Holes and islands in random point sets

Martin Balko, Manfred Scheucher, and Pavel Valtr

Faculty of Mathematics and Physics,
Charles University
Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains $k$ points in convex position.

- A $k$-hole in a point set $S$ is a $k$-tuple of points from $S$ in convex position with no points of $S$ in the interior of their convex hull.
- Every set of 3 points contains a 3-hole. Also, 5 points $\rightarrow$ 4-hole and 10 points $\rightarrow$ 5-hole (Harborth, 1978).
Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains $k$ points in convex position.
Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains $k$ points in convex position.
Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains $k$ points in convex position.
Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains $k$ points in convex position.

A $k$-hole in a point set $S$ is a $k$-tuple of points from $S$ in convex position with no points of $S$ in the interior of their convex hull.
Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains $k$ points in convex position.

- A $k$-hole in a point set $S$ is a $k$-tuple of points from $S$ in convex position with no points of $S$ in the interior of their convex hull.
Theorem (Erdős, Szekeres, 1935)
For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains $k$ points in convex position.

- A $k$-hole in a point set $S$ is a $k$-tuple of points from $S$ in convex position with no points of $S$ in the interior of their convex hull.
- Every set of 3 points contains a 3-hole. Also, 5 points $\rightarrow$ 4-hole and 10 points $\rightarrow$ 5-hole (Harborth, 1978).
Sets with no large holes

• Erdős, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a $k$-hole?

• No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).

• Every sufficiently large point set in general position contains a 6-hole (Gerken, 2008 and Nicolás, 2007).
Sets with no large holes

- Erdős, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a $k$-hole?

No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).

Every sufficiently large point set in general position contains a 6-hole (Gerken, 2008 and Nicolás, 2007).
Sets with no large holes

- **Erdős**, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a $k$-hole?  
- No. There are arbitrarily large point sets with no 7-hole (**Horton**, 1983).
Sets with no large holes

- Erdős, 1978: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a $k$-hole?
- No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).
Sets with no large holes

- **Erdős, 1978**: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a $k$-hole?
- No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).
Sets with no large holes

- **Erdős, 1978**: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a $k$-hole?
- **No.** There are arbitrarily large point sets with no 7-hole (**Horton, 1983**).
Sets with no large holes

- **Erdős, 1978**: For every $k \in \mathbb{N}$, does every large enough point set in general position contain a $k$-hole?
- No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).
- Every sufficiently large point set in general position contains a 6-hole (Gerken, 2008 and Nicolás, 2007).
Counting $k$-holes

• Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.

• How many $k$-holes do we always have?

• Let $h_k(n)$ be the minimum number of $k$-holes among all sets of $n$ points in the plane in general position.

• The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
  - $h_5(n)$ is in $\Omega(n \log^4 n)$ and $O(n^2)$.
  - $h_6(n)$ is in $\Omega(n)$ and $O(n^2)$.
  - $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).

• Holes were also considered in higher dimensions.
  - There are $d$-dimensional Horton sets not containing $k$-holes for sufficiently large $k = k(d)$ (Valtr, 1992).
  - The minimum number of $(d+1)$-holes (empty simplices) in an $n$-point set in $\mathbb{R}^d$ is in $\Theta(n^d)$ (Bárany, Füredi, 1987).
Counting $k$-holes

- Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$. 

- Holes were also considered in higher dimensions.
  - There are $d$-dimensional Horton sets not containing $k$-holes for sufficiently large $k = k(d)$ (Valtr, 1992).
  - The minimum number of $(d+1)$-holes (empty simplices) in an $n$-point set in $\mathbb{R}^d$ is in $\Theta(n^d)$ (Bárány, Füredi, 1987).
Counting $k$-holes

- Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.
- How many $k$-holes do we always have?
Counting $k$-holes

- Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.
- How many $k$-holes do we always have?
- Let $h_k(n)$ be the minimum number of $k$-holes among all sets of $n$ points in the plane in general position.
Counting $k$-holes

- Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.
- How many $k$-holes do we always have?
- Let $h_k(n)$ be the minimum number of $k$-holes among all sets of $n$ points in the plane in general position.
- The following bounds are known:
Counting $k$-holes

- Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.
- How many $k$-holes do we always have?
- Let $h_k(n)$ be the minimum number of $k$-holes among all sets of $n$ points in the plane in general position.
- The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$. 
Counting \( k \)-holes

- Every sufficiently large set of points in general position contains a \( k \)-hole for \( k \in \{3, 4, 5, 6\} \).
- How many \( k \)-holes do we always have?
- Let \( h_k(n) \) be the minimum number of \( k \)-holes among all sets of \( n \) points in the plane in general position.
- The following bounds are known:
  - \( h_3(n) \) and \( h_4(n) \) are in \( \Theta(n^2) \).
  - \( h_5(n) \) is in \( \Omega(n \log^{4/5} n) \) and \( O(n^2) \).
Counting \( k \)-holes

