there are $2^{\Theta(n^3)}$ ternary functions $f : ([n])^3 \to \{-, +\}$, but only $2^{\Theta(n \log n)}$ order types of point sets of size $n$. Is the encoding by $\Delta$ effective? Is it possible to use a binary predicate?
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- Is the encoding by $\Delta$ effective? Is it possible to use a binary predicate?
Locally consistent predicates

A binary predicate that encodes the order types exists. (Felsner, 1997). However, unlike $\Delta$, this predicate does not behave locally.

Is there a binary predicate that encodes order types and behaves locally?

A binary predicate $\Gamma$ is locally consistent on $P \in \mathcal{P}$ if, for any distinct subsets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ of $P$, having $\Gamma_P(a_i, a_j) = \Gamma_P(b_i, b_j)$ for every $(i, j) \in \{3\}^2$ implies $\Delta_P(a_1, a_2, a_3) = \Delta_P(b_1, b_2, b_3)$.

Theorem 2
For every finite set $Z$, there is a point set $P = P(Z)$ such that no binary predicate with codomain $Z$ is locally consistent on $P$.

The proof is based on Theorem 1.
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- A binary predicate $\Gamma$ is locally consistent on $P \in \mathcal{P}$ if, for any distinct subsets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ of $P$, having $\Gamma_P(a_i, a_j) = \Gamma_P(b_i, b_j)$ for every $(i, j) \in ([3])_2$ implies $\Delta_P(a_1, a_2, a_3) = \Delta_P(b_1, b_2, b_3)$.

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For every finite set $Z$, there is a point set $P = P(Z)$ such that no binary predicate with codomain $Z$ is locally consistent on $P$. 
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What can we encode with locally consistent predicates?

Codomains of size only 2 are already sufficient to encode exponentially many order types of point sets of size $n$ for every $n \in \mathbb{N}$.

**Proposition 1**
The order types of wheel sets can be encoded with a binary predicate $\Phi$ with codomain $\{-, +\}$ such that $\Phi$ is locally consistent on all wheel sets.
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The order types of wheel sets can be encoded with a binary predicate $\Phi$ with codomain $\{-, +\}$ such that $\Phi$ is locally consistent on all wheel sets.
Let $h(k)$ be the largest integer such that there is a binary predicate with codomain of size $k$ that is locally consistent on all point sets of size $h(k)$ and that encodes their order types. By Theorem 2, $h(k)$ is finite for every $k \in \mathbb{N}$.

We show a superlinear lower bound on $h(k)$.

Proposition 2

We have $h(k) \geq c \cdot k^{3/2}$ for some constant $c > 0$.

The proof is based on Lovász’s Local Lemma and the fact that there are $2^{O(k \log k)}$ order types of point sets of size $k$.

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What is the growth rate of $h(k)$?
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Recall that all point sets are ordered $(k, 1)$-Ramsey, but not ordered $(k, 2)$-Ramsey. Ordered $(k, p)$-Ramsey sets for $p \geq 3$ are caps and cups.

Signatures can be defined also for generalized point sets, where lines are replaced by pseudolines. We can thus introduce ordered $(k, p)$-Ramsey generalized point sets.

For $p = 1$ and $p \geq 3$, analogous results hold for generalized point sets. However, the case $p = 2$ is wide open.

Question 2

Is there a generalized point set that is not ordered $(2, 2)$-Ramsey?

Generalized point sets correspond to ordered 3-uniform hypergraphs with 8 forbidden induced sub-hypergraphs. However, known structural results do not seem to apply here.

All ordered 3-uniform hypergraphs are ordered $(2, 2)$-Ramsey (Neˇ setˇ ril and R¨ odl, 1983).

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