

Algorithmic game theory

Martin Balko

2nd lecture

October 7th 2025



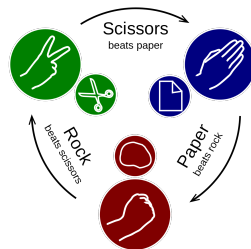
Proof of Nash's Theorem

Nash equilibria in normal-form games

Nash equilibria in normal-form games

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Sources: <https://en.wikipedia.org/>

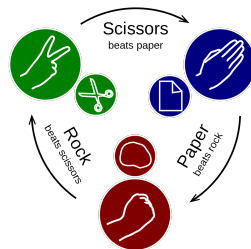


Nash equilibria in normal-form games

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Sources: <https://en.wikipedia.org/>

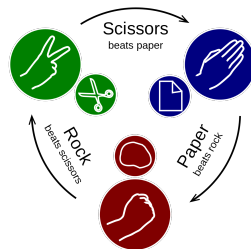
- We introduced perhaps the most influential solution concept, which captures a notion of stability.



Nash equilibria in normal-form games

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Sources: <https://en.wikipedia.org/>

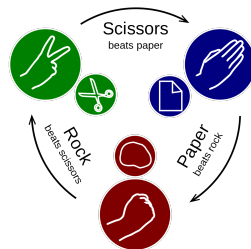


- We introduced perhaps the most influential solution concept, which captures a notion of stability.
- The **best response** of player i to a strategy profile s_{-i} is a mixed strategy s_i^* such that $u_i(s_i^*; s_{-i}) \geq u_i(s'_i; s_{-i})$ for each $s'_i \in S_i$.

Nash equilibria in normal-form games

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Sources: <https://en.wikipedia.org/>

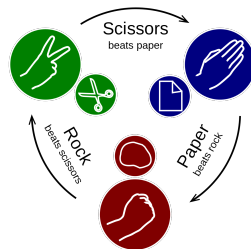


- We introduced perhaps the most influential solution concept, which captures a notion of stability.
- The **best response** of player i to a strategy profile s_{-i} is a mixed strategy s_i^* such that $u_i(s_i^*; s_{-i}) \geq u_i(s'_i; s_{-i})$ for each $s'_i \in S_i$.
- For a normal-form game $G = (P, A, u)$ of n players, a **Nash equilibrium (NE)** in G is a strategy profile $(s_1, \dots, s_n)$ such that s_i is a best response of player i to s_{-i} for every $i \in P$.

Nash equilibria in normal-form games

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Sources: <https://en.wikipedia.org/>



- We introduced perhaps the most influential solution concept, which captures a notion of stability.
- The **best response** of player i to a strategy profile s_{-i} is a mixed strategy s_i^* such that $u_i(s_i^*; s_{-i}) \geq u_i(s_i'; s_{-i})$ for each $s_i' \in S_i$.
- For a normal-form game $G = (P, A, u)$ of n players, a **Nash equilibrium (NE)** in G is a strategy profile $(s_1, \dots, s_n)$ such that s_i is a best response of player i to s_{-i} for every $i \in P$.
- Amazingly, **every normal-form game has a Nash equilibrium**.

Nash's Theorem

Nash's Theorem (Theorem 2.16)

Every normal-form game has a Nash equilibrium.

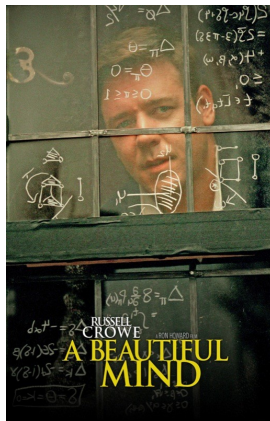


Figure: John Forbes Nash Jr. (1928–2015) and his depiction in the movie *A Beautiful mind*.

Preparations for the proof of Nash's theorem

Preparations for the proof of Nash's theorem

- The proof is essentially **topological**, as its main ingredient is a fixed-point theorem. We use a theorem due to **Brouwer**.

Preparations for the proof of Nash's theorem

- The proof is essentially **topological**, as its main ingredient is a fixed-point theorem. We use a theorem due to **Brouwer**.
- For $d \in \mathbb{N}$, a subset X of $\mathbb{R}^d$ is **compact** if X is closed and bounded.
- We say that a subset Y of $\mathbb{R}^d$ is **convex** if every line segment containing two points from Y is fully contained in Y . Formally: for all x, y from Y , $tx + (1 - t)y \in Y$ for every $t \in [0, 1]$.
- For n affinely independent points $x_1, \dots, x_n \in \mathbb{R}^d$, an **$(n - 1)$ -simplex Δ_n** on $x_1, \dots, x_n$ is the set of convex combinations of the points $x_1, \dots, x_n$. Each simplex is a compact convex set in $\mathbb{R}^d$.

Preparations for the proof of Nash's theorem

- The proof is essentially **topological**, as its main ingredient is a fixed-point theorem. We use a theorem due to **Brouwer**.
- For $d \in \mathbb{N}$, a subset X of $\mathbb{R}^d$ is **compact** if X is closed and bounded.
- We say that a subset Y of $\mathbb{R}^d$ is **convex** if every line segment containing two points from Y is fully contained in Y . Formally: for all x, y from Y , $tx + (1 - t)y \in Y$ for every $t \in [0, 1]$.
- For n affinely independent points $x_1, \dots, x_n \in \mathbb{R}^d$, an **$(n - 1)$ -simplex** Δ_n on $x_1, \dots, x_n$ is the set of convex combinations of the points $x_1, \dots, x_n$. Each simplex is a compact convex set in $\mathbb{R}^d$.

Lemma (Lemma 2.18)

For $n, d_1, \dots, d_n \in \mathbb{N}$, let $K_1, \dots, K_n$ be compact sets, each K_i lying in $\mathbb{R}^{d_i}$. Then, $K_1 \times \dots \times K_n$ is a compact set in $\mathbb{R}^{d_1 + \dots + d_n}$.

Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem

- For each $d \in \mathbb{N}$, let K be a non-empty compact convex set in $\mathbb{R}^d$ and $f: K \rightarrow K$ be a continuous mapping. Then, there exists a fixed point $x_0 \in K$ for f , that is, $f(x_0) = x_0$.



Figure: L. E. J. Brouwer (1881–1966).

Source: <https://arxiv.org/pdf/1612.06820.pdf>

- https://www.youtube.com/watch?v=csInNn6pfT4&t=268s&ab_

Proof of Nash's Theorem I

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players.

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .
- We start with K .

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .
- We start with K . Let $K = S_1 \times \cdots \times S_n$ be the set of all mixed strategies.

