A SAT attack on the Erdős–Szekeres conjecture

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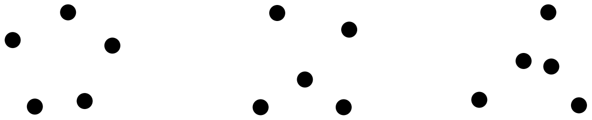

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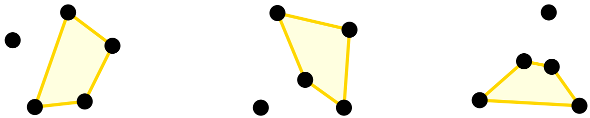

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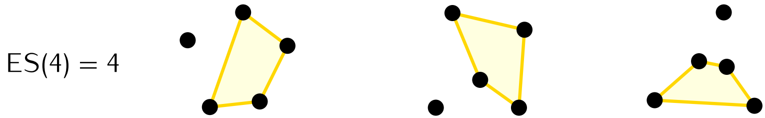

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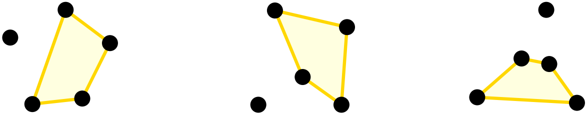

- A set of a points on a graph of a strictly concave function is an a-cap. A set of u points on a graph of a strictly convex function is a u-cup.

- In fact, they showed that every set of $\mathrm{N}(a,u)+1 = \binom{a+u-4}{a-2}+1$ points in general position contains either an a-cap or a u-cup and this is tight.

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- The Erdős–Szekeres conjecture is known to hold for $k \leq 6$. For $k = 6$ it was shown by Peters and Szekeres using an exhaustive computer search.

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- In a coloring of triples of points according to their orientation, red and blue monotone k-paths correspond to k-caps and k-cups, respectively.
- A straightforward generalization of the proof of Erdős and Szekeres gives

$$\widehat{\mathrm{N}}(a, u) = \binom{a+u-4}{a-2} = \mathrm{N}(a, u).$$

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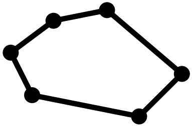

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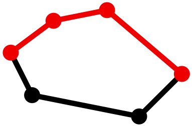

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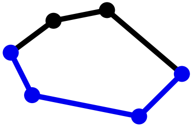

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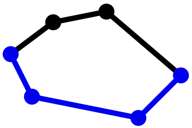

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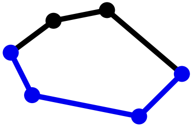

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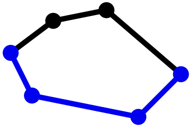

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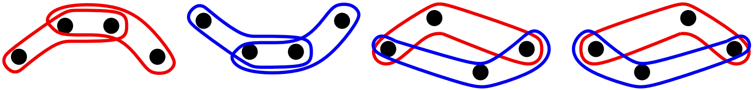

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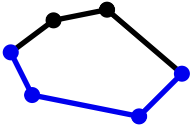

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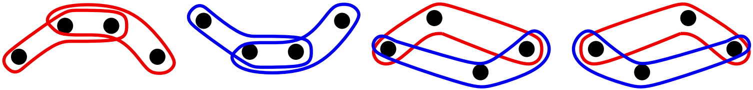

- There is exactly 2^{k-2} pairwise nonisomorphic k-gons.

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- We also tried to tackle the Erdős–Szekeres conjecture by restricting to special colorings of $\mathcal{K}_N^3$, but this conjecture remains open.

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- This conjecture is equivalent with the Erdős–Szekeres conjecture.
- In particular, showing $N(a, u, k) > \sum_{i=k-a+2}^{u} \binom{k-2}{i-2}$ for some a, u, k would refute the Erdős–Szekeres conjecture.

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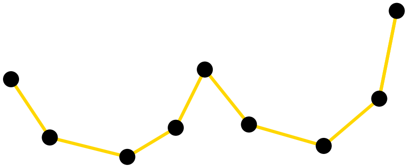

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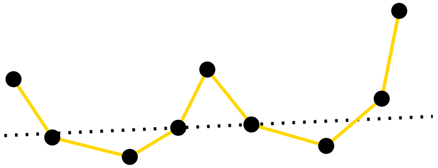

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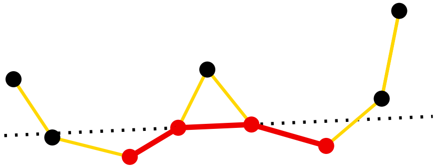

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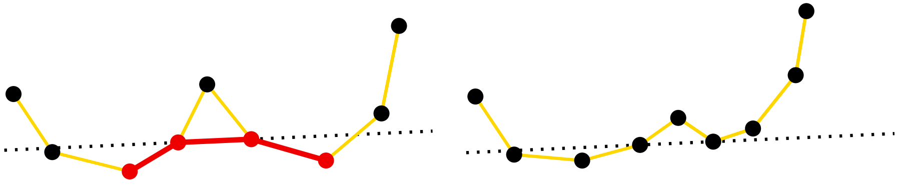

