Algorithmic game theory

Martin Balko

5th lecture

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Nash equilibria in bimatrix games

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- Is there a chance to get an efficient algorithm?
- NASH = the problem of finding NE in bimatrix games with payoffs.
- Today, we discuss the computational complexity of NASH.

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- Without proof (but you can find it in the lecture notes).

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- Let PPAD be a complexity class consisting of problems that admit a polynomial-time reduction to END-OF-THE-LINE.

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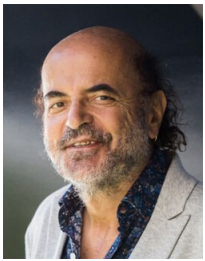


Figure: Christos Papadimitriou (born 1949).

Source: https://cs.columbia.edu

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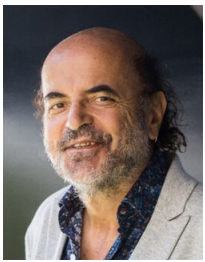


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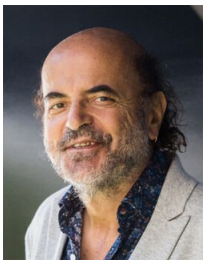


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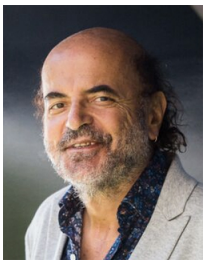


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- PPA consists of problems polynomial-time reducible to finding another degree-1 vertex in an undirected graph of maximum degree 2, given one such vertex. The graph is again given by a polynomial-time computable function.
- We have $PPAD \subseteq PPA$, but whether the classes are equal is unknown.

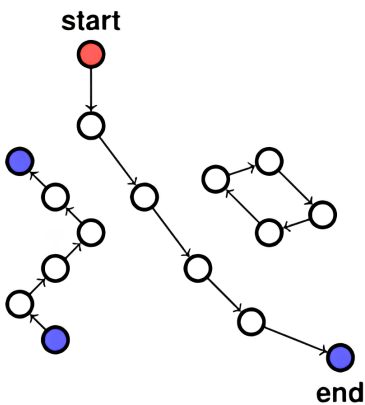
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Source: R. Savani "Polymatrix Games" Tutorial at WINE 2015

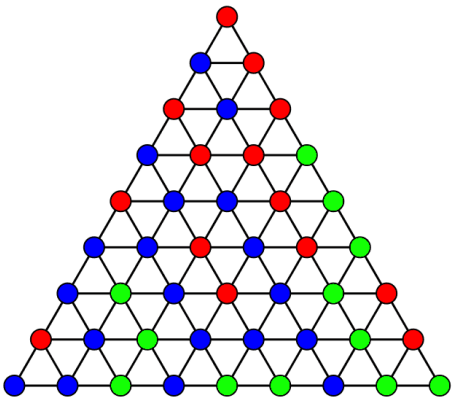
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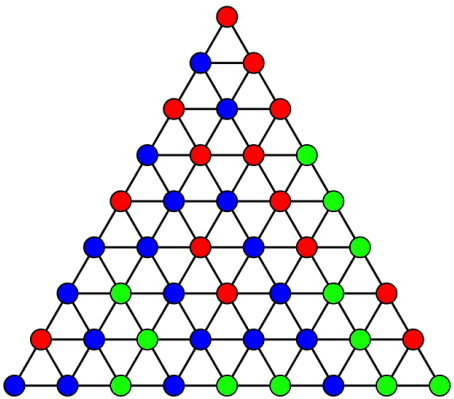
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Source: https://lesswrong.com

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Source: https://arxiv.org/pdf/1612.06820.pdf

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- An approximate version on $[0,1]^d$ in PPAD: for functions f that can be evaluated in polynomial time, with a known Lipschitz constant and given $\varepsilon > 0$, find x with $\|f(x) - x\|_\infty \leq \varepsilon$.

Problems from PPA: Ham sandwich theorem



Source: https://www.seekpng.com/

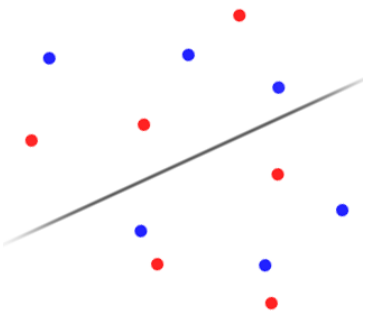
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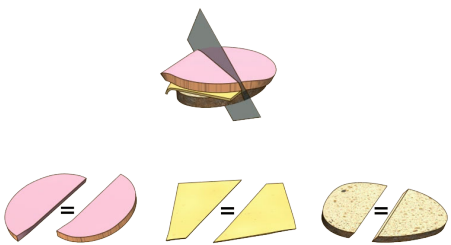
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Sources: https://ejarzo.github.io and https://curiosamathematica.tumblr.com

