Holes and islands in random point sets

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Preliminaries

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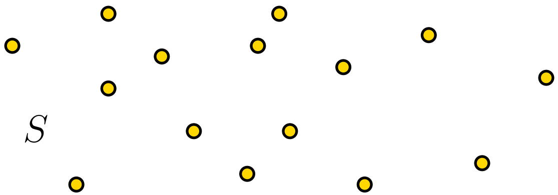

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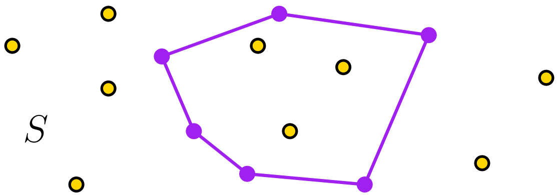

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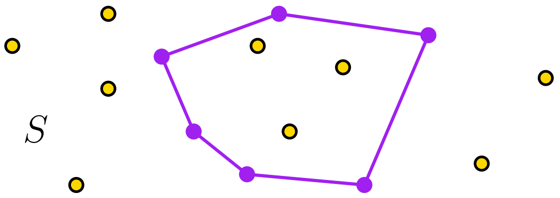

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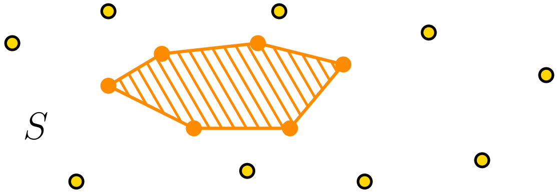

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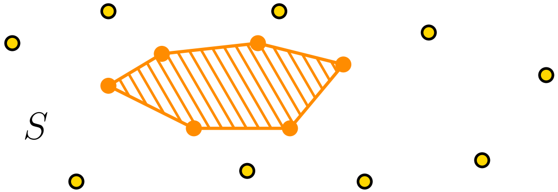

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- Every set of 3 points contains a 3-hole. Also, 5 points $\rightarrow$ 4-hole and 10 points $\rightarrow$ 5-hole (Harborth, 1978).

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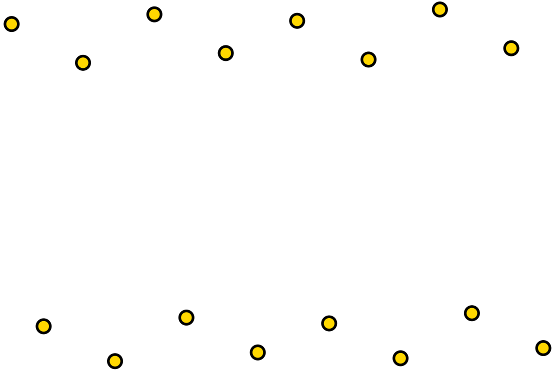

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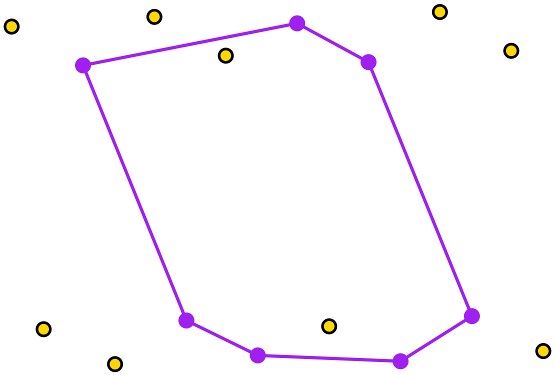

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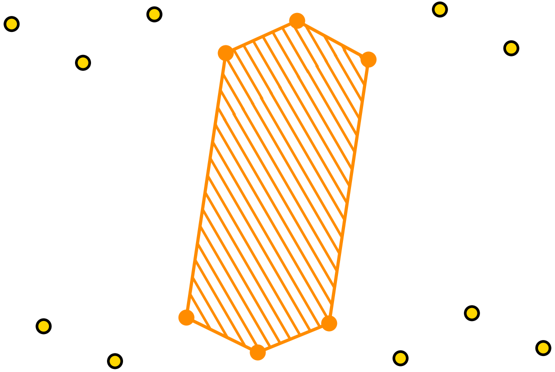

