Algorithmic game theory

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9th lecture

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Games in extensive form

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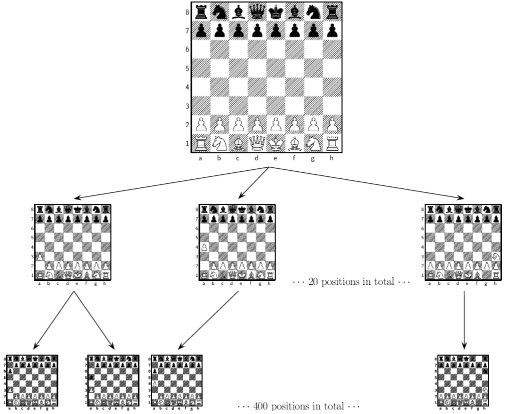

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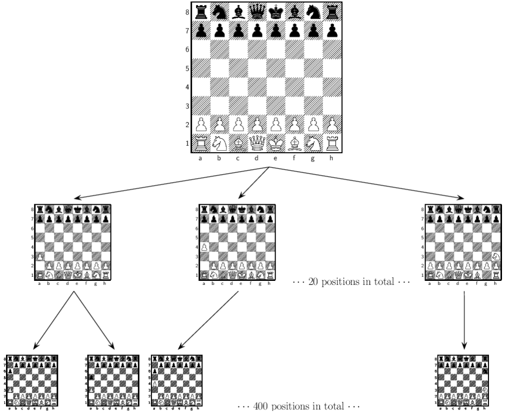

- Today, we describe strategies for such games and how to compute Nash equilibria.

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- In perfect-information games all information sets are singletons. Otherwise, we have an imperfect-information game where players have only partial knowledge of the states that they are in.

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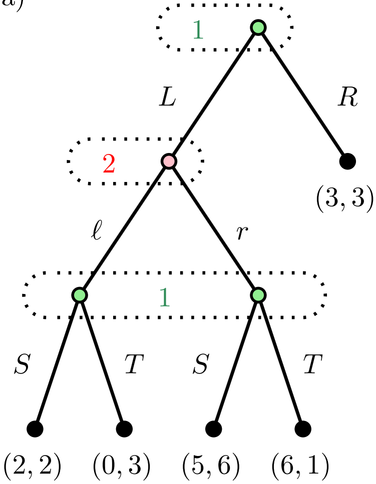
- An example of an imperfect-information game in extensive form (part (a)) and its normal-form (part (b)).

(a)



(b)

	(ℓ)	(r)
(L,S)	(2,2)	(5,6)
(L,T)	(0,3)	(6,1)
(R,S)	(3,3)	(3,3)
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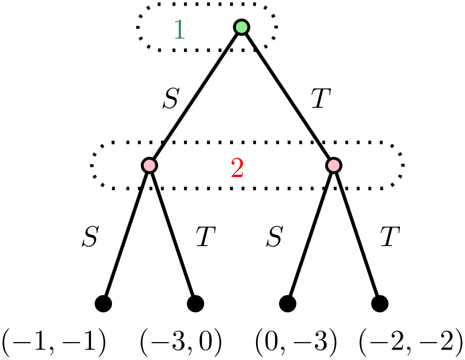

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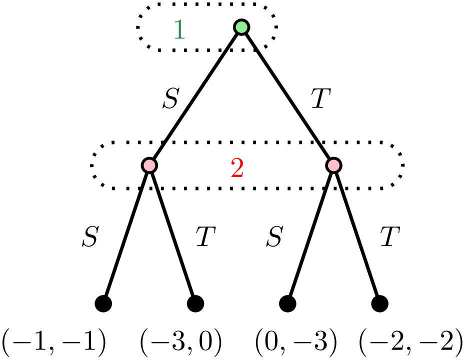

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- Every normal-form game can be expressed as an imperfect-information extensive game.

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  - Unlike in mixed strategy, here a player might play different moves in different encounters of h.