- Every sufficiently large set of points in general position contains a \( k \)-hole for \( k \in \{3, 4, 5, 6\} \).
- How many \( k \)-holes do we always have?
- Let \( h_k(n) \) be the minimum number of \( k \)-holes among all sets of \( n \) points in the plane in general position.
- The following bounds are known:
  - \( h_3(n) \) and \( h_4(n) \) are in \( \Theta(n^2) \).
  - \( h_5(n) \) is in \( \Omega(n \log^{4/5} n) \) and \( O(n^2) \).
  - \( h_6(n) \) is in \( \Omega(n) \) and \( O(n^2) \).
Counting $k$-holes

- Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.
- How many $k$-holes do we always have?
- Let $h_k(n)$ be the minimum number of $k$-holes among all sets of $n$ points in the plane in general position.
- The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
  - $h_5(n)$ is in $\Omega(n \log^{4/5} n)$ and $O(n^2)$.
  - $h_6(n)$ is in $\Omega(n)$ and $O(n^2)$.
  - $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).
Counting \( k \)-holes

- Every sufficiently large set of points in general position contains a \( k \)-hole for \( k \in \{3, 4, 5, 6\} \).
- How many \( k \)-holes do we always have?
- Let \( h_k(n) \) be the minimum number of \( k \)-holes among all sets of \( n \) points in the plane in general position.

The following bounds are known:
- \( h_3(n) \) and \( h_4(n) \) are in \( \Theta(n^2) \).
- \( h_5(n) \) is in \( \Omega(n \log^{4/5} n) \) and \( O(n^2) \).
- \( h_6(n) \) is in \( \Omega(n) \) and \( O(n^2) \).
- \( h_k(n) = 0 \) for every \( k \geq 7 \) (Horton, 1983).

- Holes were also considered in higher dimensions.
Counting $k$-holes

- Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.
- How many $k$-holes do we always have?
- Let $h_k(n)$ be the minimum number of $k$-holes among all sets of $n$ points in the plane in general position.
- The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
  - $h_5(n)$ is in $\Omega(n \log^{4/5} n)$ and $O(n^2)$.
  - $h_6(n)$ is in $\Omega(n)$ and $O(n^2)$.
  - $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).
- Holes were also considered in higher dimensions.
- There are $d$-dimensional Horton sets not containing $k$-holes for sufficiently large $k = k(d)$ (Valtr, 1992).
Counting $k$-holes

- Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.
- How many $k$-holes do we always have?
- Let $h_k(n)$ be the minimum number of $k$-holes among all sets of $n$ points in the plane in general position.
- The following bounds are known:
  - $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
  - $h_5(n)$ is in $\Omega(n \log^{4/5} n)$ and $O(n^2)$.
  - $h_6(n)$ is in $\Omega(n)$ and $O(n^2)$.
  - $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).
- Holes were also considered in higher dimensions.
- There are $d$-dimensional Horton sets not containing $k$-holes for sufficiently large $k = k(d)$ (Valtr, 1992).
- The minimum number of $(d + 1)$-holes (empty simplices) in an $n$-point set in $\mathbb{R}^d$ is in $\Theta(n^d)$ (Bárány, Füredi, 1987).
Random point sets

Random point sets give the upper bound $O(n^d)$ on the number of empty simplices.

Let $k$ be a positive integer and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.

Let $E_{K,d,k}(n)$ be the expected number of $k$-holes in sets of $n$ points chosen independently and uniformly at random from $K$.

Bárány and Füredi showed that $E_{K,d,k}(n) \leq (2^d)^{2d^2} \cdot (n^d)$. 
Random point sets

- Random point sets give the upper bound $O(n^d)$ on the number of empty simplices.
Random point sets

- Random point sets give the upper bound $O(n^d)$ on the number of empty simplices.
- Let $k$ be a positive integer and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$. 
- Bárany and Füredi showed that $\mathbb{E}H_{K,d+1}(n) \leq (2d)^{2d} \cdot (n^d)$. 

Random point sets

- Random point sets give the upper bound $O(n^d)$ on the number of empty simplices.
- Let $k$ be a positive integer and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH_{d,k}^K(n)$ be the expected number of $k$-holes in sets of $n$ points chosen independently and uniformly at random from $K$. 
Random point sets

- Random point sets give the upper bound $O(n^d)$ on the number of empty simplices.
- Let $k$ be a positive integer and let $K \subseteq \mathbb{R}^d$ be a convex body of volume $\lambda_d(K) = 1$.
- Let $EH_{d,k}^K(n)$ be the expected number of $k$-holes in sets of $n$ points chosen independently and uniformly at random from $K$.
- Bárány and Füredi showed that

$$EH_{d,d+1}^K(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}.$$
Our results

• We extend previous bounds to larger holes and even to islands.