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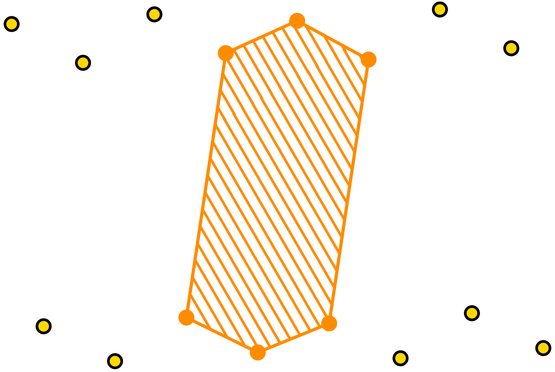

- Every sufficiently large point set in general position contains a 6-hole (Gerken, 2008 and Nicolás, 2007).

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- The minimum number of $(d+1)$-holes (empty simplices) in an n-point set in $\mathbb{R}^d$ is in $\Theta(n^d)$ (Bárány, Füredi, 1987).

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- Bárány and Füredi showed that

$$EH^K_{d,d+1}(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}.$$

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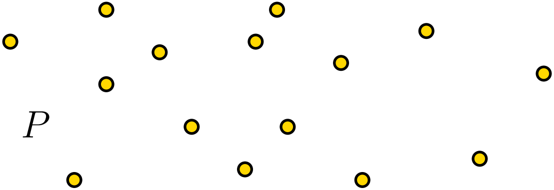

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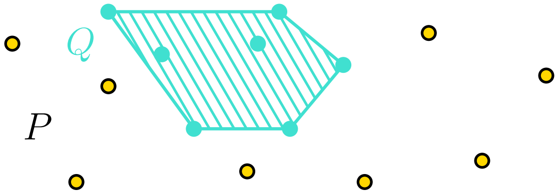

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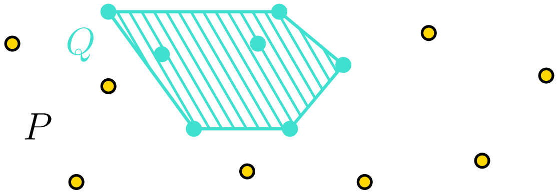

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$$2^{d-1} \cdot \left(2d^{2d-1}\binom{k}{\lfloor d/2 \rfloor}\right)^{k-d-1} \cdot (k-d) \cdot \frac{n(n-1)\cdots(n-k+2)}{(n-k+1)^{k-d-1}} \in O(n^d).$$

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  - For 4-holes in the plane, we get $EH^K_{2,4}(n) \leq 12n^2 + o(n^2)$.

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Theorem 4

Let $d \geq 2$ and k be fixed positive integers. Then every d-dimensional Horton set H with n points contains at least $\Omega(n^{\min\{2^{d-1},k\}})$ k-islands in H. If $k \leq 3 \cdot 2^{d-1}$, then H even contains at least $\Omega(n^{\min\{2^{d-1},k\}})$ k-holes in H.

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- We draw the points in the canonical order and estimate the probability in every step.
- We start by estimating the probability that the vertices p_1, p_2, p_3 of Δ with a points $p_4, \ldots, p_{3+a}$ inside Δ form an island in S.

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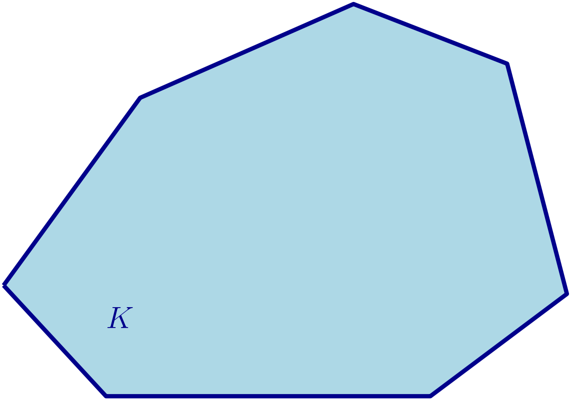