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .
- We start with K . Let $K = S_1 \times \cdots \times S_n$ be the set of all mixed strategies.
 - We verify that K is compact and convex.

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .
- We start with K . Let $K = S_1 \times \cdots \times S_n$ be the set of all mixed strategies.
 - We verify that K is compact and convex.
 - By definition, each S_i is, a simplex

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .
- We start with K . Let $K = S_1 \times \cdots \times S_n$ be the set of all mixed strategies.
 - We verify that K is compact and convex.
 - By definition, each S_i is, a simplex which is compact and convex.

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .
- We start with K . Let $K = S_1 \times \cdots \times S_n$ be the set of all mixed strategies.
 - We verify that K is compact and convex.
 - By definition, each S_i is, a simplex which is compact and convex.
 - By **Lemma 2.18**, the set $K = S_1 \times \cdots \times S_n$ is **compact**.

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .
- We start with K . Let $K = S_1 \times \cdots \times S_n$ be the set of all mixed strategies.
 - We verify that K is compact and convex.
 - By definition, each S_i is, a simplex which is compact and convex.
 - By **Lemma 2.18**, the set $K = S_1 \times \cdots \times S_n$ is **compact**.
 - For any strategy profiles $s = (s_1, \dots, s_n)$, $s' = (s'_1, \dots, s'_n) \in K$ and a number $t \in [0, 1]$, the point

$$ts + (1 - t)s' = (ts_1 + (1 - t)s'_1, \dots, ts_n + (1 - t)s'_n)$$

is also a mixed-strategy profile in K .

Proof of Nash's Theorem I

- Let $G = (P, A, u)$ be a normal-form game of n players. Recall that S_i is the set of mixed strategies of player i .
- We want to apply **Brouwer's theorem**, thus we need to find a suitable compact convex body K and a continuous mapping $f: K \rightarrow K$ whose fixed points are NE in G .
- We start with K . Let $K = S_1 \times \cdots \times S_n$ be the set of all mixed strategies.
 - We verify that K is compact and convex.
 - By definition, each S_i is, a simplex which is compact and convex.
 - By **Lemma 2.18**, the set $K = S_1 \times \cdots \times S_n$ is **compact**.
 - For any strategy profiles $s = (s_1, \dots, s_n)$, $s' = (s'_1, \dots, s'_n) \in K$ and a number $t \in [0, 1]$, the point

$$ts + (1 - t)s' = (ts_1 + (1 - t)s'_1, \dots, ts_n + (1 - t)s'_n)$$

is also a mixed-strategy profile in K . Thus, K is **convex**.

Proof of Nash's Theorem II

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.
- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}: K \rightarrow \mathbb{R}$ by setting

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}.$$

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.
- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}: K \rightarrow \mathbb{R}$ by setting

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}.$$

- $\varphi_{i,a_i}(s) > 0$ iff i can improve his payoff by using a_i instead of s_i .

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.
- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}: K \rightarrow \mathbb{R}$ by setting

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}.$$

- $\varphi_{i,a_i}(s) > 0$ iff i can improve his payoff by using a_i instead of s_i .
- By the definition of u_i , this mapping is continuous.

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.
- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}: K \rightarrow \mathbb{R}$ by setting

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}.$$

- $\varphi_{i,a_i}(s) > 0$ iff i can improve his payoff by using a_i instead of s_i .
 - By the definition of u_i , this mapping is continuous.
- Given $s \in K$, we define a new “improved” strategy profile $s' \in K$ as

$$s'_i(a_i) = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} (s_i(b_i) + \varphi_{i,b_i}(s))}$$

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.
- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}: K \rightarrow \mathbb{R}$ by setting

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}.$$

- $\varphi_{i,a_i}(s) > 0$ iff i can improve his payoff by using a_i instead of s_i .
 - By the definition of u_i , this mapping is **continuous**.
- Given $s \in K$, we define a new “improved” strategy profile $s' \in K$ as

$$s'_i(a_i) = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} (s_i(b_i) + \varphi_{i,b_i}(s))} = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}.$$

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.
- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}: K \rightarrow \mathbb{R}$ by setting

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}.$$

- $\varphi_{i,a_i}(s) > 0$ iff i can improve his payoff by using a_i instead of s_i .
 - By the definition of u_i , this mapping is **continuous**.
- Given $s \in K$, we define a new “improved” strategy profile $s' \in K$ as

$$s'_i(a_i) = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} (s_i(b_i) + \varphi_{i,b_i}(s))} = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}.$$

- “Increase probability at actions that are better responses to s_{-i} .”

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.
- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}: K \rightarrow \mathbb{R}$ by setting

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}.$$

- $\varphi_{i,a_i}(s) > 0$ iff i can improve his payoff by using a_i instead of s_i .
 - By the definition of u_i , this mapping is **continuous**.
- Given $s \in K$, we define a new “improved” strategy profile $s' \in K$ as

$$s'_i(a_i) = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} (s_i(b_i) + \varphi_{i,b_i}(s))} = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}.$$

- “Increase probability at actions that are better responses to s_{-i} .”
 - $s' \in K$ as each $s'_i(a_i)$ lies in $[0, 1]$ and $\sum_{a_i \in A_i} s'_i(a_i) = 1$.

Proof of Nash's Theorem II

- We now find the continuous mapping $f: K \rightarrow K$.
- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}: K \rightarrow \mathbb{R}$ by setting

$$\varphi_{i,a_i}(s) = \max\{0, u_i(a_i; s_{-i}) - u_i(s)\}.$$

- $\varphi_{i,a_i}(s) > 0$ iff i can improve his payoff by using a_i instead of s_i .
 - By the definition of u_i , this mapping is **continuous**.
- Given $s \in K$, we define a new “improved” strategy profile $s' \in K$ as

$$s'_i(a_i) = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} (s_i(b_i) + \varphi_{i,b_i}(s))} = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}.$$

- “Increase probability at actions that are better responses to s_{-i} .”
 - $s' \in K$ as each $s'_i(a_i)$ lies in $[0, 1]$ and $\sum_{a_i \in A_i} s'_i(a_i) = 1$.
- We then define f by setting $f(s) = s'$.

Proof of Nash's Theorem III

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G .

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$.

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$.

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$,

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$ and, since $s_i(a'_i) > 0$, the denominator in the denominator is 1.

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$ and, since $s_i(a'_i) > 0$, the denominator in the denominator is 1. This means that $\varphi_{i,b_i}(s) = 0$ for every $b_i \in A_i$.

