Algorithmic game theory

Martin Balko

3rd lecture

October 14th 2025

Proof of the Minimax Theorem

The Minimax Theorem

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The Minimax Theorem (Theorem 2.21)

For every zero-sum game, worst-case optimal strategies for both players exist and can be efficiently computed. There is a number v such that, for any worst-case optimal strategies x^* and y^*, the strategy profile (x^*, y^*) is a Nash equilibrium and $\beta(x^*) = (x^*)^\top M y^* = \alpha(y^*) = v$.

Figure: John von Neumann (1903–1957) and Oskar Morgenstern (1902–1977).

Sources: https://en.wikiquote.org and https://austriainusa.org

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- We prove the theorem using linear programming.

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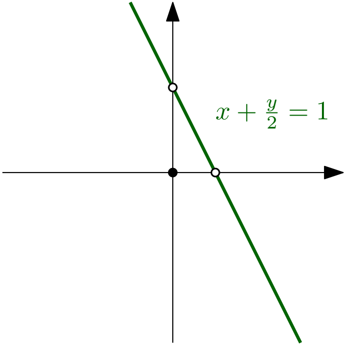

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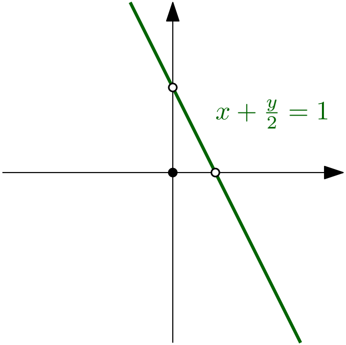

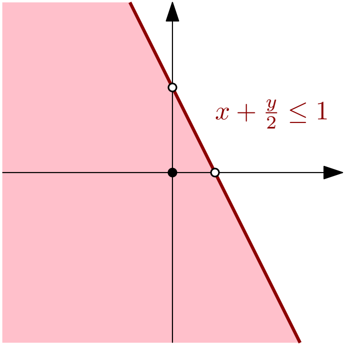

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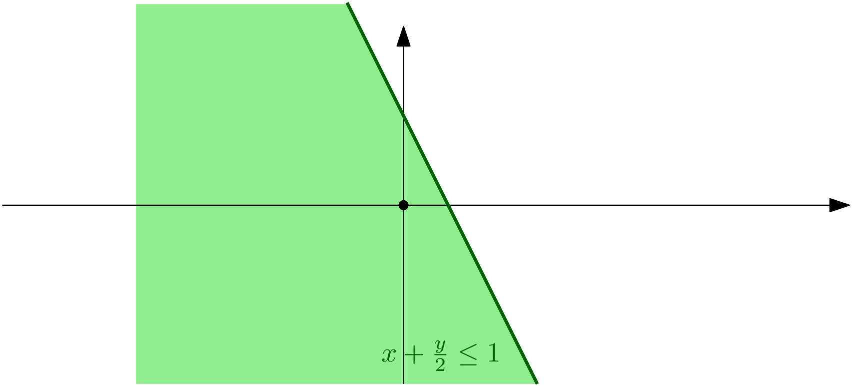

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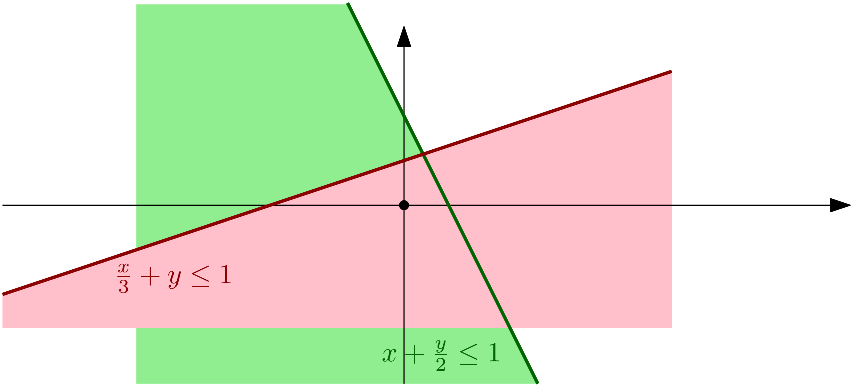