An island in a point set \( P \) is a subset \( Q \) of \( P \) with \( P \cap \text{conv}(Q) = Q \).

Theorem 1

Let \( d \geq 2 \) and \( k \geq d + 1 \) be integers and let \( K \) be a convex body in \( \mathbb{R}^d \) with \( \lambda_d(K) = 1 \). If \( S \) is a set of \( n \geq k \) points chosen uniformly and independently at random from \( K \), then the expected number of \( k \)-islands in \( S \) is at most

\[
2^{d-1} \cdot \left(2^d - 1\right) \cdot \left(k - d - 1\right) \cdot \left(n - 1\right) \cdot \ldots \cdot \left(n - k + 2\right) \cdot \left(n - k + 1\right)^{k - d - 1} \in O(n^d).
\]
Our results

- We extend previous bounds to larger holes and even to *islands*.
Our results

- We extend previous bounds to larger holes and even to *islands*.
- An *island* in a point set $P$ is a subset $Q$ of $P$ with $P \cap \text{conv}(Q) = Q$.

Theorem 1

Let $d \geq 2$ and $k \geq d + 1$ be integers and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. If $S$ is a set of $n \geq k$ points chosen uniformly and independently at random from $K$, then the expected number of $k$-islands in $S$ is at most $2^{d-1} \cdot \left(2^{d^2-d} \cdot k^{k-d-1} \cdot n \cdot (n-1) \cdot \ldots \cdot (n-k+2) / (n-k+1)^{k-d-1} \right) \in O(n^d)$. 
Our results

• We extend previous bounds to larger holes and even to *islands*.
• An island in a point set $P$ is a subset $Q$ of $P$ with $P \cap \text{conv}(Q) = Q$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{island.png}
\caption{An example of an island in a point set $P$.}
\end{figure}
Our results

- We extend previous bounds to larger holes and even to *islands*.
- An *island* in a point set $P$ is a subset $Q$ of $P$ with $P \cap \text{conv}(Q) = Q$. 

\begin{equation}
\text{Theorem 1}
\end{equation}

Let $d \geq 2$ and $k \geq d + 1$ be integers and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. If $S$ is a set of $n \geq k$ points chosen uniformly and independently at random from $K$, then the expected number of $k$-islands in $S$ is at most

\[
2^{d-1} \cdot \left(2^{\frac{d}{2}} - 1\right)^{k-d-1} \cdot n(n-1) \cdots (n-k+2) \cdot k-d-1 \in O\left(n^d\right).
\]
Our results

- We extend previous bounds to larger holes and even to *islands*.
- An island in a point set $P$ is a subset $Q$ of $P$ with $P \cap \text{conv}(Q) = Q$.

**Theorem 1**

Let $d \geq 2$ and $k \geq d + 1$ be integers and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. If $S$ is a set of $n \geq k$ points chosen uniformly and independently at random from $K$, then the expected number of $k$-islands in $S$ is at most

$$2^{d-1} \cdot \left(2d^{2d-1} \left(\frac{k}{\lfloor d/2 \rfloor}\right)\right)^{k-d-1} \cdot (k-d) \cdot \frac{n(n-1) \cdots (n-k+2)}{(n-k+1)^{k-d-1}} \in O(n^d).$$
Our results II

• The bound from Theorem 1 is asymptotically optimal, but the leading constant can be improved for $k$-holes.

**Theorem 2**

Let $d \geq 2$ and $k \geq d + 1$ be integers and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. If $S$ is a set of $n \geq k$ points chosen uniformly and independently at random from $K$, then the expected number $EH_{K^d,k}(n)$ of $k$-holes is at most

$$2^{d-1} \cdot \left(2^d - 1 \right)^{k-d-1} \cdot \binom{n}{k-1} \cdot \prod_{i=0}^{k-d-1} \binom{n-i}{i} \in O(n^d).$$

• Theorem 2 even gives better bounds than earlier results.

  ◦ For empty simplices in $\mathbb{R}^d$ Theorem 2 gives the estimate
    $$EH_{K^d,d+1}(n) \leq 2^{d-1} \cdot d! \cdot (n^d).$$

  ◦ For 4-holes in the plane, we get
    $$EH_{K^2,4}(n) \leq 12n^2 + o(n^2).$$
Our results II

- The bound from Theorem 1 is asymptotically optimal, but the leading constant can be improved for $k$-holes.
Our results II

- The bound from Theorem 1 is asymptotically optimal, but the leading constant can be improved for $k$-holes.