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$ and, since $s_i(a'_i) > 0$, the denominator in the denominator is 1. This means that $\varphi_{i,b_i}(s) = 0$ for every $b_i \in A_i$. It follows that s is NE as

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$ and, since $s_i(a'_i) > 0$, the denominator in the denominator is 1. This means that $\varphi_{i,b_i}(s) = 0$ for every $b_i \in A_i$. It follows that s is NE as

$$u_i(s''_i; s_{-i})$$

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$ and, since $s_i(a'_i) > 0$, the denominator in the denominator is 1. This means that $\varphi_{i,b_i}(s) = 0$ for every $b_i \in A_i$. It follows that s is NE as

$$u_i(s''_i; s_{-i}) = \sum_{b_i \in A_i} s''_i(b_i) u_i(b_i; s_{-i})$$

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$ and, since $s_i(a'_i) > 0$, the denominator in the denominator is 1. This means that $\varphi_{i,b_i}(s) = 0$ for every $b_i \in A_i$. It follows that s is NE as

$$u_i(s''_i; s_{-i}) = \sum_{b_i \in A_i} s''_i(b_i) u_i(b_i; s_{-i}) \leq \sum_{b_i \in A_i} s''_i(b_i) u_i(s)$$

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$ and, since $s_i(a'_i) > 0$, the denominator in the denominator is 1. This means that $\varphi_{i,b_i}(s) = 0$ for every $b_i \in A_i$. It follows that s is NE as

$$u_i(s''_i; s_{-i}) = \sum_{b_i \in A_i} s''_i(b_i) u_i(b_i; s_{-i}) \leq \sum_{b_i \in A_i} s''_i(b_i) u_i(s) = u_i(s).$$

Proof of Nash's Theorem III

- Then, f is **continuous**, since the mappings φ_{i,a_i} are.
- It remains to show that **fixed points of f are exactly NE** in G . Then, **Brouwer's theorem** gives us a fixed point of f , which is NE in G .
- **First**, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$. So s is a fixed point for f .
- **Second**, assume that $s = (s_1, \dots, s_n) \in K$ is a fixed point for f .
 - For any player i , there is $a'_i \in A_i$ with $s_i(a_i) > 0$ such that $u_i(a'_i; s_{-i}) \leq u_i(s)$. Otherwise, $u_i(s) < \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$, which is impossible by the **linearity of the expected payoff**.
 - Then, $\varphi_{i,a'_i}(s) = 0$ and we get $s'_i(a'_i) = \frac{s_i(a'_i)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$.
 - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$ and, since $s_i(a'_i) > 0$, the denominator in the denominator is 1. This means that $\varphi_{i,b_i}(s) = 0$ for every $b_i \in A_i$. It follows that s is NE as

$$u_i(s''_i; s_{-i}) = \sum_{b_i \in A_i} s''_i(b_i) u_i(b_i; s_{-i}) \leq \sum_{b_i \in A_i} s''_i(b_i) u_i(s) = u_i(s).$$



Nash's Theorem: remarks

Nash's Theorem: remarks

- Two pages worth of Nobel prize!

45 MATHEMATICS: J. F. NASH, JR. Proc. N. A. S.

This follows from the arguments used in a forthcoming paper.¹⁰ It is proved by constructing an "abstract" mapping cylinder of λ and translating into algebraic terms the proof of the endgame theorem on CNL-congruence.

¹⁰ This was done from considerations during the tenure of a John Simon Guggenheim Memorial Fellowship by Madison.

¹¹ Kuhn, L. H. G., "Combinatorial Homotopy I and II," *Ann. d. M. S.*, 38, 284-309 and 401-497 (1960). We refer to these papers as CH I and CH II, respectively.

¹² For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

¹³ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

¹⁴ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

¹⁵ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

¹⁶ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

¹⁷ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

¹⁸ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

¹⁹ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

²⁰ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

²¹ For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

²² For a complete and self-contained treatment of CNL-congruence, see the book by Nash, J. F., "The Theory of Games and Economic Behavior," Princeton University Press, Princeton, 1950.

EQUILIBRIUM POINTS IN n -PERSON GAMES

By JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Slichter, November 15, 1950

One may define a concept of an n -person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n -tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

46 MATHEMATICS: G. POLYA

distribution over the pure strategies, the payoff functions are the expectations of the players, thus becoming probabilistic forms in the probabilities with which the various players play their various pure strategies.

Any n -tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n -tuple contains another if the strategy of each player in the containing n -tuple yields the highest obtainable expectation for its player against the $n-1$ strategies of the other players in the contained n -tuple. A self-containing n -tuple is called an equilibrium point.

The correspondence of such n -tuples with its set of containing n -tuples gives a one-to-one mapping of the product space into itself. From the definition of containing we see that the set of containing points of a point is convex. By using the continuity of the payoff functions we see that the graph of the mapping is closed.

The elements in the image of the mapping are equivalent to saying $P_1, P_2, \dots, P_n$ and $Q_1, Q_2, \dots, Q_n$ are sequences of points in the product space where $Q_i \rightarrow P_i$ and Q_i contains P_i then Q contains P .

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem¹ that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person case one can see the "main theorem" and the existence of an equilibrium point are equivalent. In this case any two equilibria points lead to the same expectations for the players, but this need not occur in general.

¹ The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the U. S. R. C. for financial support.

* Kuhn, L. H. G., *Ann. d. M. S.*, 38, 284-309 and 401-497 (1960).

² Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behavior*, Princeton University Press, Princeton, 1947.

REMARKS ON WEYL'S NOTE "INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION"

By GEORGE POLYA

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated by H. Weyl, November 15, 1949

In the note quoted above H. Weyl proved a theorem involving a function $\psi(\lambda)$ and concerning the eigenvalues α_i of a linear transformation A and those, α_i of A^*A . If the α_i and $\lambda_i = |\alpha_i|^2$ are arranged in descending order,

Sources: J. F. Nash: Equilibrium points in n -person games (1950).

Nash's Theorem: remarks

- Two pages worth of **Nobel prize!**

45 MATHEMATICS: J. F. NASH, JR. PRIZE: N. A. S.

This follows from the arguments used in a forthcoming paper.¹⁰ It is proved by constructing an "abstract" mapping cylinder of Γ and translating into algebraic terms the proof of the endgame theorem on CNL compactness.

¹⁰ This was done from considerations during the tenure of a John Simon Guggenheim Memorial Fellowship by Madison.

¹¹ Kuhn, J. R. C., "Combinatorial Homotopy I and II," *Ann. of Math.*, 68, 284-306 and 489-490 (1958). We refer to these papers as CH I and CH II, respectively.

¹² For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

¹³ For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

¹⁴ For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

¹⁵ For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

¹⁶ For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

¹⁷ For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

¹⁸ For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

¹⁹ For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

²⁰ For a complete proof of the existence of a Nash equilibrium, see the book by Nash, J. F., *Non-cooperative Games*, (Cambridge, Mass., 1950).

