The Erdős–Szekeres Theorem

Theorem (Erdős and Szekeres, 1935) For every $k$ there is a least number $ES(k) = \binom{k+1}{2} - k$ such that every set of $ES(k) + 1$ points in the plane in general position contains $k$ points in convex position.

A set of $a$ points on a graph of a strictly concave function is an $a$-cap.

A set of $u$ points on a graph of a strictly convex function is a $u$-cup.

In fact, they showed that every set of $N(a, u) + 1 = \frac{(a+u-4)(a-2)}{2} + 1$ points in general position contains either an $a$-cap or a $u$-cup and this is tight.
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The Erdős–Szekeres Conjecture

Trivially, we have $\text{ES}(k) \leq N(k, k) = (2k^2 - 4k - 2)$.

In 1960, Erdős and Szekeres showed $\text{ES}(k) \geq 2k - 2$ for every $k \geq 2$.

Conjecture (Erdős and Szekeres, 1935)
For every $k \geq 2$, $\text{ES}(k) = 2k - 2$.

In 2005, Tóth and Valtr showed current best upper bound $\text{ES}(k) \leq (2k^2 - 5k - 2)$.

The Erdős–Szekeres conjecture is known to hold for $k \leq 6$. For $k = 6$ it was shown by Peters and Szekeres using an exhaustive computer search.
The Erdős–Szekeres Conjecture

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General setting

Fox, Pach, Sudakov, and Suk introduced the following abstract setting. Let $K^3_N$ be the complete 3-uniform hypergraph with the vertex set $[N]$. For vertices $v_1 < \cdots < v_k$ of $K^3_N$, the edges $\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \ldots, \{v_{k-2}, v_{k-1}, v_k\}$ form a (monotone) $k$-path.

A coloring of $K^3_N$ assigns either a red or a blue color to every edge of $K^3_N$. Let $\hat{N}(a, u)$ be the maximum number $N$ such that there is a coloring of $K^3_N$ with no red $a$-path and no blue $u$-path.

In a coloring of triples of points according to their orientation, red and blue monotone $k$-paths correspond to $k$-caps and $k$-cups, respectively. A straightforward generalization of the proof of Erd˝ os and Szekeres gives $\hat{N}(a, u) = \binom{a + u - 4}{a - 2}$. \[\hat{N}(a, u) = N(a, u).\]
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![Monotone 5-path diagram]

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$$\hat{N}(a, u) = \binom{a + u - 4}{a - 2} = N(a, u).$$
There are point sets in convex position that are not a cap nor a cup.

Every point set in convex position is a union of a cap and a cup.

Peters and Szekeres generalized the notion of convex position as follows.

A (convex) $k$-gon is an ordered 3-uniform hypergraph on $k$ vertices consisting of a red and a blue monotone path that are vertex disjoint except for the common end-vertices.

There is exactly $2^{k-2}$ pairwise nonisomorphic $k$-gons.
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For $k \geq 2$, let $\hat{ES}(k)$ be the maximum number $N$ such that there is a coloring of $K_3^N$ with no $k$-gon. By the results of Erdős and Szekeres, we have $2^k - 2 \leq \hat{ES}(k) \leq (2^k - 4)^2 - 2$.

Peters and Szekeres proved $\hat{ES}(k) = 2^k - 2$ for $k \leq 5$ using exhaustive computer search.

Conjecture (Peters and Szekeres, 2006) For every $k \geq 2$, $\hat{ES}(k) = 2^k - 2$.

As our main result we refute this conjecture.

Theorem We have $\hat{ES}(7) > 32$ and $\hat{ES}(8) > 64$.

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The Erdős–Szekeres Conjecture revisited

In 1996, Erdős, Tuza, and Valtr refined the Erdős-Szekeres conjecture by adding a third parameter in $N(a, u, k)$. For integers $a, u, k$ with $2 \leq a, u \leq k \leq a + u - 2$, let $N(a, u, k)$ be the maximum $N$ such that there is a set of $N$ points in the plane in general position with no $a$-cap, no $u$-cup, and no $k$ points in convex position.

Conjecture (Erdős, Tuza, and Valtr, 1996)

For all integers $a, u, k$ with $2 \leq a, u \leq k \leq a + u - 2$, we have $N(a, u, k) = u \sum_{i = k - a + 2}^{k} N(i, k + 2 - i) = u \sum_{i = k - a + 2}^{k} (k - 2i - 2)$. This conjecture is equivalent with the Erdős–Szekeres conjecture. In particular, showing $N(a, u, k) > \sum_{i = k - a + 2}^{k} (k - 2i - 2)$ for some $a, u, k$ would refute the Erdős–Szekeres conjecture.
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- In particular, showing \(N(a, u, k) > \sum_{i=k-a+2}^{u} \binom{k - 2}{i - 2}\) for some \(a, u, k\) would refute the Erdős–Szekeres conjecture.
Bounds for $N(a, u, k)$

Erdős, Tuza, and Valtr showed $N(a, u, k) \geq \sum_{i=1}^{u} i = k - a + 2$ for all $a, u, k$ with $2 \leq a, u \leq k \leq a + u - 2$.

The best known upper bound for $N(a, u, k)$ is $N(a, u, k) \leq (a + u - 4)$ obtained from $N(a, u, k) \leq N(a, u)$.

The conjecture is true for $k = a + u - 2$ and $k = a + u - 3$.

Proposition

For every integer $k \geq 3$, we have $N(4, k, k) \leq (k^2 - 1)$.
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We find an analogous refinement for the Peters–Szekeres conjecture. For integers \(a, u, k\) with \(2 \leq a, u \leq k \leq a + u - 2\), let \(\hat{N}(a, u, k)\) be the maximum number \(N\) such that there is a coloring of \(K_3^N\) with no red \(a\)-path, no blue \(u\)-path, and no \(k\)-gon.

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This allows us to employ computer experiments for larger values of \(k\).
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The SAT attack

In our experiments we use the Glucose SAT solver. We found a coloring of $K_{17}$ with no red 4-path and no 7-gon and proved $\hat{N}(4,7,7) = 17$. By the lemma, we refute the Peters–Szekeres conjecture. We also have $\hat{N}(4,8,8) \geq 23$.

Further counterexamples:

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To tackle the Erdős–Szekeres conjecture, we consider only special colorings of $K_{3N}$. A coloring of $K_{3N}$ is pseudolinear if every 4-tuple of vertices of $K_{3N}$ induces a coloring that is an order type of a set of 4 points in the plane.


For pseudolinear colorings, all our results matched the values from the refined Erdős–Szekeres conjecture. We verified the refined Erdős–Szekeres conjecture for some cases. We have $N(4,7,7) = 16$ and $N(4,8,8) = 22$. 
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Open problems

Problem (Peters and Szekeres, 2006)
For every $k \geq 2$, is it true that every pseudolinear coloring of $K_{3N}$ with $N = 2^{k-2} + 1$ contains a $k$-gon?

Conjecture (Goodman and Pollack, 1981)
For every $k \geq 2$ the number $ES(k)$ equals the maximum $N$ for which there is a pseudolinear coloring of $K_{3N}$ with no $k$-gon.

Is there some structure behind the found colorings?
Is there a general construction of colorings of $K_{3N}$ with no $k$-gon for arbitrarily large $k$ and $N > 2^{k-2} + 1$. Thank you.
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