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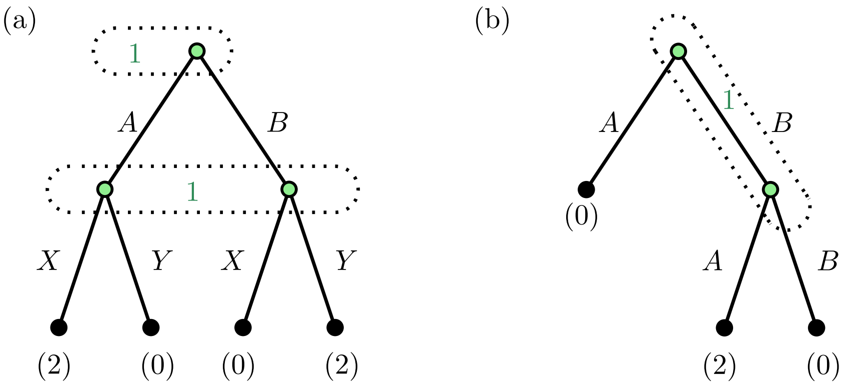

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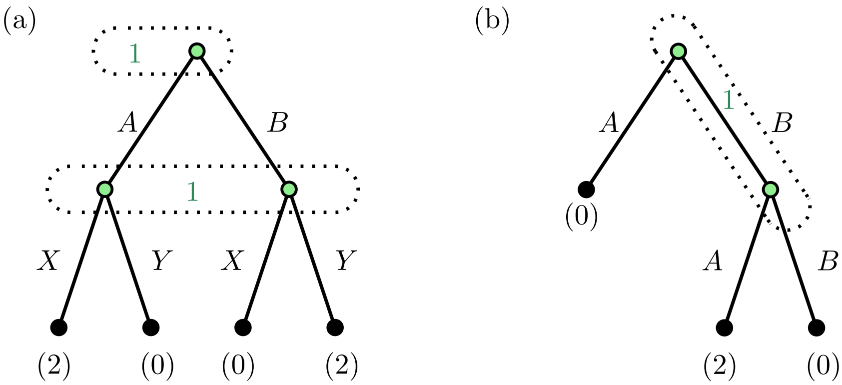

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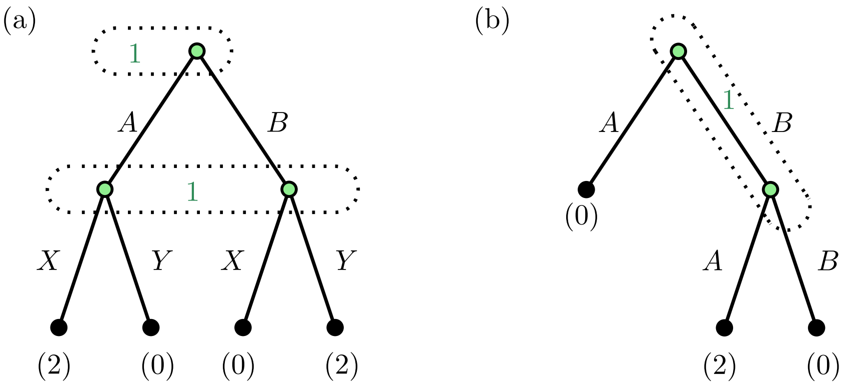

- (a) No behavioral strategy can remember whether A or B was played, while the mixed strategy $(\frac{1}{2}(A, X), \frac{1}{2}(B, Y))$ can employ this.
- (b) A mixed strategy has to commit to A or B both times, while the behavioral strategy $\frac{1}{2}A + \frac{1}{2}B$ can give a different move each time and yields a better payoff.

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  - Every perfect-information game is a game of perfect recall.

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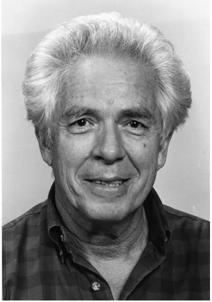

Figure: Harold William Kuhn (1925–2014).