**Theorem 2**

Let $d \geq 2$ and $k \geq d + 1$ be integers and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. If $S$ is a set of $n \geq k$ points chosen uniformly and independently at random from $K$, then the expected number $EH_{d,k}^K(n)$ of $k$-holes is at most

$$2^{d-1} \cdot \left(2d^{2d-1}\left(\begin{array}{c} k \\ \lfloor d/2 \rfloor \end{array}\right)\right)^{k-d-1} \cdot \frac{n(n-1) \cdots (n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}} \in O(n^d).$$
Our results II

- The bound from Theorem 1 is asymptotically optimal, but the leading constant can be improved for $k$-holes.

**Theorem 2**

Let $d \geq 2$ and $k \geq d + 1$ be integers and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. If $S$ is a set of $n \geq k$ points chosen uniformly and independently at random from $K$, then the expected number $EH_{d,k}^K(n)$ of $k$-holes is at most

$$2^{d-1} \cdot \left(2d^{2d-1} \binom{k}{\lfloor d/2 \rfloor} \right)^{k-d-1} \cdot \frac{n(n-1) \cdots (n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}} \in O(n^d).$$

- Theorem 2 even gives better bounds than earlier results.
Our results II

- The bound from Theorem 1 is asymptotically optimal, but the leading constant can be improved for $k$-holes.

**Theorem 2**

Let $d \geq 2$ and $k \geq d + 1$ be integers and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. If $S$ is a set of $n \geq k$ points chosen uniformly and independently at random from $K$, then the expected number $EH_{d,k}^K(n)$ of $k$-holes is at most

$$2^{d-1} \cdot \left(2d^{2d-1}\left(\begin{array}{c} k \\ \lfloor d/2 \rfloor \end{array}\right)\right)^{k-d-1} \frac{n(n-1) \cdots (n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}} \in O(n^d).$$

- Theorem 2 even gives better bounds than earlier results.
  - For empty simplices in $\mathbb{R}^d$ Theorem 2 gives the estimate
    $$EH_{d,d+1}^K(n) \leq 2^{d-1} \cdot d! \cdot \binom{n}{d}.$$
Our results II

- The bound from Theorem 1 is asymptotically optimal, but the leading constant can be improved for $k$-holes.

**Theorem 2**

Let $d \geq 2$ and $k \geq d + 1$ be integers and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. If $S$ is a set of $n \geq k$ points chosen uniformly and independently at random from $K$, then the expected number $EH_{d,k}^K(n)$ of $k$-holes is at most

$$2^{d-1} \cdot \left(2d^{2d-1} \binom{k}{\lfloor d/2 \rfloor}\right)^{k-d-1} \cdot \frac{n(n-1) \cdots (n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}} \in O(n^d).$$

- Theorem 2 even gives better bounds than earlier results.
  - For empty simplices in $\mathbb{R}^d$ Theorem 2 gives the estimate
    $$EH_{d,d+1}^K(n) \leq 2^{d-1} \cdot d! \cdot \binom{n}{d}.$$
  - For 4-holes in the plane, we get $EH_{2,4}^K(n) \leq 12n^2 + o(n^2)$. 

Our results III

• We cannot have the bound $O(n^d)$ for $k$-islands if $k$ is not fixed.

Theorem 3

Let $d \geq 2$ be an integer and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. Then, for every set $S$ of $n$ points chosen uniformly and independently at random from $K$, the expected number of islands in $S$ is in $\Theta(n^{(d-1)/(d+1)})$.

• Theorem 1 is the first nontrivial bound for $k$-islands in $\mathbb{R}^d$ for $d > 2$.

• In the plane, the $O(n^2)$ bound can be achieved by Horton sets (Fabila-Monroy and Huemer, 2012).

• $d$-dimensional Horton sets with $d > 2$ do not give the $O(n^d)$ bound.

Theorem 4

Let $d \geq 2$ and $k$ be fixed positive integers. Then every $d$-dimensional Horton set $H$ with $n$ points contains at least $\Omega(n \min\{2^{d-1}, k\})$ $k$-islands in $H$. If $k \leq 3 \cdot 2^{d-1}$, then $H$ even contains at least $\Omega(n \min\{2^{d-1}, k\})$ $k$-holes in $H$.
Our results III

- We cannot have the bound \( O(n^d) \) for \( k \)-islands if \( k \) is not fixed.
Our results III

- We cannot have the bound $O(n^d)$ for $k$-islands if $k$ is not fixed.

**Theorem 3**

Let $d \geq 2$ be an integer and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. Then, for every set $S$ of $n$ points chosen uniformly and independently at random from $K$, the expected number of islands in $S$ is in $2^{\Theta(n^{(d-1)/(d+1)})}$. 
Our results III

• We cannot have the bound $O(n^d)$ for $k$-islands if $k$ is not fixed.

**Theorem 3**

Let $d \geq 2$ be an integer and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. Then, for every set $S$ of $n$ points chosen uniformly and independently at random from $K$, the expected number of islands in $S$ is in $2^{\Theta(n^{(d-1)/(d+1)})}$.

