A superlinear lower bound on the number of 5-holes

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Theorem (Erdős, Szekeres, 1935)

For each $k \in \mathbb{N}$, every sufficiently large point set in general position (no 3 points are collinear) in the plane contains $k$ points in convex position.

A $k$-hole in a point set $S$ is a convex polygon with $k$ vertices from $S$ and with no points of $S$ in its interior.

Every set of 3 points contains a 3-hole. Also, 5 points $\rightarrow$ 4-hole and 10 points $\rightarrow$ 5-hole (Harborth, 1978).
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Preliminaries

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Sets with no large holes

Erdős, 1978: For every \( k \in \mathbb{N} \), does every large enough point set in general position contain a \( k \)-hole?

No. There are arbitrarily large point sets with no 7-hole (Horton, 1983).

Every sufficiently large point set in general position contains a 6-hole (Gerken, 2008 and Nicolás, 2007).
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Every sufficiently large set of points in general position contains a $k$-hole for $k \in \{3, 4, 5, 6\}$.

How many $k$-holes do we always have?

Let $h_k(n)$ be the minimum number of $k$-holes among all sets of $n$ points in the plane in general position.

The following bounds are known:

- $h_3(n)$ and $h_4(n)$ are in $\Theta(n^2)$.
- $h_k(n) = 0$ for every $k \geq 7$ (Horton, 1983).
- $h_5(n)$ and $h_6(n)$ are in $\Omega(n)$ and $O(n^2)$.

We focus on estimating $h_5(n)$. 
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It is widely conjectured that $h_5(n)$ is quadratic in $n$.

**Conjecture 1**
We have $h_5(n) = \Theta(n^2)$.

However, even the following problem was open since the 1980's.

**Conjecture 2**
The function $h_5(n)$ is superlinear in $n$.

Several attempts to improve the bounds:

- $h_5(n) \leq 1.0207n^2 + o(n^2)$ (Bárány and Valtr, 2004),
- $h_5(n) \geq \lfloor n/10 \rfloor$ (Bárány and Füredi, 1987, Harborth, 1978),
- $h_5(n) \geq n/6 - O(1)$ (Bárány and Károlyi, 2001),
- $h_5(n) \geq 3\lfloor n - 48/7 \rfloor$ (García, 2012),
- $h_5(n) \geq \lceil 3/7(n - 11) \rceil$ (Aichholzer, Hackl, Vogtenhuber, 2012),
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Our results

We show that $h_5(n)$ is superlinear in $n$.

Theorem 1

There is a fixed constant $c > 0$ such that for every integer $n \geq 10$ we have $h_5(n) \geq cn \log \frac{4}{5} n$.

This proves Conjecture 2. Conjecture 1 is still open.

A point set $P = A \cup B$ is $\ell$-divided if the line $\ell$ contains no point of $P$ and partitions $P$ into two non-empty subsets $A$ and $B$. 
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![Diagram showing a point set $P$ divided by line $\ell$ into two subsets $A$ and $B$.]
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Theorem 2

Let $P = A \cup B$ be an $\ell$-divided set with $|A|, |B| \geq 5$ and with neither $A$ nor $B$ in convex position. Then there is a 5-hole in $P$ with points in both $A$ and $B$ (so-called $\ell$-divided 5-hole).

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An island in a point set $P$ is a subset $Q$ of $P$ with $P \cap \operatorname{conv}(Q) = Q$.

Note that $k$-holes in an island of $P$ are also $k$-holes in $P$.

We proceed by induction on $t = \log_2 n$.