46 MATHEMATICS: G. POLYA

distribution over the pure strategies, the pay-off function are the expectations of the players, then bounding probability forms in the probability with which the various players play their various pure strategies.

Any n -tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n -tuple contains another if the strategy of each player in the containing n -tuple yields the highest obtainable expectation for its player against the $n-1$ strategies of the other players in the contained n -tuple. A self-containing n -tuple is called an equilibrium point. The correspondence of such n -tuples with its set of containing n -tuples gives a one-to-one mapping of the product space into itself. From the definition of containing we see that the set of containing points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: If $P_1, P_2, \dots$ and $Q_1, Q_2, \dots$ are sequences of points in the product space where $Q_i \rightarrow Q$, $P_i \rightarrow P$ and Q_i contains P_i , then Q contains P . Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person case we have the "main theorem" and the existence of an equilibrium point are equivalent. In this case any two equilibria points lead to the same expectations for the players, but this need not occur in general.

¹ The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof of the main theorem.

² Kuhn, J. R. C., *Ann. of Math.*, 68, 489-490 (1958).

³ Nash, J. F., and Harsanyi, J., *The Theory of Games and Economic Behavior*, Chap. 5, Princeton University Press, Princeton, 1950.

EQUILIBRIUM POINTS IN n -PERSON GAMES

By JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Slichter, November 15, 1950

One may define a concept of an n -person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n -tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

REMARKS ON WEYL'S NOTE "INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION"

By GEORGE POLYA

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated by H. Weyl, November 15, 1950

In the note quoted above H. Weyl proved a theorem involving a function $\psi(\lambda)$ and concerning the eigenvalues α_i of a linear transformation A and those, α_i of A^*A . If the α_i and $\lambda_i = |\alpha_i|^2$ are arranged in descending order,

Sources: J. F. Nash: Equilibrium points in n -person games (1950).

- Requires **finite numbers of players and actions**, both assumptions are necessary.

Nash's Theorem: remarks

- Two pages worth of **Nobel prize!**

45 MATHEMATICS: J. F. NASH, JR. Proc. N. A. S.

This follows from the arguments used in a forthcoming paper.¹⁰ It is proved by constructing an "abstract" mapping cylinder of I and transcribing into algebraic form the proof of the endogenous theorem on CNL mappings.

¹⁰ This was done from considerations during the tenure of a John Simon Guggenheim Memorial Fellowship by Madison.

¹¹ Villard, J. R. Co., "Combinatorial Homotopy I and II," *Adv. M.S.*, 18, 261-280 and 401-402 (1965). We refer to these papers as CH I and CH II respectively.

¹² For a complete and self-contained treatment of CNL mappings, see Villard, J. R. Co., "Combinatorial Homotopy I and II," *Adv. M.S.*, 18, 261-280 and 401-402 (1965). We refer to these papers as CH I and CH II respectively.

¹³ For a complete and self-contained treatment of CNL mappings, see Villard, J. R. Co., "Combinatorial Homotopy I and II," *Adv. M.S.*, 18, 261-280 and 401-402 (1965). We refer to these papers as CH I and CH II respectively.

¹⁴ For a complete and self-contained treatment of CNL mappings, see Villard, J. R. Co., "Combinatorial Homotopy I and II," *Adv. M.S.*, 18, 261-280 and 401-402 (1965). We refer to these papers as CH I and CH II respectively.

¹⁵ For a complete and self-contained treatment of CNL mappings, see Villard, J. R. Co., "Combinatorial Homotopy I and II," *Adv. M.S.*, 18, 261-280 and 401-402 (1965). We refer to these papers as CH I and CH II respectively.

46 MATHEMATICS: G. POLYA

distribution over the pure strategies, the p_i functions are the expectations of the players, then bounding probability forms in the probability simplex with which the various players play their various pure strategies.

Any n -tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n -tuple contains another if the strategy of each player in the containing n -tuple yields the highest obtainable expectation for its player against the $n-1$ strategies of the other players in the contained n -tuple. A self-containing n -tuple is called an equilibrium point. The correspondence of each n -tuple with its set of containing n -tuples gives a one-to-many mapping of the product space into itself. From the definition of containing we see that the set of containing points of a point is convex. By using the continuity of the p_i functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: If $P_1, P_2, \dots$ and $Q_1, Q_2, \dots$ are sequences of points in the product space where $Q_i \rightarrow Q$, $P_i \rightarrow P$ and Q_i contains P_i , then Q contains P . Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem¹¹ that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person case we have the "minimax theorem" and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

¹¹ The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof of the n -person case.

¹² Kakutani, S., *Duke Math. J.*, 8, 407-415 (1941).

¹³ Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behavior*, Chap. 3, Princeton University Press, Princeton, 1944.

EQUILIBRIUM POINTS IN n -PERSON GAMES

By JOHN F. NASH, JR.*

Princeton University

Communicated by S. Slichter, November 15, 1950

One may define a concept of an n -person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n -tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

REMARKS ON WEYL'S NOTE "INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION"

By GEORGE POLYA

Department of Mathematics, Princeton University

Communicated by H. Weyl, November 15, 1949

In the note quoted above H. Weyl proved a theorem involving a function $\psi(\lambda)$ and concerning the eigenvalues α_i of a linear transformation A and those, α_i of A^*A . If the α_i and $\lambda_i = |\alpha_i|^2$ are arranged in descending order,

Sources: J. F. Nash: Equilibrium points in n -person games (1950).

- Requires **finite numbers of players and actions**, both assumptions are necessary. (Consider 2-player game "who guesses larger number wins".)

Pareto optimality

Pareto optimality

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.
- We want to capture “the best” state of a game.

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.
- We want to capture “the best” state of a game. Might be difficult, consider the Battle of sexes.

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.
- We want to capture “the best” state of a game. Might be difficult, consider the Battle of sexes.
- A strategy profile s in G Pareto dominates s' , written $s' \prec s$, if, for every player i , $u_i(s) \geq u_i(s')$, and there exists a player j such that $u_j(s) > u_j(s')$.

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.
- We want to capture “the best” state of a game. Might be difficult, consider the Battle of sexes.
- A strategy profile s in G Pareto dominates s' , written $s' \prec s$, if, for every player i , $u_i(s) \geq u_i(s')$, and there exists a player j such that $u_j(s) > u_j(s')$.
 - The relation $\prec$ is a partial ordering of the set S of all strategy profiles of G .

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.
- We want to capture “the best” state of a game. Might be difficult, consider the Battle of sexes.
- A strategy profile s in G Pareto dominates s' , written $s' \prec s$, if, for every player i , $u_i(s) \geq u_i(s')$, and there exists a player j such that $u_j(s) > u_j(s')$.
 - The relation $\prec$ is a partial ordering of the set S of all strategy profiles of G .
 - The outcomes of G that are considered best are the maximal elements of S in $\prec$.