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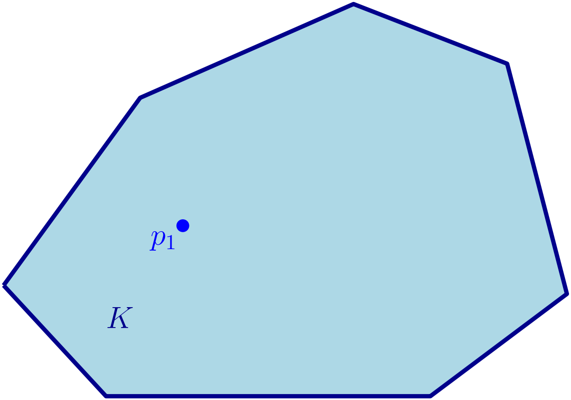

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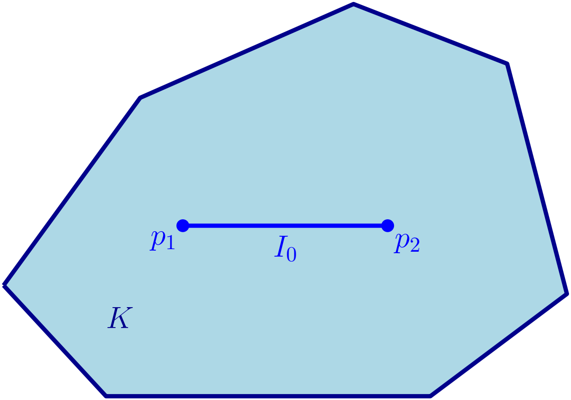

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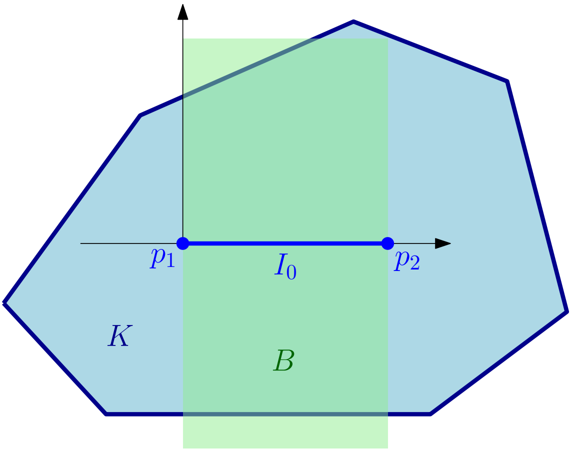

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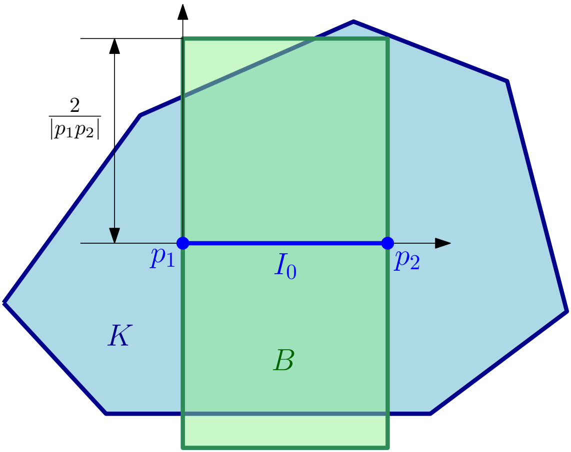

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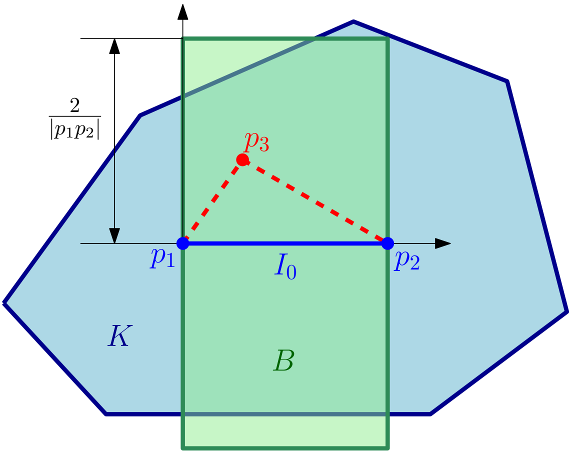