• Theorem 1 is the first nontrivial bound for $k$-islands in $\mathbb{R}^d$ for $d > 2$. 

• In the plane, the $O(n^2)$ bound can be achieved by Horton sets (Fabila-Monroy and Huemer, 2012).

• $d$-dimensional Horton sets with $d > 2$ do not give the $O(n^d)$ bound.

**Theorem 4**

Let $d \geq 2$ and $k$ be fixed positive integers. Then every $d$-dimensional Horton set $H$ with $n$ points contains at least $\Omega(n \min\{2^{d-1}, k\})$ $k$-islands in $H$. If $k \leq 3 \cdot 2^{d-1}$, then $H$ even contains at least $\Omega(n \min\{2^{d-1}, k\})$ $k$-holes in $H$. 

Our results III

• We cannot have the bound $O(n^d)$ for $k$-islands if $k$ is not fixed.

Theorem 3

Let $d \geq 2$ be an integer and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. Then, for every set $S$ of $n$ points chosen uniformly and independently at random from $K$, the expected number of islands in $S$ is in $2^{\Theta(n^{(d-1)/(d+1)})}$.

• Theorem 1 is the first nontrivial bound for $k$-islands in $\mathbb{R}^d$ for $d > 2$.

• In the plane, the $O(n^2)$ bound can be achieved by Horton sets (Fabila-Monroy and Huemer, 2012).
Our results III

- We cannot have the bound $O(n^d)$ for $k$-islands if $k$ is not fixed.

**Theorem 3**

Let $d \geq 2$ be an integer and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. Then, for every set $S$ of $n$ points chosen uniformly and independently at random from $K$, the expected number of islands in $S$ is in $2^{\Theta(n^{(d-1)/(d+1)})}$.

- Theorem 1 is the first nontrivial bound for $k$-islands in $\mathbb{R}^d$ for $d > 2$.
- In the plane, the $O(n^2)$ bound can be achieved by Horton sets (Fabila-Monroy and Huemer, 2012).
- $d$-dimensional Horton sets with $d > 2$ do not give the $O(n^d)$ bound.
Our results III

- We cannot have the bound $O(n^d)$ for $k$-islands if $k$ is not fixed.

**Theorem 3**

Let $d \geq 2$ be an integer and let $K$ be a convex body in $\mathbb{R}^d$ with $\lambda_d(K) = 1$. Then, for every set $S$ of $n$ points chosen uniformly and independently at random from $K$, the expected number of islands in $S$ is in $2^{\Theta(n^{(d-1)/(d+1)})}$.

- Theorem 1 is the first nontrivial bound for $k$-islands in $\mathbb{R}^d$ for $d > 2$.
- In the plane, the $O(n^2)$ bound can be achieved by Horton sets (Fabila-Monroy and Huemer, 2012).
- $d$-dimensional Horton sets with $d > 2$ do not give the $O(n^d)$ bound.

**Theorem 4**

Let $d \geq 2$ and $k$ be fixed positive integers. Then every $d$-dimensional Horton set $H$ with $n$ points contains at least $\Omega(n^{\min\{2^{d-1},k\}})$ $k$-islands in $H$. If $k \leq 3 \cdot 2^{d-1}$, then $H$ even contains at least $\Omega(n^{\min\{2^{d-1},k\}})$ $k$-holes in $H$. 
Sketch of the proof of Theorem 1: the plane case

We want to prove the bound $O(n^2)$ on the number of $k$-islands in sets of $n$ points in the plane.

We assume that the drawn points are in a canonical order $p_1, \ldots, p_k$:

- $\Delta = \text{conv}(\{p_1 p_2 p_3\})$ is the triangle of the largest volume,
- $p_1 p_2$ is its longest edge,
- points outside of $\Delta$ have increasing distances to the convex hull of the previously placed points and the points inside $\Delta$ are uniquely ordered.

We draw the points in the canonical order and estimate the probability in every step.

We start by estimating the probability that the vertices $p_1, p_2, p_3$ of $\Delta$ with $a$ points $p_4, \ldots, p_{3+a}$ inside $\Delta$ form an island in $S$. 
We want to prove the bound $O(n^2)$ on the number of $k$-islands in sets of $n$ points in the plane.
We want to prove the bound $O(n^2)$ on the number of $k$-islands in sets of $n$ points in the plane.