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.
- We want to capture “the best” state of a game. Might be difficult, consider the Battle of sexes.
- A strategy profile s in G Pareto dominates s' , written $s' \prec s$, if, for every player i , $u_i(s) \geq u_i(s')$, and there exists a player j such that $u_j(s) > u_j(s')$.
 - The relation $\prec$ is a partial ordering of the set S of all strategy profiles of G .
 - The outcomes of G that are considered best are the maximal elements of S in $\prec$.
- A strategy profile $s \in S$ is Pareto optimal if there does not exist another strategy profile $s' \in S$ that Pareto dominates s .

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.
- We want to capture “the best” state of a game. Might be difficult, consider the Battle of sexes.
- A strategy profile s in G Pareto dominates s' , written $s' \prec s$, if, for every player i , $u_i(s) \geq u_i(s')$, and there exists a player j such that $u_j(s) > u_j(s')$.
 - The relation $\prec$ is a partial ordering of the set S of all strategy profiles of G .
 - The outcomes of G that are considered best are the maximal elements of S in $\prec$.
- A strategy profile $s \in S$ is Pareto optimal if there does not exist another strategy profile $s' \in S$ that Pareto dominates s .
 - In zero-sum games, all strategy profiles are Pareto-optimal.

Pareto optimality

- A brief detour: another example of an interesting solution concept, other than NE.
- We want to capture “the best” state of a game. Might be difficult, consider the Battle of sexes.
- A strategy profile s in G Pareto dominates s' , written $s' \prec s$, if, for every player i , $u_i(s) \geq u_i(s')$, and there exists a player j such that $u_j(s) > u_j(s')$.
 - The relation $\prec$ is a partial ordering of the set S of all strategy profiles of G .
 - The outcomes of G that are considered best are the maximal elements of S in $\prec$.
- A strategy profile $s \in S$ is Pareto optimal if there does not exist another strategy profile $s' \in S$ that Pareto dominates s .
 - In zero-sum games, all strategy profiles are Pareto-optimal.
 - Not all NE are Pareto-optimal (the NE in Prisoner's dilemma)

Vilfredo Pareto

Vilfredo Pareto

- an Italian engineer, sociologist, economist, political scientist, and philosopher.

Vilfredo Pareto

- an Italian engineer, sociologist, economist, political scientist, and philosopher.

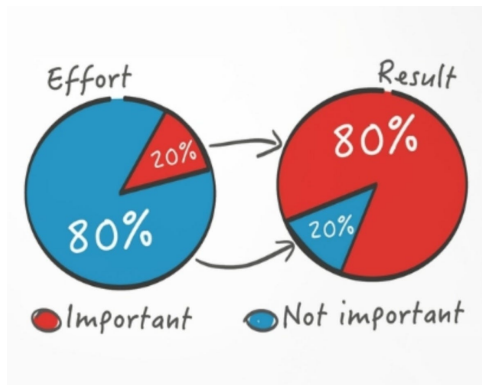
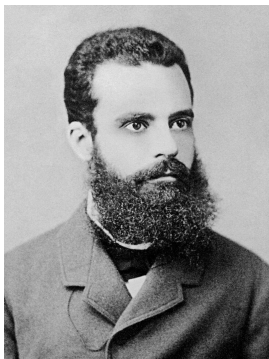


Figure: Vilfredo Pareto (1848–1923).

Sources: <https://en.wikipedia.org> and <https://medium.com/>

Vilfredo Pareto

- an Italian engineer, sociologist, economist, political scientist, and philosopher.

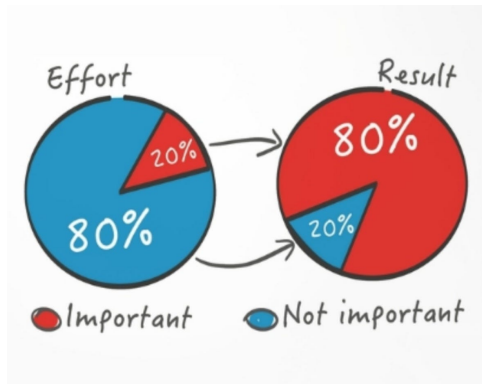
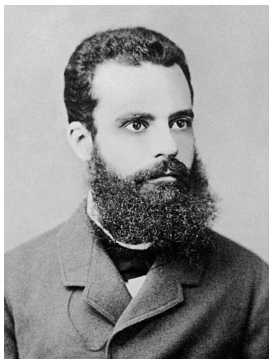


Figure: Vilfredo Pareto (1848–1923).

Sources: <https://en.wikipedia.org> and <https://medium.com/>

- Pareto principle:** for many outcomes roughly 80% of consequences come from 20% of the causes.

Finding Nash equilibria in zero-sum games

Finding Nash equilibria

Finding Nash equilibria

- We know that NE exist in every normal-form game (**Nash's theorem**).

Finding Nash equilibria

- We know that NE exist in every normal-form game (**Nash's theorem**).
- However, **we do not have any algorithm** for how to find them yet.

Finding Nash equilibria

- We know that NE exist in every normal-form game (**Nash's theorem**).
- However, **we do not have any algorithm** for how to find them yet.
- We start with a simple class of 2-player games, so-called **zero-sum games**.

Finding Nash equilibria

- We know that NE exist in every normal-form game (**Nash's theorem**).
- However, **we do not have any algorithm** for how to find them yet.
- We start with a simple class of 2-player games, so-called **zero-sum games**.
- We show that **we can find NE efficiently in this case**.

Finding Nash equilibria

- We know that NE exist in every normal-form game (**Nash's theorem**).
- However, **we do not have any algorithm** for how to find them yet.
- We start with a simple class of 2-player games, so-called **zero-sum games**.
- We show that **we can find NE efficiently in this case**. In fact, we show that NE “solves” zero-sum games completely.

Finding Nash equilibria

- We know that NE exist in every normal-form game (**Nash's theorem**).
- However, **we do not have any algorithm** for how to find them yet.
- We start with a simple class of 2-player games, so-called **zero-sum games**.
- We show that **we can find NE efficiently in this case**. In fact, we show that NE “solves” zero-sum games completely.
- Historically, zero-sum games were considered first in game theory (by **Morgenstern** and **Von Neumann** in the 1940s).

Zero-sum games

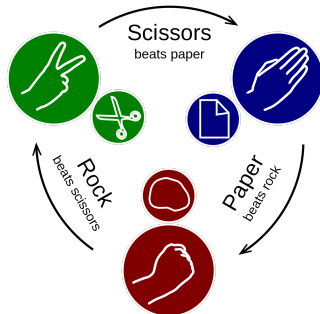
Zero-sum games

- Two-player games (P, A, u) where $u_1(a) = -u_2(a)$ for every $a \in A$.