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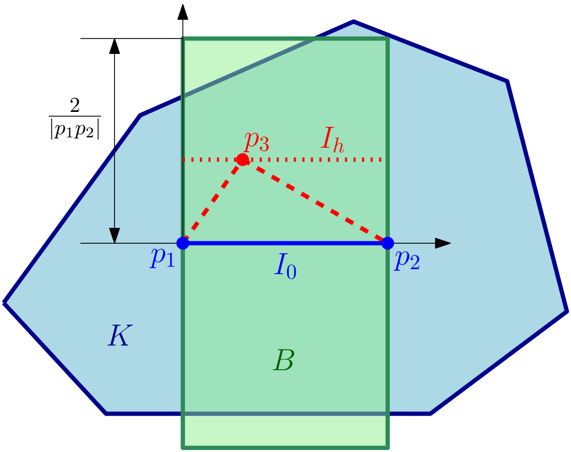

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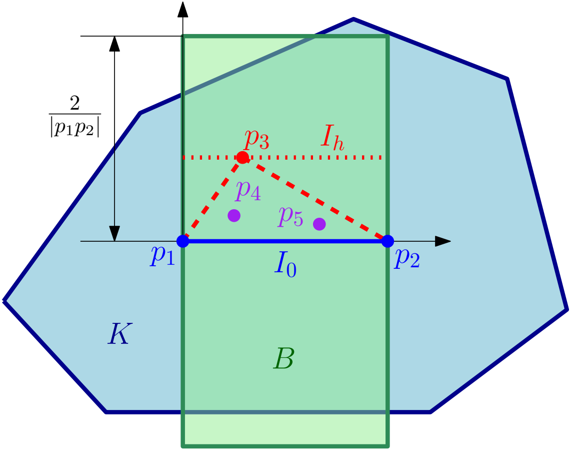

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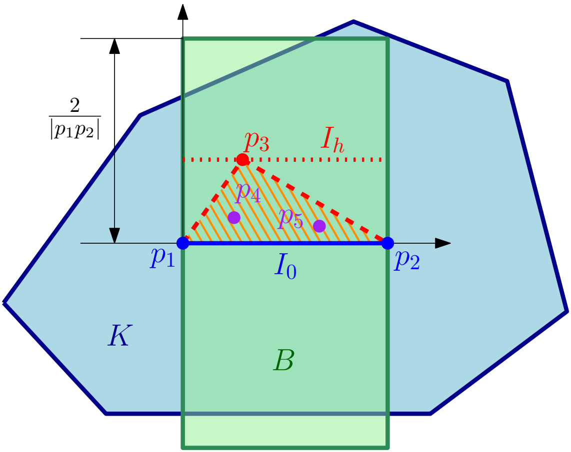

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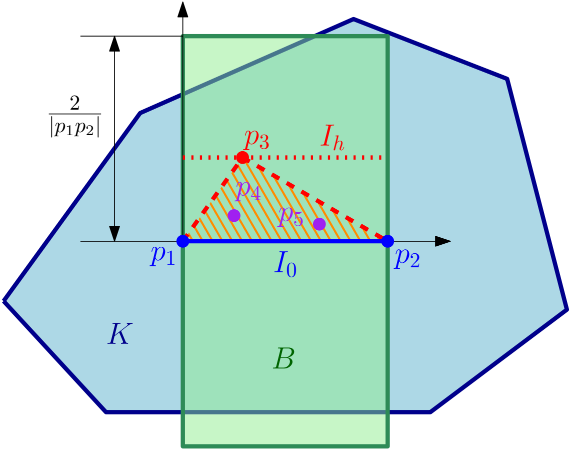

- The probability that the drawn points in Δ are an island in S is at most

$$\int_{-2/|I_0|}^{2/|I_0|} \frac{|I_h \cap K|}{a! \cdot (k-a-3)!} \cdot \left(\frac{|I_0| \cdot |h|}{2}\right)^{a} \cdot \left(1 - \frac{|I_0| \cdot |h|}{2}\right)^{n-a-3} \mathrm{d}h.$$

