Algorithmic game theory

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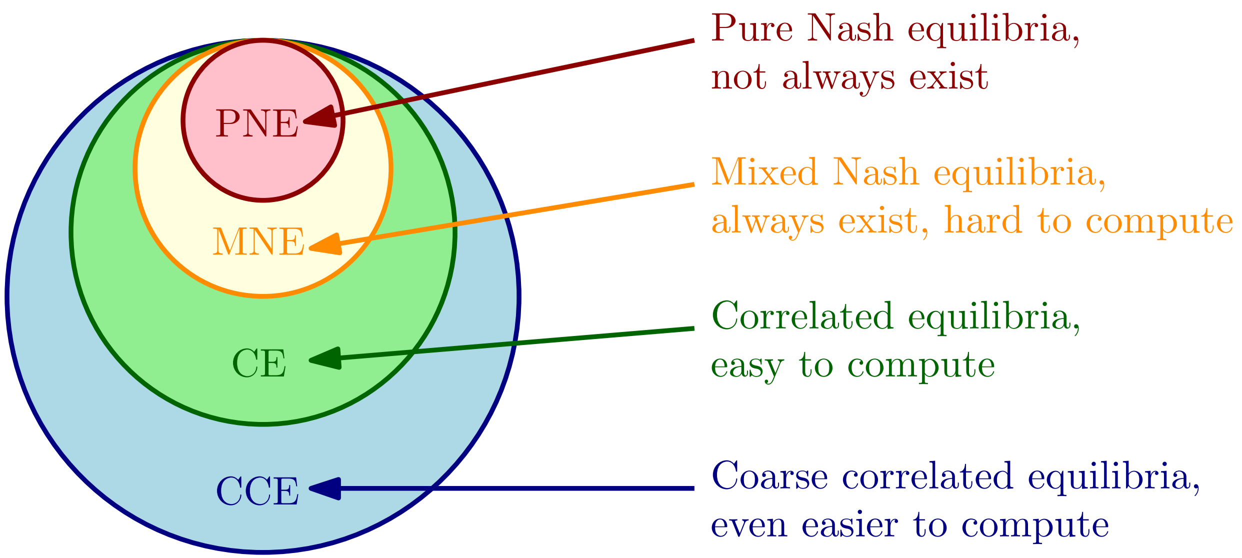
Applications of regret minimization

Concluding the story of NE

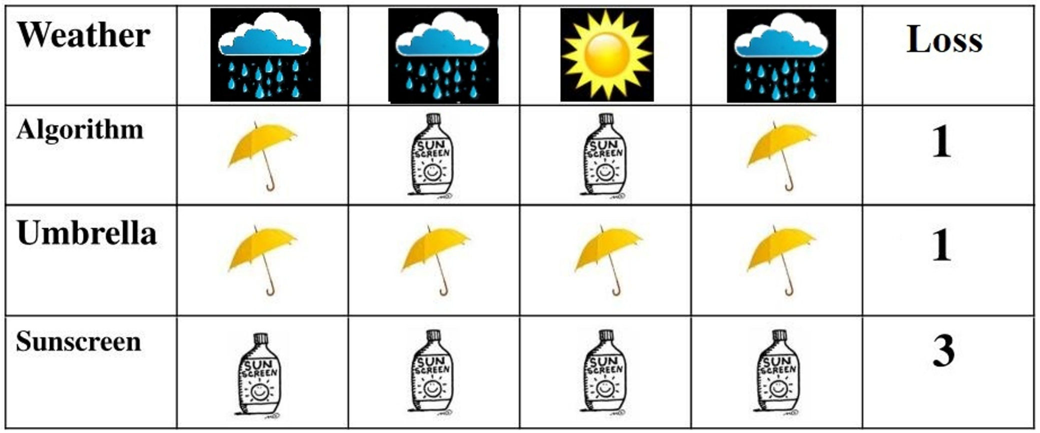
- We learned that Nash equilibria (NE) always exist. However, there seem to be no polynomial-time algorithm for computing NE.
- Therefore we came up with "relaxations" of NE. Correlated equilibria (CE) look particularly interesting as they are natural and we can compute them in polynomial time using linear programming.
- Using external regret minimization, we can apply No-regret dynamics to converge to more general coarse correlated equilibria (CCE).