Base case: For $t = 5$, we have $n = 2^t > 10$ and $h_5(10) = 1$ gives at least $c \cdot n \log_4/5 n$ $5$-holes in $P$ for $c$ small enough.
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- **Base case:** For $t = 5^5$, we have $n = 2^t > 10$ and $h_5(10) = 1$ gives at least $c \cdot n \log_2^{4/5} n$ 5-holes in $P$ for $c$ small enough.
Theorem 2 implies Theorem 1 – induction step

We choose $\ell$ to be a line partitioning $P$ into $A$ and $B$ of sizes $n/2$. For a parameter $r \in \mathbb{N}$, we partition $P$ into $n/(2^r) \ell$-divided islands $P_1, \ldots, P_{n/(2^r)}$ with $|P_i \cap A|, |P_i \cap B| = r$ for every $i$. We apply Theorem 2 to each island $P_i$. 
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We apply Theorem 2 to each island $P_i$. 
Theorem 2 implies Theorem 1 – counting

We choose $r = \log \frac{1}{5} \leq \frac{1}{5} \geq 5$.

If $P_i \cap A$ or $P_i \cap B$ is in convex position for at least half of the islands:

Since $|P_i \cap A|, |P_i \cap B| = r$, each such island gives $(r^5 \cdot 5)$-holes in $P$.

In total, the number of 5-holes in $P$ is at least $\frac{1}{2} \cdot n^2 \cdot r \cdot (r^5 \cdot 5) \geq c \cdot n \log \frac{4}{5} \cdot 2n$.

If $P_i \cap A$ and $P_i \cap B$ are not in convex position for at least half of the islands:

Each such $P_i$ gives an $\ell$-divided 5-hole in $P$.

We proceed inductively on $A$ and $B$ and obtain at least $h(\frac{n}{2}) + h(\frac{n}{2}) + \frac{n}{4^r} \geq c \cdot n \log \frac{4}{5} \cdot 2n$ 5-holes in $P$. 
Theorem 2 implies Theorem 1 – counting

We choose $r = \log_{2^{1/5}} n = t^{1/5} \geq 5$. 
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  - Since $|P_i \cap A|, |P_i \cap B| = r$, each such island gives $\binom{r}{5}$ 5-holes in $P$. 

- If $P_i \cap A$ and $P_i \cap B$ are not in convex position for at least half of the islands:
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- We choose \( r = \log_{2/5} n = t^{1/5} \geq 5 \).

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    \[
    h_5(n/2) + h_5(n/2) + n/(4r) \geq c \cdot n \log_{\frac{4}{5}} n
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    5-holes in \( P \).
Let $P = A \cup B$ be an $\ell$-divided set with $|A|, |B| \geq 5$ and with neither $A$ nor $B$ in convex position. Suppose for contradiction, that there is no $\ell$-divided 5-hole in $P$. The case $|A| = 5 = |B|$ follows from $h_5(10) = 1$ (Harborth, 1978). We reduce $P$ to an island $Q$ by removing extremal points until either: $|Q \cap A| = 5$ or $|Q \cap B| = 5$, or $|Q \cap A|, |Q \cap B| \geq 6$ and $Q$ is $\ell$-critical, i.e., for each extremal point $x$ of $Q$ either $(Q \cap A) \setminus \{x\}$ or $(Q \cap B) \setminus \{x\}$ is in convex position. The first case is handled by computer.
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Suppose for contradiction, that there is no $\ell$-divided 5-hole in $P$.
The case $|A| = 5 = |B|$ follows from $h_5(10) = 1$ (Harborth, 1978).
Sketch of the proof of Theorem 2 – peeling off extremal points

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Sketch of the proof of Theorem 2 – second case

Let $a^*$ be the rightmost inner point of $Q \cap A$ and $b^*$ be the leftmost inner point of $Q \cap B$. Rays from $a^*$ to $(Q \cap A) \{a^*\}$ partition the plane into $a^*$-wedges.

Since $Q$ is \ell-critical, it has a special structure: There are at most two extremal points of $Q$ in $Q \cap A$. If there are two, then $a^*$ is the unique interior point in $Q \cap A$.

By symmetry, analogous statements hold for $Q \cap B$.