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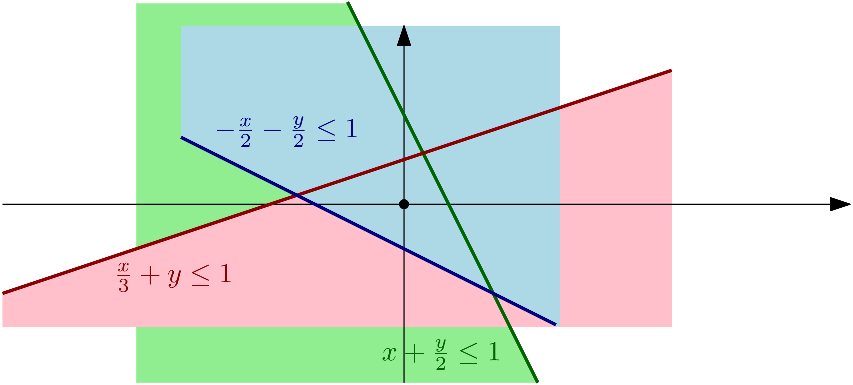

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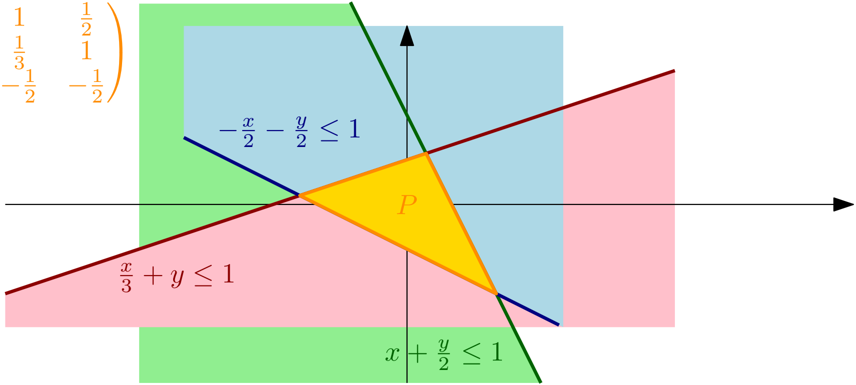

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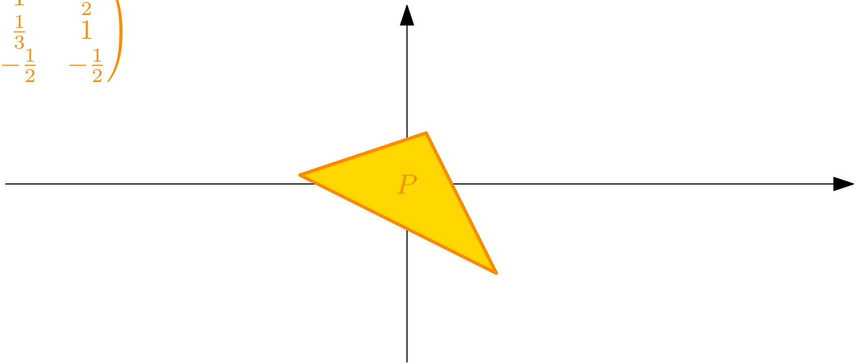

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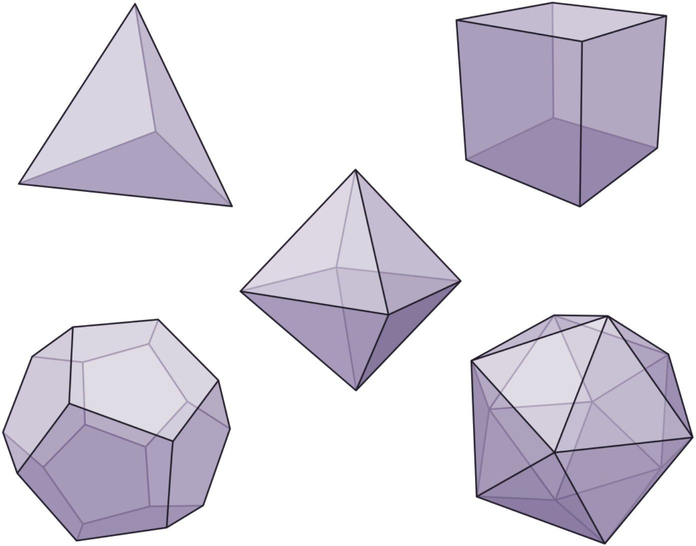

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- Every linear program P can be expressed in the canonical form: given $c \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, and $A^{n \times m}$, we want to maximize $c^\top x$ subject to the constraints $Ax \leq b$ and $x \geq \mathbf{0}$.