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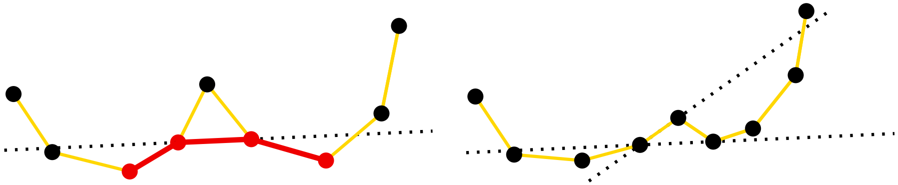

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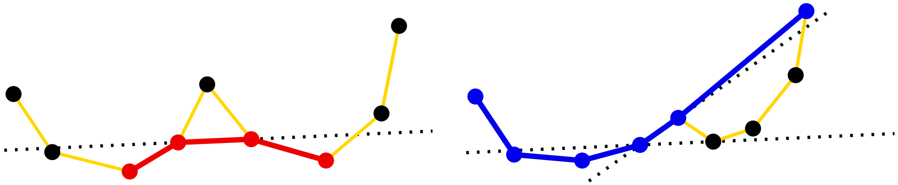

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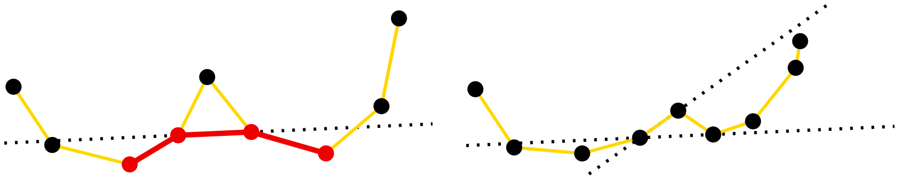

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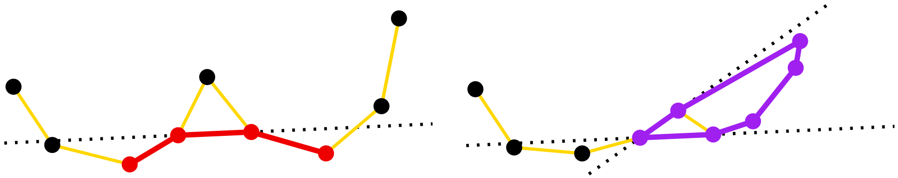

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The following statement is equivalent with the Peters–Szekeres conjecture. For all integers a, u, k with $2 \leq a, u \leq k \leq a + u - 2$, we have

$$\widehat{\mathrm{N}}(a, u, k) = \sum_{i=k-a+2}^{u} \widehat{\mathrm{N}}(i, k + 2 - i) = \sum_{i=k-a+2}^{u} \binom{k-2}{i-2}.$$

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- This allows us to employ computer experiments for larger values of k.

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- Further counterexamples:

$\widehat{\mathrm{N}}(a,u,7)$	2	3	4	5	6	7
2						1
3					5	6
4				10	15	**17**
5			10	20	[**26**,35]	[**27**,56]
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- For $k = 6$, we verified the refined Peters–Szekeres conjecture in all cases, except $a = u = k$.

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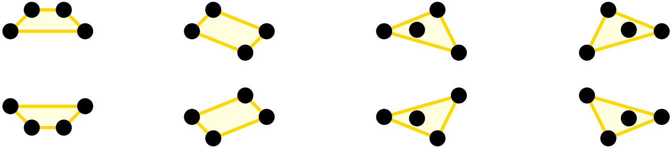

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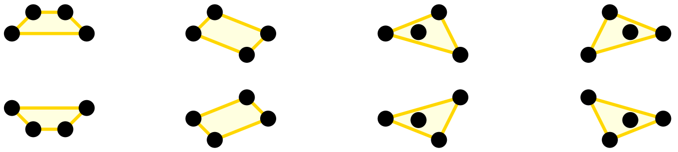

Theorem (Streinu, 1997, Felsner and Weil, 2001, B., Fulek, and Kynčl, 2013)

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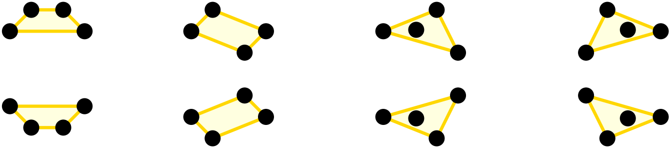

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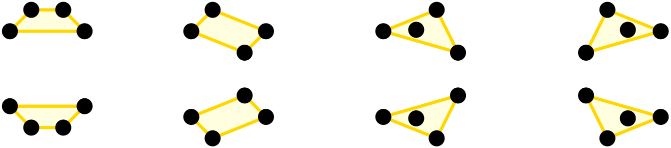

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- We verified the refined Erdős–Szekeres conjecture for some cases. We have $\mathrm{N}(4,7,7) = 16$ and $\mathrm{N}(4,8,8) = 22$.

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