Zero-sum games

- Two-player games (P, A, u) where $u_1(a) = -u_2(a)$ for every $a \in A$.

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)



Sources: <https://en.wikipedia.org/>

Zero-sum games examples: chess



Source: <https://edition.cnn.com/>

Zero-sum games examples: table tennis



Source: <https://www.reddit.com/>

Zero-sum games examples: derivative trading



Source: <https://www.linkedin.com/>

Zero-sum games examples: elections



Source: <https://youtube.com/>

Zero-sum games examples: many more



Source: <https://lhongtortai.com/collection/what-is-a-non-zero-sum-game>

Representing zero-sum games

Representing zero-sum games

- With zero-sum games, our notation simplifies.

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game.

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.
- If $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, then G can be represented with an $m \times n$ payoff matrix M where $M_{i,j} = u_1(i, j)$

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.
- If $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, then G can be represented with an $m \times n$ payoff matrix M where $M_{i,j} = u_1(i, j) = -u_2(i, j)$.

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.
- If $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, then G can be represented with an $m \times n$ payoff matrix M where $M_{i,j} = u_1(i, j) = -u_2(i, j)$.
- For a strategy profile (s_1, s_2) , we write $x_i = s_1(i)$ and $y_j = s_2(j)$,

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.
- If $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, then G can be represented with an $m \times n$ payoff matrix M where $M_{i,j} = u_1(i, j) = -u_2(i, j)$.
- For a strategy profile (s_1, s_2) , we write $x_i = s_1(i)$ and $y_j = s_2(j)$, representing (s_1, s_2) with mixed strategy vectors $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ that satisfy $\sum_{i=1}^m x_i = 1$ and $\sum_{j=1}^n y_j = 1$.

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.
- If $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, then G can be represented with an $m \times n$ payoff matrix M where $M_{i,j} = u_1(i, j) = -u_2(i, j)$.
- For a strategy profile (s_1, s_2) , we write $x_i = s_1(i)$ and $y_j = s_2(j)$, representing (s_1, s_2) with mixed strategy vectors $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ that satisfy $\sum_{i=1}^m x_i = 1$ and $\sum_{j=1}^n y_j = 1$.
- The expected payoff of player 1 then equals

$$u_1(s) = \sum_{a=(i,j) \in A} u_1(a) s_1(i) s_2(j)$$

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.
- If $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, then G can be represented with an $m \times n$ payoff matrix M where $M_{i,j} = u_1(i, j) = -u_2(i, j)$.
- For a strategy profile (s_1, s_2) , we write $x_i = s_1(i)$ and $y_j = s_2(j)$, representing (s_1, s_2) with mixed strategy vectors $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ that satisfy $\sum_{i=1}^m x_i = 1$ and $\sum_{j=1}^n y_j = 1$.
- The expected payoff of player 1 then equals

$$u_1(s) = \sum_{a=(i,j) \in A} u_1(a) s_1(i) s_2(j) = \sum_{i=1}^m \sum_{j=1}^n M_{i,j} x_i y_j$$

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.
- If $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, then G can be represented with an $m \times n$ payoff matrix M where $M_{i,j} = u_1(i, j) = -u_2(i, j)$.
- For a strategy profile (s_1, s_2) , we write $x_i = s_1(i)$ and $y_j = s_2(j)$, representing (s_1, s_2) with mixed strategy vectors $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ that satisfy $\sum_{i=1}^m x_i = 1$ and $\sum_{j=1}^n y_j = 1$.
- The expected payoff of player 1 then equals

$$u_1(s) = \sum_{a=(i,j) \in A} u_1(a) s_1(i) s_2(j) = \sum_{i=1}^m \sum_{j=1}^n M_{i,j} x_i y_j = x^\top M y$$

Representing zero-sum games

- With zero-sum games, our notation simplifies.
- Let $G = (P, A = A_1 \times A_2, u)$ be a zero-sum game. That is, $u_1(a) + u_2(a) = 0$ for every $a \in A$.
- If $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, then G can be represented with an $m \times n$ payoff matrix M where $M_{i,j} = u_1(i, j) = -u_2(i, j)$.
- For a strategy profile (s_1, s_2) , we write $x_i = s_1(i)$ and $y_j = s_2(j)$, representing (s_1, s_2) with mixed strategy vectors $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ that satisfy $\sum_{i=1}^m x_i = 1$ and $\sum_{j=1}^n y_j = 1$.
- The expected payoff of player 1 then equals

$$u_1(s) = \sum_{a=(i,j) \in A} u_1(a) s_1(i) s_2(j) = \sum_{i=1}^m \sum_{j=1}^n M_{i,j} x_i y_j = x^\top M y = -u_2(s).$$

Worst-case optimal strategies

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$.

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$. Player's 1 best response to a strategy y of 2 is $x \in S_1$ that maximizes $x^\top My$.

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$. Player's 1 best response to a strategy y of 2 is $x \in S_1$ that maximizes $x^\top My$.
- Let $\beta(x) = \min_{y \in S_2} x^\top My$ be the best expected payoff of 2 against x .

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$. Player's 1 best response to a strategy y of 2 is $x \in S_1$ that maximizes $x^\top My$.
- Let $\beta(x) = \min_{y \in S_2} x^\top My$ be the best expected payoff of 2 against x . Let $\alpha(y) = \max_{x \in S_1} x^\top My$ be the best expected payoff of 1 to y .

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$. Player's 1 best response to a strategy y of 2 is $x \in S_1$ that maximizes $x^\top My$.
- Let $\beta(x) = \min_{y \in S_2} x^\top My$ be the best expected payoff of 2 against x . Let $\alpha(y) = \max_{x \in S_1} x^\top My$ be the best expected payoff of 1 to y .
- A strategy profile (x, y) is then a NE if and only if it satisfies $\beta(x) = x^\top My = \alpha(y)$.

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$. Player's 1 best response to a strategy y of 2 is $x \in S_1$ that maximizes $x^\top My$.
- Let $\beta(x) = \min_{y \in S_2} x^\top My$ be the best expected payoff of 2 against x . Let $\alpha(y) = \max_{x \in S_1} x^\top My$ be the best expected payoff of 1 to y .
- A strategy profile (x, y) is then a NE if and only if it satisfies $\beta(x) = x^\top My = \alpha(y)$.
- Assume player 1 expects player 2 to select a best response to every strategy x he can come up with.