We assume that the drawn points are in a **canonical order** $p_1, \ldots, p_k$: 

$\Delta = \text{conv}(\{p_1p_2p_3\})$ is the triangle of the largest volume, $p_1p_2$ is its longest edge, points outside of $\Delta$ have increasing distances to the convex hull of the previously placed points and the points inside $\Delta$ are uniquely ordered.
Sketch of the proof of Theorem 1: the plane case

• We want to prove the bound $O(n^2)$ on the number of $k$-islands in sets of $n$ points in the plane.
• We assume that the drawn points are in a canonical order $p_1, \ldots, p_k$: $\Delta = \text{conv}(\{p_1p_2p_3\})$ is the triangle of the largest volume, $p_1p_2$ is its longest edge, points outside of $\Delta$ have increasing distances to the convex hull of the previously placed points and the points inside $\Delta$ are uniquely ordered.
• We draw the points in the canonical order and estimate the probability in every step.
Sketch of the proof of Theorem 1: the plane case

- We want to prove the bound \( O(n^2) \) on the number of \( k \)-islands in sets of \( n \) points in the plane.
- We assume that the drawn points are in a canonical order \( p_1, \ldots, p_k \): \( \Delta = \text{conv}(\{p_1p_2p_3\}) \) is the triangle of the largest volume, \( p_1p_2 \) is its longest edge, points outside of \( \Delta \) have increasing distances to the convex hull of the previously placed points and the points inside \( \Delta \) are uniquely ordered.
- We draw the points in the canonical order and estimate the probability in every step.
- We start by estimating the probability that the vertices \( p_1, p_2, p_3 \) of \( \Delta \) with \( a \) points \( p_4, \ldots, p_{3+a} \) inside \( \Delta \) form an island in \( S \).
Sketch of the proof of Theorem 1: the plane case

\[ \int_2^{-2} \frac{|I_0|}{|I_0|} \cdot (k - a - 3)! \cdot \left( |I_0| \cdot |h| \right)^2 a \cdot \left( 1 - |I_0| \cdot |h| \right)^{n - a - 3} dh. \]
Sketch of the proof of Theorem 1: the plane case

The probability that the drawn points in $\triangle$ are an island in $\mathcal{S}$ is at most

$$\int \frac{2}{|I_0|} - \frac{2}{|I_0| |I_h \cap K|} a! \cdot (k - a - 3)! \cdot (|I_0| \cdot |I_h|)^{a} \cdot (1 - |I_0| \cdot |I_h|)^{n - a - 3} dh.$$
Sketch of the proof of Theorem 1: the plane case

The probability that the drawn points in $\Delta$ are an island in $S$ is at most

$$\int \frac{2}{|I_0|} - \frac{2}{|I_0|} |I_h \cap K| a (k - a - 3)! \cdot \left(\frac{|I_0| \cdot |h|}{2}\right)^a \cdot \left(1 - \frac{|I_0| \cdot |h|}{2}\right)^{n-a-3} dh.$$

$\Delta$, $I_0$, $K$, $p_1$
The probability that the drawn points in $\Delta$ are an island in $S$ is at most 
$$\int \frac{2}{|I_0|} - \frac{2}{|I_0|} d\mu \cdot \left( |I_0| \cdot |h| \right)^{a} \cdot \left( 1 - |I_0| \cdot |h| \right)^{n-a-3}.$$
Sketch of the proof of Theorem 1: the plane case

The probability that the drawn points in $\Delta$ are an island in $S$ is at most

\[ \int \frac{2}{|I_0|} - \frac{2}{|I_0|} |I_0| \cdot |h| \cdot (|I_0| \cdot |h|^2)^a \cdot (1 - |I_0| \cdot |h|^2)^{n-a-3} \, dh. \]
Sketch of the proof of Theorem 1: the plane case

The probability that the drawn points in $\Delta$ are an island in $S$ is at most

$$\int \frac{2}{|p_1p_2|} \cdot \frac{2}{|I_0|} \cdot |h| \cdot |I|_{\cap K} \cdot a \cdot (k - a - 3)! \cdot \left(\frac{|I_0| \cdot |h|}{2}\right)^2 \cdot \left(1 - \frac{|I_0| \cdot |h|}{2}\right)^{n - a - 3} \, dh.$$
Sketch of the proof of Theorem 1: the plane case

The probability that the drawn points in $\triangle$ are an island in $S$ is at most

$$\int \frac{2}{|p_1 p_2|} - \frac{2}{|p_1 p_2|} |I_0| \cdot |p_3 - (k - a - 3)! \cdot \left(\frac{|I_0| \cdot |h|}{2}\right)^a \cdot \left(1 - \frac{|I_0| \cdot |h|}{2}\right)^{n - a - 3} \, dh.$$
Sketch of the proof of Theorem 1: the plane case

The probability that the drawn points in $\Delta$ are an island in $S$ is at most

$$\int \frac{2}{|p_1p_2|} - \frac{2}{|I_0|} |I_h \cap K|^{a} \cdot \left( |I_0| \cdot |h|^{2} \right)^{a} \cdot \left( 1 - |I_0| \cdot |h|^{2} \right)^{n-a-3} \cdot \frac{d}{h}.$$
Sketch of the proof of Theorem 1: the plane case

