

Algorithmic game theory – Tutorial 2*

1 Nash equilibria

A *normal-form game* is a triple (P, A, u) , where P is a finite set of n *players*, $A = A_1 \times \dots \times A_n$ is a set of *action profiles*, where A_i is a set of *actions* available to player i , and $u = (u_1, \dots, u_n)$ is an n -tuple, where each $u_i: A \rightarrow \mathbb{R}$ is the *utility function* for player i .

The set of *pure strategies* of player i is the set A_i of available actions for i . The set S_i of *mixed strategies* of player i is the set of all probability distributions on A_i . The *expected payoff* for player i of the mixed-strategy profile $s = (s_1, \dots, s_n)$ is

$$u_i(s) = \sum_{a=(a_1, \dots, a_n) \in A} u_i(a) \prod_{j=1}^n s_j(a_j).$$

We use the notation $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ and, for a strategy $s'_i \in S_i$ of player i , we use $u_i(s'_i; s_{-i})$ to denote the number $u_i(s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$.

The *best response* of player i to the strategy profile s_{-i} is a mixed strategy s_i^* such that $u_i(s_i^*; s_{-i}) \geq u_i(s'_i; s_{-i})$ for each strategy $s'_i \in S_i$ of i . A *Nash equilibrium* in G is a strategy profile $(s_1, \dots, s_n)$ such that s_i is a best response of player i to s_{-i} for every $i \in P$.

Observation 1 (Best response condition). *In a normal-form game $G = (P, A, u)$ of n players, for every player $i \in P$, a mixed strategy s_i is a best response to s_{-i} if and only if all pure strategies in the support of s_i are best responses to s_{-i} .*

Exercise 1. *Verify that the expected payoff of a mixed strategy in a normal-form game $G = (P, A, u)$ of n players is linear. That is, prove that $u_i(s) = \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$ for every player $i \in P$ and every mixed-strategy profile $s = (s_1, \dots, s_n)$.*

Exercise 2. *Compute mixed Nash equilibria in the following games:*

(a) *Prisoner's dilemma,*

	Testify	Remain silent
Testify	(-2,-2)	(0,-3)
Remain silent	(-3,0)	(-1,-1)

Table 1: A normal form of the game Prisoner's dilemma.

(b) *Rock-Paper-Scissors.*

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Table 2: A normal form of the game Rock-paper-scissors.

and formally show that no other Nash equilibria exist in these games.

*Information about the course can be found at <http://kam.mff.cuni.cz/~balko/>

Exercise 3 (Iterated dominance equilibrium). Let $G = (P, A, u)$ be a normal-form game of n players. For player i , we say that a strategy $s_i \in S_i$ is strictly dominated by a strategy $s'_i \in S_i$ if, for every $s_{-i} \in S_{-i}$, we have $u_i(s_i; s_{-i}) < u_i(s'_i; s_{-i})$. Consider the following iterated process that will help us find Nash equilibria in some games.

Set $A_i^0 = A_i$ and $S_i^0 = S_i$ for every player $i \in P$. For $t \geq 1$ and $i \in P$, let A_i^t be the set of pure strategies from A_i^{t-1} that are not strictly dominated by a strategy from S_i^{t-1} and let S_i^t be the set of mixed strategies with support contained in A_i^t . Let T be the first step, when the sets A_i^T and S_i^T are no longer shrinking for any $i \in P$. If each player $i \in P$ is left with one strategy $a_i \in A_i^T$, we call $a_1 \times \dots \times a_n$ an iterated dominance equilibrium of G .

- (a) Show that every iterated dominance equilibrium is a Nash equilibrium.
- (b) Find an example of a two-person game in normal form game with a pure Nash equilibrium that is not iterated dominance equilibrium.

Exercise 4. Use iterated elimination of strictly dominated strategies (introduced in Exercise 3) to find the unique Nash equilibrium in the following normal-form game of 2 players (see Table 3) by first reducing the game to a 2×2 game.

	c_1	c_2	c_3	c_4
r_1	(5, 2)	(22, 4)	(4, 9)	(7, 6)
r_2	(16, 4)	(18, 5)	(1, 10)	(10, 2)
r_3	(15, 12)	(16, 9)	(18, 10)	(11, 3)
r_4	(9, 15)	(23, 9)	(11, 5)	(5, 13)

Table 3: A game from Exercise 4.

Exercise 5 (Weekly challenge 1). Watch the scene from *A Beautiful Mind* where the John Nash character, played by Russell Crowe, explains what a Nash equilibrium is. Assume that the scenario described is modeled as a game with four players (the men), each with the same five actions (the women). Explain why the solution proposed in the movie is not a Nash equilibrium.

Exercise 6 (Weekly challenge 2). There are $n \geq 2$ students in the class, and the teacher has just asked a question. Each student has two possible actions: either responding or not. If nobody responds, all students get a payoff 0. If somebody responds, all get a payoff 1, but everybody who responded has to subtract c from their payoff, where $0 < c < 1$. Find a symmetric Nash equilibrium of this game. What is the probability that somebody responds? Is it good for the teacher to have more students in the class?