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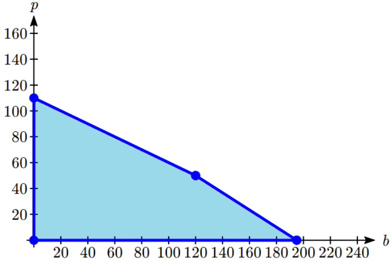

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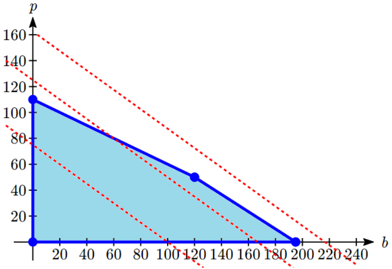

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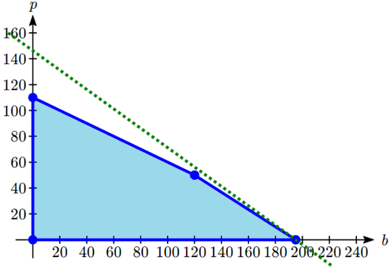

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- Dual programs can be constructed for any linear program.

General recipe for duality

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	Primal linear program	Dual linear program
Variables	$x_1, \ldots, x_m$	$y_1, \ldots, y_n$
Matrix	$A \in \mathbb{R}^{n \times m}$	$A^\top \in \mathbb{R}^{m \times n}$
Right-hand side	$b \in \mathbb{R}^n$	$c \in \mathbb{R}^m$
Objective function	$\max c^\top x$	$\min b^\top y$
Constraints	ith constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	jth constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

Table: A recipe for making dual programs.

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The Minimax Theorem (Theorem 2.21)

For every zero-sum game, worst-case optimal strategies for both players exist and can be efficiently computed. There is a number v such that, for any worst-case optimal strategies x^* and y^*, the strategy profile (x^*, y^*) is a Nash equilibrium and $\beta(x^*) = (x^*)^\top M y^* = \alpha(y^*) = v$.

- We want to compute x^* such that $\beta(x^*) = \max_{x \in S_1} \beta(x)$ where $\beta(x) = \min_{y \in S_2} x^\top M y$ using LP. We first show how not to do it.
- Naive straightforward approach with variables $x_1, \ldots, x_m$:

$$\text{maximize } \beta(x) \text{ subject to the constraints } \sum_{i=1}^{m} x_i = 1 \text{ and } x \geq \mathbf{0}.$$

- This is not LP! (the objective function $\beta(x) = \min_{y \in S_2} x^\top M y$ is not linear in x) What can we compute with LP?

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- The optimum x^* of D' is a worst-case optimum strategy for 1!

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Nash equilibria in bimatrix games

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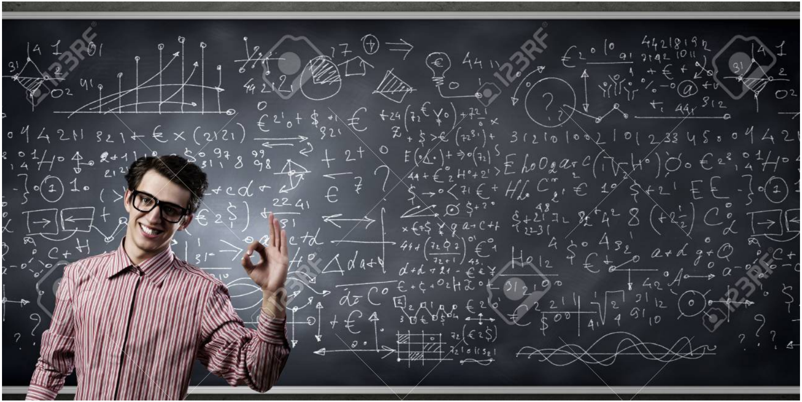
	Testify	Remain silent
Testify	(-2,-2)	(-3,0)
Remain silent	(0,-3)	(-1,-1)

Sources: https://sciworthy.com/

Bimatrix games examples: collaborative projects

Source: https://filestage.io/

Bimatrix games examples: education, knowledge sharing



Source: https://www.123rf.com/

Bimatrix games examples: the battle for Gotham's soul

	Cooperate	Detonate
Cooperate	(0,0)	(0,1)
Detonate	(1,0)	(0,0)