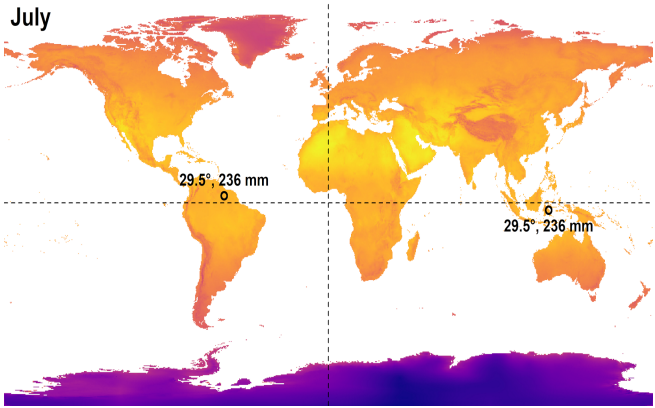
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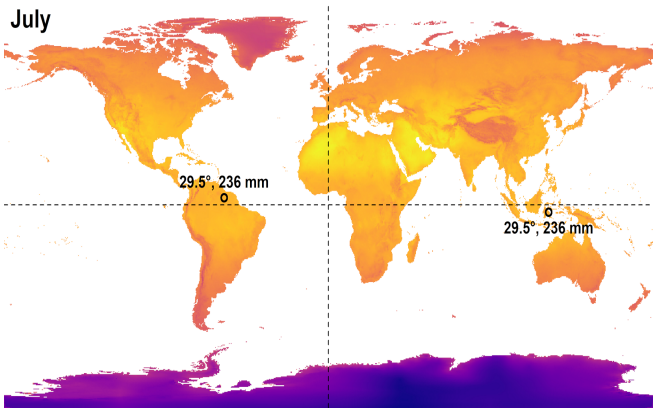
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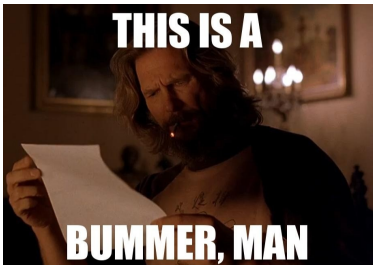
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- We omit the proof, as it is complicated (the paper has over 50 pages).

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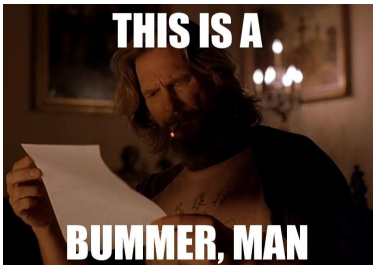
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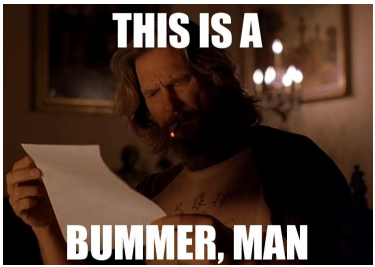
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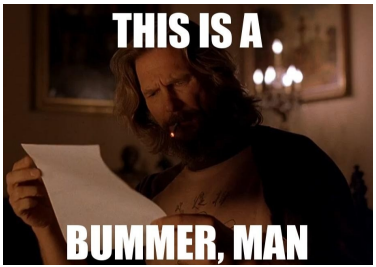
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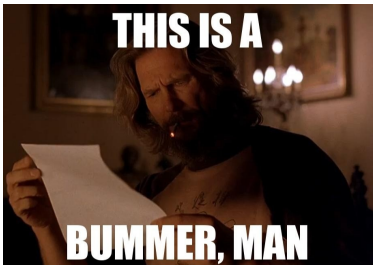


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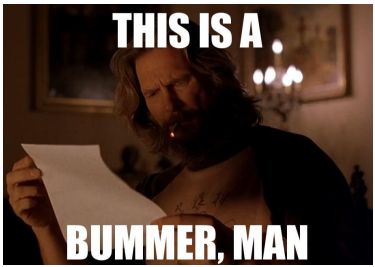


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- This seems to be a problem with the concept of NE. "How can we expect the players to find a Nash equilibrium, if our computers cannot?"

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- Finding profiles close to an exact NE in games with at least three players appears to be strictly harder than PPAD.
- Deciding whether an NE satisfying additional constraints exists is often NP-complete.

- This seems to be a problem with the concept of NE. "How can we expect the players to find a Nash equilibrium, if our computers cannot?"
- We introduce other solution concepts that possess some qualities of NE and may be easier to compute.