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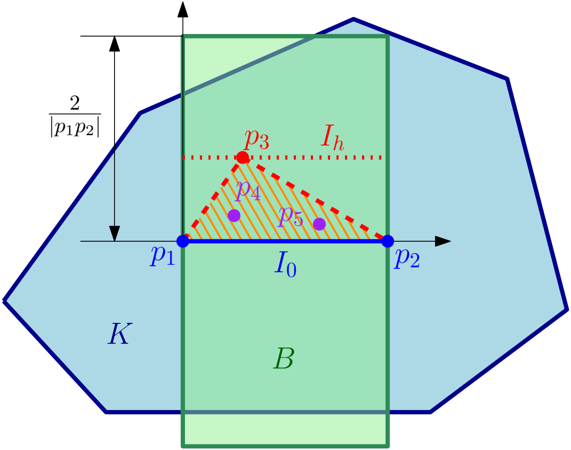

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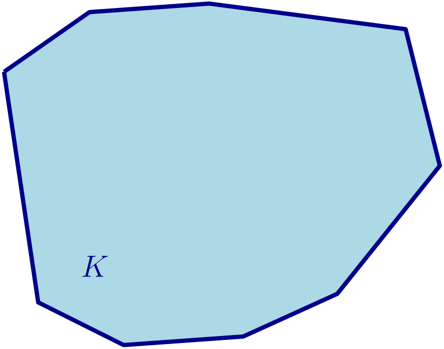

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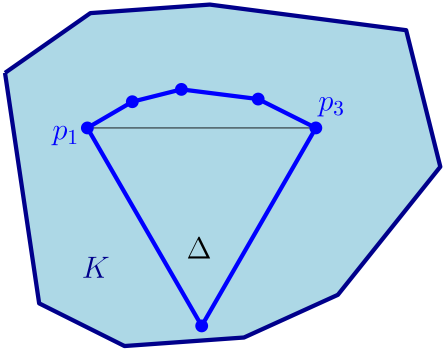

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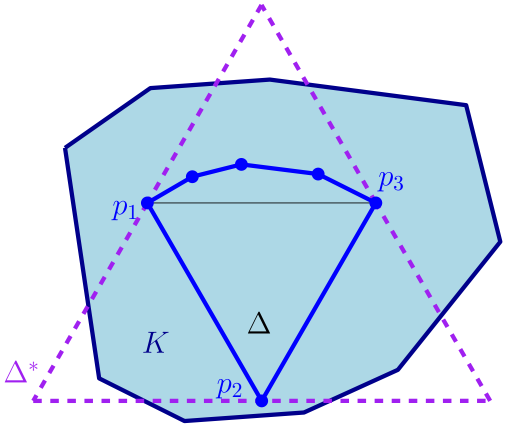

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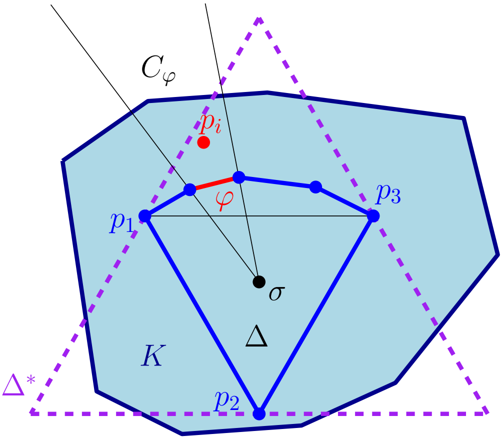

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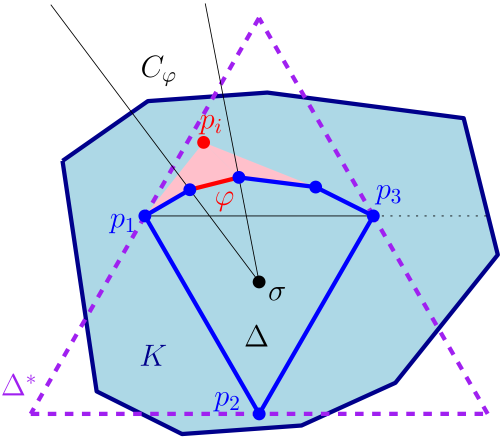

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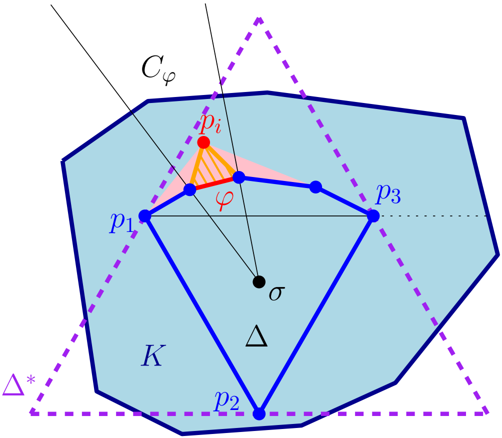