Sources: https://www.cbr.com/

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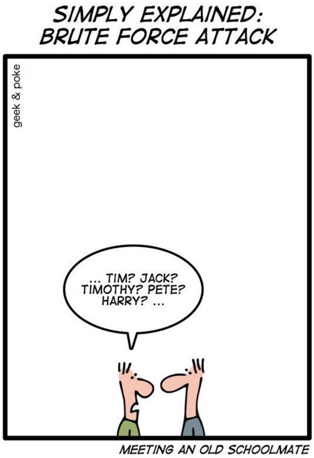

Source: https://pinterest.com

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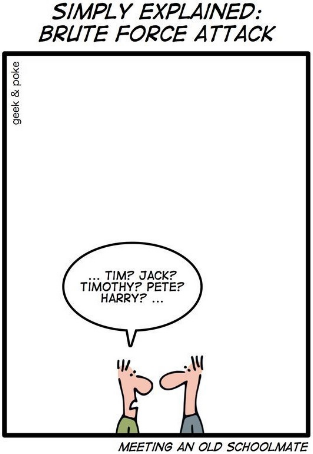

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- Later, we show the currently best known algorithm for this problem.

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In a normal-form game $G = (P, A, u)$ of n players, for every player $i \in P$, a mixed strategy s_i is a best response to s_{-i} if and only if all pure strategies in the support of s_i are best responses to s_{-i}.

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- Once we have the right supports, the precise mixed strategies can be computed by solving a system of algebraic equations (which are linear in the case of two players).

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- The running time is then about 4^n for $m = n$.

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Opera (2)	(0,0)	(1,2)

- That is, we have $M = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) = N^\top$.
- If $I = \{1,2\}$ and $J = \{1,2\}$, then we want to solve the following system of 6 equations with 6 variables x_1, x_2, y_1, y_2, u, v:

$$\begin{aligned} x_1 &= v, \quad 2x_2 = v, \quad x_1 + x_2 = 1 \\ 2y_1 &= u, \quad y_2 = u, \ ; \ y_1 + y_2 = 1 \end{aligned}$$

- This yields a unique solution $(x_1, x_2) = (\frac{2}{3}, \frac{1}{3})$ and $(y_1, y_2) = (\frac{1}{3}, \frac{2}{3})$. Since $x, y > \mathbf{0}$ and there is no better pure strategy, we have NE.

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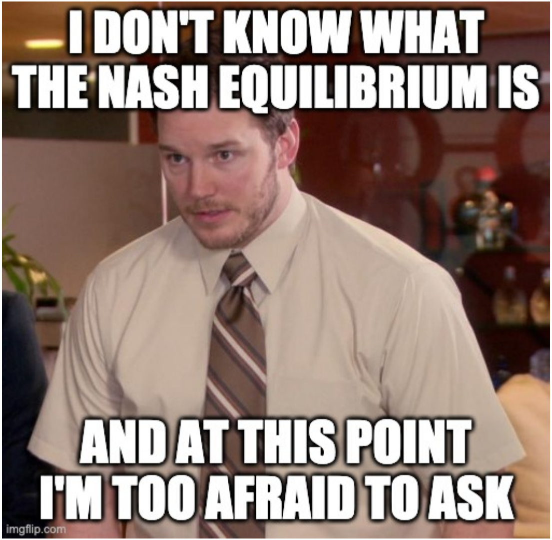

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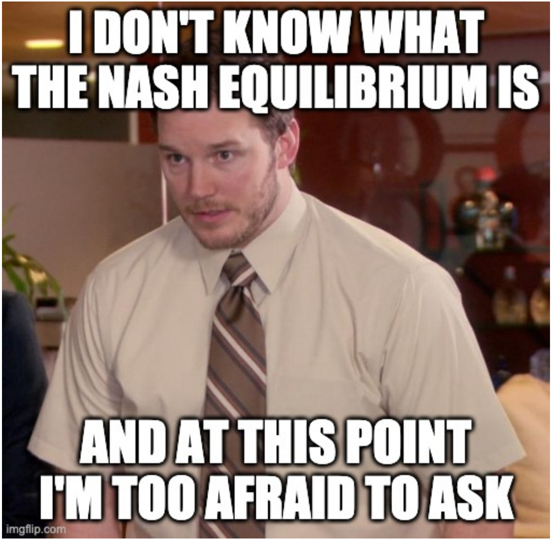

Thank you for your attention.