The probability that the drawn points in \( \Delta \) are an island in \( S \) is at most

\[
\int \frac{2}{|p_1p_2|} - \frac{2}{|I_0|} |I_h \cap K| \cdot (|I_0| \cdot |h|^2) a \cdot \left( 1 - \frac{|I_0| \cdot |h|}{2} \right)^{n-a-3} dh.
\]
Sketch of the proof of Theorem 1: the plane case

The probability that the drawn points in \( \Delta \) are an island in \( S \) is at most

\[
\int \frac{2}{|p_1p_2|} - \frac{2}{|I_0|} |I_h \cap K| \cdot \left( \frac{|I_0| \cdot |h|}{2} \right)^a \cdot \left( 1 - \frac{|I_0| \cdot |h|}{2} \right)^{n-a-3} dh.
\]
Sketch of the proof of Theorem 1: the plane case

- The probability that the drawn points in $\Delta$ are an island in $S$ is at most

$$\int_{-2/|l_0|}^{2/|l_0|} \frac{|l_h \cap K|}{a! \cdot (k-a-3)!} \cdot \left( \frac{|l_0| \cdot |h|}{2} \right)^a \cdot \left( 1 - \frac{|l_0| \cdot |h|}{2} \right)^{n-a-3} \, dh.$$
The probability that the drawn points in $\Delta$ are an island in $S$ is at most

$$\frac{4}{(n - k + 1)^{a+1}}.$$
Sketch of the proof of Theorem 1: the plane case

For $i = a + 3, \ldots, k$, let $E_{a,i}$ be the event that \{p$_1$, p$_2$, \ldots, p$_i$\} is an island in S. We estimate the probability $\Pr[E_{a,i} | E_{a,i-1}]$.

$\Pr[E_{a,i} | E_{a,i-1}] \leq 16k^{n-i+1}$. 
Sketch of the proof of Theorem 1: the plane case

- For \( i = a + 3, \ldots, k \), let \( E_{a,i} \) be the event that \( \{p_1, \ldots, p_i\} \) is an island in \( S \). We estimate the probability \( \Pr[E_{a,i} \mid E_{a,i-1}] \).
Sketch of the proof of Theorem 1: the plane case

- For $i = a + 3, \ldots, k$, let $E_{a,i}$ be the event that $\{p_1, \ldots, p_i\}$ is an island in $S$. We estimate the probability $\Pr[E_{a,i} | E_{a,i-1}]$. 

\[ \Pr[E_{a,i} | E_{a,i-1}] \leq \frac{16}{k/n - i + 1} \]
Sketch of the proof of Theorem 1: the plane case

- For $i = a + 3, \ldots, k$, let $E_{i,a}$ be the event that $\{p_1, \ldots, p_i\}$ is an island in $S$. We estimate the probability $\Pr[E_{a,i} \mid E_{a,i-1}]$. 

![Diagram of a polygon with points labeled p1 and p3, and a triangle labeled Delta within a larger shape labeled K.]
Sketch of the proof of Theorem 1: the plane case

- For $i = a + 3, \ldots, k$, let $E_{a,i}$ be the event that $\{p_1, \ldots, p_i\}$ is an island in $S$. We estimate the probability $\Pr[E_{a,i} \mid E_{a,i-1}]$. 

![Diagram](image-url)
Sketch of the proof of Theorem 1: the plane case

For $i = a + 3, \ldots, k$, let $E_{a,i}$ be the event that $\{p_1, \ldots, p_i\}$ is an island in $S$. We estimate the probability $\Pr[E_{a,i} \mid E_{a,i-1}]$. 

\[ \Pr[E_{a,i} \mid E_{a,i-1}] \leq \frac{16}{k} n - i + 1. \]
Sketch of the proof of Theorem 1: the plane case

- For $i = a + 3, \ldots, k$, let $E_{a,i}$ be the event that $\{p_1, \ldots, p_i\}$ is an island in $S$. We estimate the probability $\Pr[E_{a,i} \mid E_{a,i-1}]$. 

\[ \Pr[E_{a,i} \mid E_{a,i-1}] \leq 16 \frac{k}{n} - i + 1. \]
Sketch of the proof of Theorem 1: the plane case

• For $i = a + 3, \ldots, k$, let $E_{a,i}$ be the event that $\{p_1, \ldots, p_i\}$ is an island in $S$. We estimate the probability $\Pr[E_{a,i} \mid E_{a,i-1}]$. 
Sketch of the proof of Theorem 1: the plane case

- For $i = a + 3, \ldots, k$, let $E_{a,i}$ be the event that \{p_1, \ldots, p_i\} is an island in $S$. We estimate the probability $\Pr[E_{a,i} | E_{a,i-1}]$. 
Sketch of the proof of Theorem 1: the plane case

- For $i = a + 3, \ldots, k$, let $E_{a,i}$ be the event that $\{p_1, \ldots, p_i\}$ is an island in $S$. We estimate the probability $\Pr[E_{a,i} \mid E_{a,i-1}]$. 