No \ell-divided 5-hole in $Q$ forces several restrictions on numbers of points from $Q \cap B$ in $a^*$-wedges.
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![Diagram of pentagons and points](image.URL)
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- No $\ell$-divided 5-hole in $Q$ forces several restrictions on numbers of points from $Q \cap B$ in $a^*$-wedges.
Sketch of the proof of Theorem 2 – obtaining the contradiction

Proposition 1
Let \( Q \) be an \( \ell \)-critical set with no \( \ell \)-divided 5-hole in \( Q \), with \( |Q \cap A|, |Q \cap B| \geq 6 \), and \( |Q \cap A \cap \partial \text{conv}(Q)| = 2 \). Then \( |Q \cap B| < |Q \cap A| \).

Proposition 2
Let \( Q \) be an \( \ell \)-critical set with no \( \ell \)-divided 5-hole in \( Q \), with \( |Q \cap A|, |Q \cap B| \geq 6 \), and \( |Q \cap A \cap \partial \text{conv}(Q)| = 2 \). Then \( |Q \cap A| \leq |Q \cap B| \).

Without loss of generality, we assume \( |Q \cap A \cap \partial \text{conv}(Q)| = 2 \).

Propositions 1 and 2 thus give \( |Q \cap B| < |Q \cap A| \leq |Q \cap B| \), a contradiction.
The restrictions on $a^*$-wedges imply the following result.
Sketch of the proof of Theorem 2 – obtaining the contradiction

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**Proposition 1**

Let $Q$ be an $\ell$-critical set with no $\ell$-divided 5-hole in $Q$, with $|Q \cap A|, |Q \cap B| \geq 6$, and $|Q \cap A \cap \partial \text{conv}(Q)| = 2$. Then $|Q \cap B| < |Q \cap A|$. 
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Considering $b^*$-wedges, we obtain the following statement.
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Considering $b^*$-wedges, we obtain the following statement.

**Proposition 2**

Let $Q$ be an $\ell$-critical set with no $\ell$-divided 5-hole in $Q$, with $|Q \cap A|, |Q \cap B| \geq 6$, and $|Q \cap A \cap \partial \text{conv}(Q)| = 2$. Then $|Q \cap A| \leq |Q \cap B|$.

Without loss of generality, we assume $|Q \cap A \cap \partial \text{conv}(Q)| = 2$. Propositions 1 and 2 thus give $|Q \cap B| < |Q \cap A| \leq |Q \cap B|$, a contradiction.
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Sketch of the proof of Theorem 2 – obtaining the contradiction
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- Propositions 1 and 2 thus give $|Q \cap B| < |Q \cap A| \leq |Q \cap B|$, a contradiction.
Computer assisted results

We use four computer assisted results in the proof of Theorem 2. In each of them, we verify certain statement for sets of $\leq 11$ points.

**Computer Lemma 1**

Let $P = A \cup B$ be an $\ell$-divided set with $|A| = 5$, $|B| = 6$, and with $A$ not in convex position. Then there is an $\ell$-divided 5-hole in $P$.

The search is done by considering all order types of such point sets. We wrote two independent implementations: First implementation uses Aichholzer's database of order types (96 GB of data). Running time: hours. Second implementation does not use the database, but running time can take up to weeks (if not run in parallel).
We use four computer assisted results in the proof of Theorem 2.

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Final remarks

The assumption $|A|, |B| \geq 5$ in Theorem 2 is necessary. There are arbitrarily large $\ell$-divided point sets $P = A \cup B$ with $|A| = 4$ and with no $\ell$-divided 5-hole. The current approach does not work for 6-holes. Since $h_6(29) = 0$ (Overmars, 2002), the reduction would have to be to at least 30-point sets, which cannot be handled by computer.

Theorem 2 can be used to improve lower bounds on $h_3(n)$ and $h_4(n)$:

**Theorem 3**

We have $h_3(n) \geq n^2 + \Omega(n \log^2 n / n)$ and $h_4(n) \geq n^2 + \Omega(n \log^3 n / n)$.

Is it possible to obtain stronger bounds on $h_5(n)$ from Theorem 2?

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