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$. Player's 1 best response to a strategy y of 2 is $x \in S_1$ that maximizes $x^\top My$.
- Let $\beta(x) = \min_{y \in S_2} x^\top My$ be the best expected payoff of 2 against x . Let $\alpha(y) = \max_{x \in S_1} x^\top My$ be the best expected payoff of 1 to y .
- A strategy profile (x, y) is then a NE if and only if it satisfies $\beta(x) = x^\top My = \alpha(y)$.
- Assume player 1 expects player 2 to select a best response to every strategy x he can come up with. Player 1 then chooses a mixed strategy $\bar{x}$ from S_1 that maximizes his expected payoff under this, rather pessimistic, assumption.

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$. Player's 1 best response to a strategy y of 2 is $x \in S_1$ that maximizes $x^\top My$.
- Let $\beta(x) = \min_{y \in S_2} x^\top My$ be the best expected payoff of 2 against x . Let $\alpha(y) = \max_{x \in S_1} x^\top My$ be the best expected payoff of 1 to y .
- A strategy profile (x, y) is then a NE if and only if it satisfies $\beta(x) = x^\top My = \alpha(y)$.
- Assume player 1 expects player 2 to select a best response to every strategy x he can come up with. Player 1 then chooses a mixed strategy $\bar{x}$ from S_1 that maximizes his expected payoff under this, rather pessimistic, assumption.
- This worst-case optimal strategy for 1 satisfies $\beta(\bar{x}) = \max_{x \in S_1} \beta(x)$.

Worst-case optimal strategies

- Thus, player's 2 best response to a strategy x of 1, is a vector $y \in S_2$ that minimizes $x^\top My$. Player's 1 best response to a strategy y of 2 is $x \in S_1$ that maximizes $x^\top My$.
- Let $\beta(x) = \min_{y \in S_2} x^\top My$ be the best expected payoff of 2 against x . Let $\alpha(y) = \max_{x \in S_1} x^\top My$ be the best expected payoff of 1 to y .
- A strategy profile (x, y) is then a NE if and only if it satisfies $\beta(x) = x^\top My = \alpha(y)$.
- Assume player 1 expects player 2 to select a best response to every strategy x he can come up with. Player 1 then chooses a mixed strategy $\bar{x}$ from S_1 that maximizes his expected payoff under this, rather pessimistic, assumption.
- This worst-case optimal strategy for 1 satisfies $\beta(\bar{x}) = \max_{x \in S_1} \beta(x)$. The worst-case optimal strategy for 2 is a mixed strategy $\bar{y} \in S_2$ that satisfies $\alpha(\bar{y}) = \min_{y \in S_2} \alpha(y)$.

Worst-case optimal strategies and NE

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

(a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^\top M y \leq \alpha(y)$.

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^T M y \leq \alpha(y)$.
- (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^T M y \leq \alpha(y)$.
- (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.
- (c) Any strategies $x^* \in S_1$ and $y^* \in S_2$ satisfying $\beta(x^*) = \alpha(y^*)$ form NE (x^*, y^*) .

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^\top M y \leq \alpha(y)$.
- (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.
- (c) Any strategies $x^* \in S_1$ and $y^* \in S_2$ satisfying $\beta(x^*) = \alpha(y^*)$ form NE (x^*, y^*) .
- (a) This follows immediately from the definitions of β and α .

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^\top M y \leq \alpha(y)$.
 - (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.
 - (c) Any strategies $x^* \in S_1$ and $y^* \in S_2$ satisfying $\beta(x^*) = \alpha(y^*)$ form NE (x^*, y^*) .
-
- (a) This follows immediately from the definitions of β and α .
 - (b) Part (a) implies that $\beta(x) \leq \alpha(y^*)$ for every $x \in S_1$.

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^\top M y \leq \alpha(y)$.
 - (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.
 - (c) Any strategies $x^* \in S_1$ and $y^* \in S_2$ satisfying $\beta(x^*) = \alpha(y^*)$ form NE (x^*, y^*) .
-
- (a) This follows immediately from the definitions of β and α .
 - (b) Part (a) implies that $\beta(x) \leq \alpha(y^*)$ for every $x \in S_1$. Since (x^*, y^*) is NE, we have $\beta(x^*) = \alpha(y^*)$ and thus $\beta(x) \leq \beta(x^*)$ for every $x \in S_1$.

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^\top M y \leq \alpha(y)$.
 - (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.
 - (c) Any strategies $x^* \in S_1$ and $y^* \in S_2$ satisfying $\beta(x^*) = \alpha(y^*)$ form NE (x^*, y^*) .
-
- (a) This follows immediately from the definitions of β and α .
 - (b) Part (a) implies that $\beta(x) \leq \alpha(y^*)$ for every $x \in S_1$. Since (x^*, y^*) is NE, we have $\beta(x^*) = \alpha(y^*)$ and thus $\beta(x) \leq \beta(x^*)$ for every $x \in S_1$. Thus, x^* is a worst-case optimal for 1.

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^\top M y \leq \alpha(y)$.
 - (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.
 - (c) Any strategies $x^* \in S_1$ and $y^* \in S_2$ satisfying $\beta(x^*) = \alpha(y^*)$ form NE (x^*, y^*) .
-
- (a) This follows immediately from the definitions of β and α .
 - (b) Part (a) implies that $\beta(x) \leq \alpha(y^*)$ for every $x \in S_1$. Since (x^*, y^*) is NE, we have $\beta(x^*) = \alpha(y^*)$ and thus $\beta(x) \leq \beta(x^*)$ for every $x \in S_1$. Thus, x^* is a worst-case optimal for 1. Analogously for player 2.

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^\top M y \leq \alpha(y)$.
 - (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.
 - (c) Any strategies $x^* \in S_1$ and $y^* \in S_2$ satisfying $\beta(x^*) = \alpha(y^*)$ form NE (x^*, y^*) .
-
- (a) This follows immediately from the definitions of β and α .
 - (b) Part (a) implies that $\beta(x) \leq \alpha(y^*)$ for every $x \in S_1$. Since (x^*, y^*) is NE, we have $\beta(x^*) = \alpha(y^*)$ and thus $\beta(x) \leq \beta(x^*)$ for every $x \in S_1$. Thus, x^* is a worst-case optimal for 1. Analogously for player 2.
 - (c) If $\beta(x^*) = \alpha(y^*)$, then (a) implies $\beta(x^*) = (x^*)^\top M y^* = \alpha(y^*)$.

Worst-case optimal strategies and NE

- To achieve NE in a zero-sum game, both players must select their worst-case optimal strategies.