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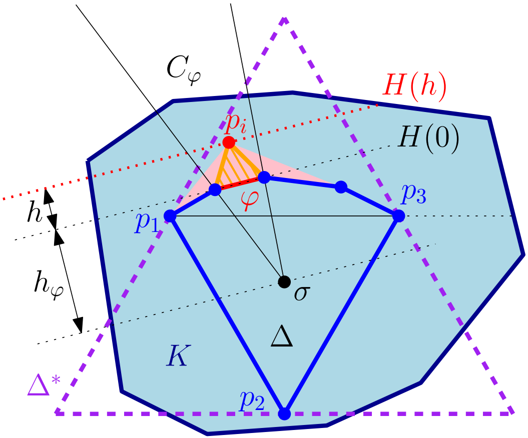

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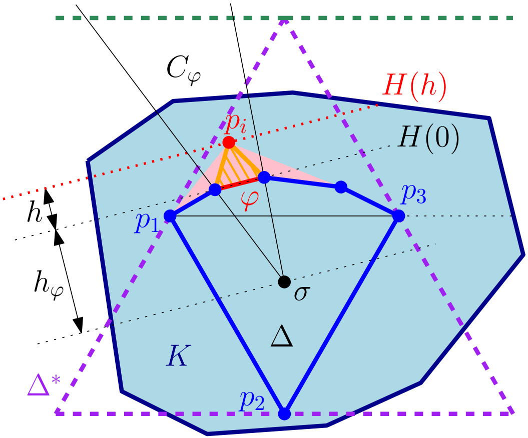

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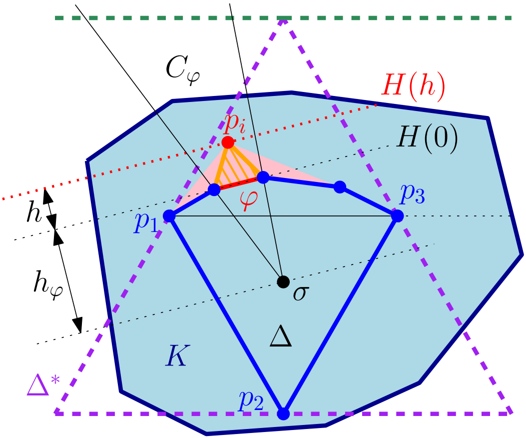

$$\Pr[E_{a,i} \mid E_{a,i-1}] \le \frac{16k}{n - i + 1}.$$

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- The expected number of k-islands in S is then at most

$$n(n-1)\cdots(n-k+1) \cdot P/2.$$

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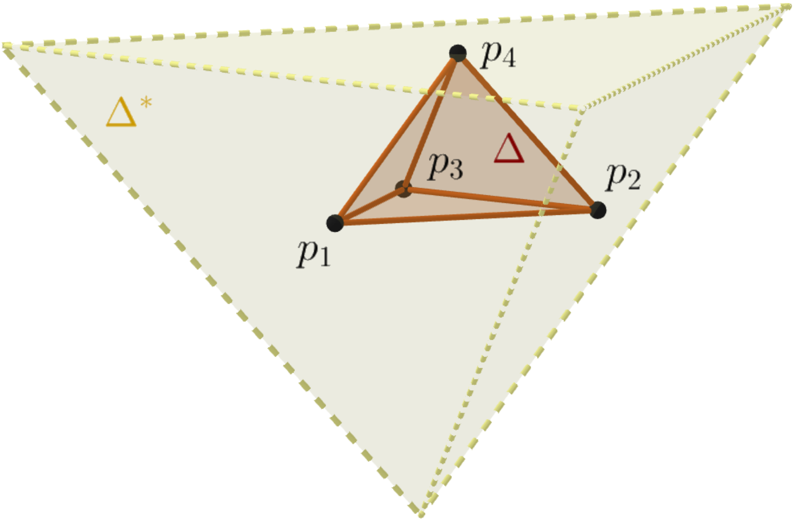

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Thank you for your attention.