Sources: https://alchetron.com/Harold-W-Kuhn and https://www.cantorsparadise.com/

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- An example of an imperfect-information game in extensive form (part (a)) and its sequence form payoff matrices (part (b)).

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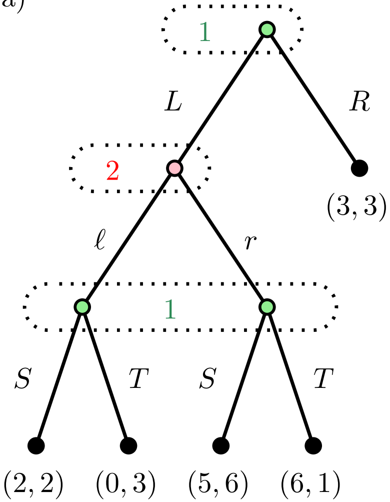

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$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} & \begin{array}{ccc} \emptyset & \ell & r \end{array} \\ & \begin{pmatrix} & & \\ & & \\ 3 & & \\ & 2 & 5 \\ & 0 & 6 \end{pmatrix} \begin{array}{l} \emptyset \\ L \\ R \\ LS \\ LT \end{array} \end{array}$$

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- An example of an imperfect-information game in extensive form (part (a)) and linear constraints in its sequence form (part (b)).

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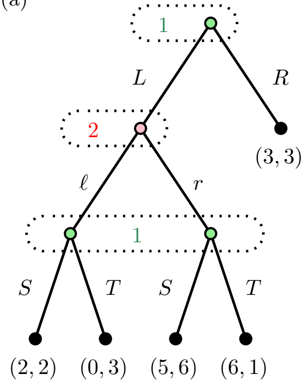

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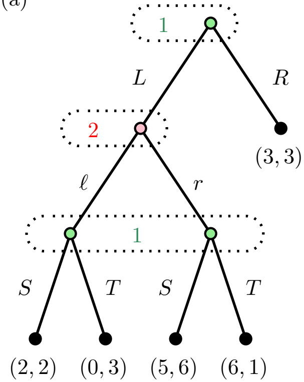

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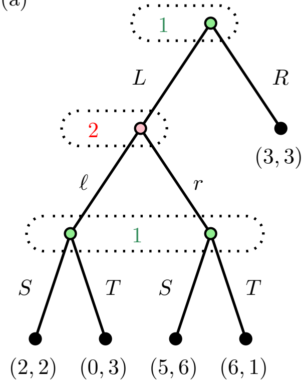

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- Analogous LPs can be used to compute best responses of player 2.

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NE of a 2-player zero-sum extensive game of perfect recall are solutions of the following LP:

$$\min_{u,y} e^\top u \text{ subject to } Fy = f, E^\top u - Ay \geq \mathbf{0}, y \geq \mathbf{0}.$$

- The dual of this program has variables x and v and is of the form

$$\max_{v,x} f^\top v \text{ subject to } Ex = e, F^\top v - A^\top x \leq \mathbf{0}, x \geq \mathbf{0}.$$

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A pair (x, y) of realization plans in a 2-player game in the extensive form of perfect recall is NE iff there are vectors u and v such that:

$$\begin{aligned} x^\top(E^\top u - Ay) = 0, &\quad y^\top(F^\top v - B^\top x) = 0, \\ Ex = e, x \geq \mathbf{0}, &\quad Fy = f, y \geq \mathbf{0}, \\ E^\top u - Ay \geq \mathbf{0}, &\quad F^\top v - B^\top x \geq \mathbf{0}. \end{aligned}$$

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- These can be solved with Lemke's algorithm, which can take exponentially many steps, similarly to the Lemke–Howson algorithm.
- Still, it is exponentially faster than to run the Lemke–Howson algorithm on the induced normal-form game.

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Thank you for your attention.