![Diagram]
Sketch of the proof of Theorem 1: the plane case

- For \( i = a + 3, \ldots, k \), let \( E_{a,i} \) be the event that \( \{p_1, \ldots, p_i\} \) is an island in \( S \). We estimate the probability \( \Pr[E_{a,i} \mid E_{a,i-1}] \).

\[
\Pr[E_{a,i} \mid E_{a,i-1}] \leq \frac{16k}{n - i + 1}.
\]
Sketch of the proof of Theorem 1: the plane case

• Now, we just put the estimates together.
• Since,
  \[ E_a, a+3 \supseteq E_a, a+4 \supseteq \ldots \supseteq E_a, k \]
  the probability that \( k \) points form an island in the canonical order is
  \[
  P \leq k - 3 \sum_{a=0}^{4} (n-k+1) a+1 \cdot k \prod_{i=a+4}^{16} k n - i + 1 \leq (16k) k - 3 \cdot (k-2) \cdot 4 (n-k+1) k - 2.
  \]
• The expected number of \( k \)-islands in \( S \) is then at most
  \[
  n (n-1) \ldots (n-k+1) \cdot P/2.
  \]
• Now, we just put the estimates together.
Sketch of the proof of Theorem 1: the plane case

- Now, we just put the estimates together.
- Since,
  \[ E_{a,a+3} \supseteq E_{a,a+4} \supseteq \cdots \supseteq E_{a,k} \]

  the probability that \( k \) points form an island in the canonical order is
Sketch of the proof of Theorem 1: the plane case

• Now, we just put the estimates together.
• Since,

\[ E_{a,a+3} \supseteq E_{a,a+4} \supseteq \cdots \supseteq E_{a,k} \]

the probability that \( k \) points form an island in the canonical order is

\[
P \leq \sum_{a=0}^{k-3} \frac{4}{(n-k+1)^{a+1}} \cdot \prod_{i=a+4}^{k} \frac{16k}{n-i+1}
\]
Now, we just put the estimates together.

Since,

\[ E_{a,a+3} \supseteq E_{a,a+4} \supseteq \cdots \supseteq E_{a,k} \]

the probability that \( k \) points form an island in the canonical order is

\[
P \leq \sum_{a=0}^{k-3} \frac{4}{(n-k+1)^{a+1}} \cdot \prod_{i=a+4}^{k} \frac{16k}{n-i+1}
\]

\[
\leq (16k)^{k-3} \cdot (k-2) \cdot \frac{4}{(n-k+1)^{k-2}}.
\]
• Now, we just put the estimates together.

• Since,

\[ E_{a,a+3} \supseteq E_{a,a+4} \supseteq \cdots \supseteq E_{a,k} \]

the probability that \( k \) points form an island in the canonical order is

\[
P \leq \sum_{a=0}^{k-3} \frac{4}{(n-k+1)^{a+1}} \cdot \prod_{i=a+4}^{k} \frac{16k}{n-i+1}
\]

\[
\leq (16k)^{k-3} \cdot (k-2) \cdot \frac{4}{(n-k+1)^{k-2}}.
\]

• The expected number of \( k \)-islands in \( S \) is then at most

\[
n(n-1) \cdots (n-k+1) \cdot P/2.
\]
Sketch of the proof of Theorem 1: higher dimensions

• Some steps become a bit more involved, but we use the same ideas.
Sketch of the proof of Theorem 1: higher dimensions

- Some steps become a bit more involved, but we use the same ideas.
Sketch of the proof of Theorem 1: higher dimensions

- Some steps become a bit more involved, but we use the same ideas.
Future work

• How good is the estimate?
• We believe that the leading constant is optimal for empty triangles in the plane.
• We plan to improve the bounds for 4-holes.
• Is there a better lower/upper bound for \((d+2)\)-holes?

Thank you for your attention.
Future work

- How good is the estimate?
Future work

- How good is the estimate?
- We believe that the leading constant is optimal for empty triangles in the plane.
Future work

- How good is the estimate?
- We believe that the leading constant is optimal for empty triangles in the plane.
- We plan to improve the bounds for 4-holes.
Future work

- How good is the estimate?
- We believe that the leading constant is optimal for empty triangles in the plane.
- We plan to improve the bounds for 4-holes.
- Is there a better lower/upper bound for \((d + 2)\)-holes?
Future work

- How good is the estimate?
- We believe that the leading constant is optimal for empty triangles in the plane.
- We plan to improve the bounds for 4-holes.
- Is there a better lower/upper bound for $(d + 2)$-holes?

Thank you for your attention.