Other notions of equilibria

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  - We will see that this concept is also computationally difficult.

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- I no longer present the proof (see the lecture notes).

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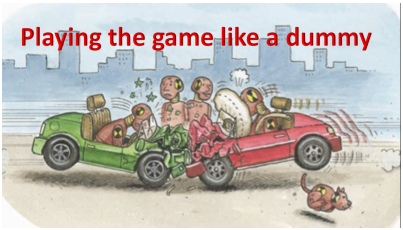
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Example of correlated equilibria: Game of Chicken

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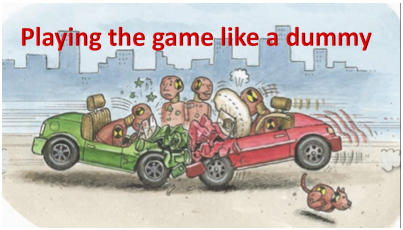
	Stop	Go
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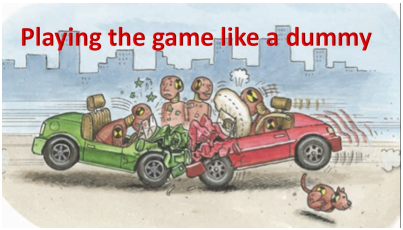


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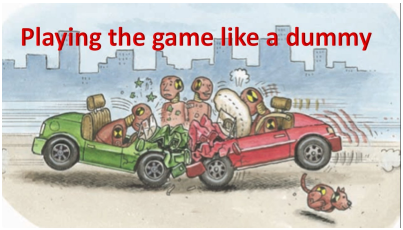


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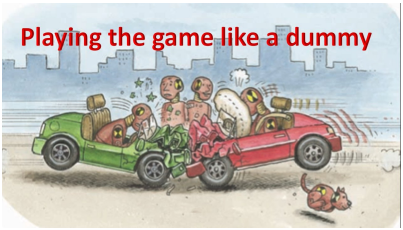


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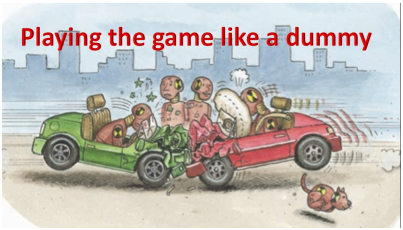


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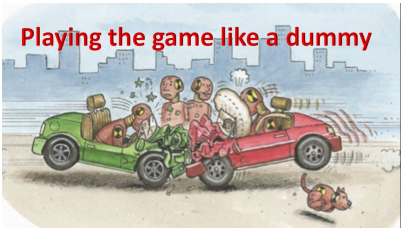


Sources: https://peakd.com/

- There are two pure NE with $(s_1(S), s_2(S)) = (1, 0)$ and $(s_1(S), s_2(S)) = (0, 1)$, and one mixed NE with $(s_1(S), s_2(S)) = (9/10, 9/10)$.
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Example of correlated equilibria: Game of Chicken

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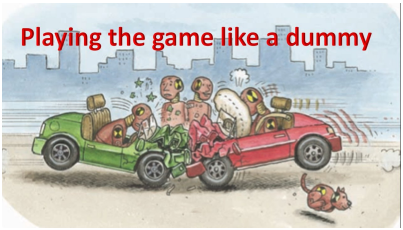


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  - By symmetry, driver 2 does not deviate as well.



Source: Students of MFF UK

Example of correlated equilibria: Battle of sexes

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	Football	Opera
Football	(2,1)	(0,0)
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  - The objective function can be arbitrary as long as it is linear.

- The concept of correlated equilibria was introduced by Robert Aumann, who received a Nobel prize in economics for his work in game theory.

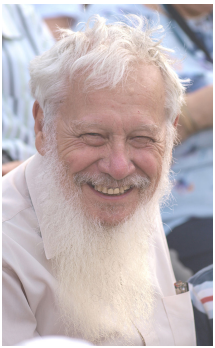




Figure: Robert Aumann (born 1930).

Sources: https://en.wikipedia.org and https://slideslive.com/38910863/strategic-information-theory

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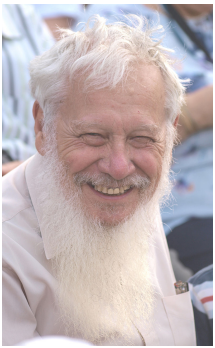




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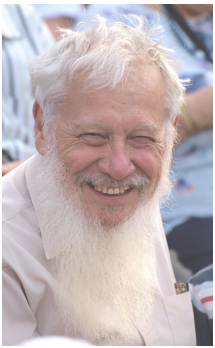




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Thank you for your attention.