Lemma 2.20

- (a) For all $x \in S_1$ and $y \in S_2$, we have $\beta(x) \leq x^\top M y \leq \alpha(y)$.
 - (b) If a strategy profile (x^*, y^*) is NE, then both strategies x^* and y^* are worst-case optimal.
 - (c) Any strategies $x^* \in S_1$ and $y^* \in S_2$ satisfying $\beta(x^*) = \alpha(y^*)$ form NE (x^*, y^*) .
-
- (a) This follows immediately from the definitions of β and α .
 - (b) Part (a) implies that $\beta(x) \leq \alpha(y^*)$ for every $x \in S_1$. Since (x^*, y^*) is NE, we have $\beta(x^*) = \alpha(y^*)$ and thus $\beta(x) \leq \beta(x^*)$ for every $x \in S_1$. Thus, x^* is a worst-case optimal for 1. Analogously for player 2.
 - (c) If $\beta(x^*) = \alpha(y^*)$, then (a) implies $\beta(x^*) = (x^*)^\top M y^* = \alpha(y^*)$. □

The Minimax Theorem

The Minimax Theorem

The Minimax Theorem (Theorem 2.21)

For every zero-sum game, worst-case optimal strategies for both players exist and can be efficiently computed.

The Minimax Theorem

The Minimax Theorem (Theorem 2.21)

For every zero-sum game, worst-case optimal strategies for both players exist and can be efficiently computed. There is a number v such that, for any worst-case optimal strategies x^* and y^* , the strategy profile (x^*, y^*) is a Nash equilibrium and $\beta(x^*) = (x^*)^\top M y^* = \alpha(y^*) = v$.

The Minimax Theorem

The Minimax Theorem (Theorem 2.21)

For every zero-sum game, worst-case optimal strategies for both players exist and can be efficiently computed. There is a number v such that, for any worst-case optimal strategies x^* and y^* , the strategy profile (x^*, y^*) is a Nash equilibrium and $\beta(x^*) = (x^*)^\top M y^* = \alpha(y^*) = v$.



Figure: John von Neumann (1903–1957) and Oskar Morgenstern (1902–1977).

The Minimax Theorem: remarks

The Minimax Theorem: remarks

- It was a starting point of game theory.

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by Von Neumann in 1928 (predates Nash's Theorem).

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by **Von Neumann** in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by **Von Neumann** in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).
- The Minimax theorem tells us everything about zero-sum games: **there is NE** and it can be found **efficiently**.

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by Von Neumann in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).
- The Minimax theorem tells us everything about zero-sum games: there is NE and it can be found efficiently. Moreover, there is a unique value of the game $v = (x^*)^\top M(y^*)$ of the payoff attained in any NE (x^*, y^*) .

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by **Von Neumann** in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).
- The Minimax theorem tells us everything about zero-sum games: **there is NE** and it can be found **efficiently**. Moreover, there is a unique **value of the game** $v = (x^*)^\top M(y^*)$ of the payoff attained in any NE (x^*, y^*) .
- There are **no secrets in zero-sum games**: strategies known in advance change nothing, each player can choose a worst-case optimal strategy and get payoff $\geq v$.

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by **Von Neumann** in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).
- The Minimax theorem tells us everything about zero-sum games: **there is NE** and it can be found **efficiently**. Moreover, there is a unique **value of the game** $v = (x^*)^\top M(y^*)$ of the payoff attained in any NE (x^*, y^*) .
- There are **no secrets in zero-sum games**: strategies known in advance change nothing, each player can choose a worst-case optimal strategy and get payoff $\geq v$. If the opponent chooses his worst-case optimal strategy, then his payoff is always $\leq v$.

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by **Von Neumann** in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).
- The Minimax theorem tells us everything about zero-sum games: **there is NE** and it can be found **efficiently**. Moreover, there is a unique **value of the game** $v = (x^*)^\top M(y^*)$ of the payoff attained in any NE (x^*, y^*) .
- There are **no secrets in zero-sum games**: strategies known in advance change nothing, each player can choose a worst-case optimal strategy and get payoff $\geq v$. If the opponent chooses his worst-case optimal strategy, then his payoff is always $\leq v$.
- **The name**: the expanded equality $\beta(x^*) = v = \alpha(y^*)$ becomes

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by **Von Neumann** in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).
- The Minimax theorem tells us everything about zero-sum games: **there is NE** and it can be found **efficiently**. Moreover, there is a unique **value of the game** $v = (x^*)^\top M(y^*)$ of the payoff attained in any NE (x^*, y^*) .
- There are **no secrets in zero-sum games**: strategies known in advance change nothing, each player can choose a worst-case optimal strategy and get payoff $\geq v$. If the opponent chooses his worst-case optimal strategy, then his payoff is always $\leq v$.
- **The name**: the expanded equality $\beta(x^*) = v = \alpha(y^*)$ becomes

$$\max_{x \in S_1} \min_{y \in S_2} x^\top M y = v = \min_{y \in S_2} \max_{x \in S_1} x^\top M y.$$

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by **Von Neumann** in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).
- The Minimax theorem tells us everything about zero-sum games: **there is NE** and it can be found **efficiently**. Moreover, there is a unique **value of the game** $v = (x^*)^\top M(y^*)$ of the payoff attained in any NE (x^*, y^*) .
- There are **no secrets in zero-sum games**: strategies known in advance change nothing, each player can choose a worst-case optimal strategy and get payoff $\geq v$. If the opponent chooses his worst-case optimal strategy, then his payoff is always $\leq v$.
- **The name**: the expanded equality $\beta(x^*) = v = \alpha(y^*)$ becomes

$$\max_{x \in S_1} \min_{y \in S_2} x^\top M y = v = \min_{y \in S_2} \max_{x \in S_1} x^\top M y.$$

- Original proof uses **Brouwer's theorem**.

The Minimax Theorem: remarks

- It was a starting point of game theory.
- Proved by **Von Neumann** in 1928 (predates Nash's Theorem).
- *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved."* (Von Neumann).
- The Minimax theorem tells us everything about zero-sum games: **there is NE** and it can be found **efficiently**. Moreover, there is a unique **value of the game** $v = (x^*)^\top M(y^*)$ of the payoff attained in any NE (x^*, y^*) .
- There are **no secrets in zero-sum games**: strategies known in advance change nothing, each player can choose a worst-case optimal strategy and get payoff $\geq v$. If the opponent chooses his worst-case optimal strategy, then his payoff is always $\leq v$.
- **The name**: the expanded equality $\beta(x^*) = v = \alpha(y^*)$ becomes

$$\max_{x \in S_1} \min_{y \in S_2} x^\top M y = v = \min_{y \in S_2} \max_{x \in S_1} x^\top M y.$$

- Original proof uses **Brouwer's theorem**. We will use **linear programming**.



Source: <https://czthomas.files.wordpress.com>

Thank you for your attention.