- Today, we show that the No-swap-regret dynamics converges to CE.

Our notation

- At each step $t = 1, \ldots, T$:
  - Our agent A selects a probability distribution $p^t = (p_1^t, \ldots, p_N^t)$ over X, where p_i^t is the probability that A selects i in step t.
  - Then, the adversary chooses a loss vector $\ell^t = (\ell_1^t, \ldots, \ell_N^t)$, where $\ell_i^t \in [-1, 1]$ is the loss of action i in step t.
- The agent A receives loss $\ell_A^t = \sum_{i=1}^N p_i^t \ell_i^t$ at step t. The cumulative loss of A is $L_A^T = \sum_{t=1}^T \ell_A^t$. The cumulative loss of i is $L_i^T = \sum_{t=1}^T \ell_i^t$.



- We modify a sequence $(p^t)_{t=1}^T$ with $F\colon X \to X$ by replacing it with a sequence $(f^t)_{t=1}^T$, where $f^t = (f_1^t, \ldots, f_N^t)$ and $f_i^t = \sum_{j\colon F(j)=i} p_j^t$.
- The cumulative loss of A modified by F is $L_{A,F}^T = \sum_{t=1}^T \sum_{i=1}^N f_i^t \ell_i^t$.

All the regrets we have

- We also recall all variants of regret that we discussed.
- For a set $\mathcal{F}^{ex} = \{F_i \colon i \in X\}$ of rules where F_i always outputs action i, we obtain exactly the external regret:

$$R^T_{A,\mathcal{F}^{ex}} = \max_{F \in \mathcal{F}^{ex}} \left\{ L^T_A - L^T_{A,F} \right\}.$$

  - "Agent A vs. agents who always play the same action."
- For $\mathcal{F}^{in} = \{F_{i,j} \colon (i,j) \in X \times X, i \neq j\}$ where $F_{i,j}$ is defined by $F_{i,j}(i) = j$ and $F_{i,j}(i') = i'$ for each $i' \neq i$, we get the internal regret:

$$R^T_{A,\mathcal{F}^{in}} = \max_{F \in \mathcal{F}^{in}} \left\{ L^T_A - L^T_{A,F} \right\}.$$

  - "Agent A vs. agents who play j whenever A plays i."
- For the set $\mathcal{F}^{sw}$ of all modification rules, we get the swap regret:

$$R^T_{A,\mathcal{F}^{sw}} = \max_{F \in \mathcal{F}^{sw}} \left\{ L^T_A - L^T_{A,F} \right\}.$$

  - "Agent A vs. all his modifications." Note: $R^T_{A,\mathcal{F}^{ex}}, R^T_{A,\mathcal{F}^{in}} \leq R^T_{A,\mathcal{F}^{sw}}$.

Reduction from external regret to swap regret

- We have the PW algorithm with arbitrarily small external regret.
- Can we design such an algorithm also for swap regret? Yes, using a clever black-box reduction!
- An R-external regret algorithm A guarantees that for every sequence $(\ell^t)_{t=1}^T$ of loss vectors and for every action $j \in X$, we have

$$L_A^T = \sum_{t=1}^{T} \ell_A^t \leq \sum_{t=1}^{T} \ell_j^t + R = L_j^T + R.$$

Theorem 2.55

For every R-external regret algorithm A, there exists an algorithm $M = M(A)$ such that, for every $F\colon X \to X$ and $T \in \mathbb{N}$, we have

$$L_M^T \leq L_{M,F}^T + NR.$$

That is, the swap regret of M is at most NR.

Proof of the reduction I

- Assume that $A_1, \ldots, A_N$ are copies of the algorithm A. In every time step t, each A_i outputs a probability distribution $q_i^t = (q_{i,1}^t, \ldots, q_{i,N}^t)$, where $q_{i,j}^t$ is the fraction A_i assigns to an action $j \in X$.
- We construct the master algorithm M by combining these copies of A.
- We construct a single probability distribution $p^t = (p_1^t, \ldots, p_N^t)$ by letting $p_j^t = \sum_{i=1}^N p_i^t q_{i,j}^t$ for every $j \in X$. That is, $(p^t)^\top = (p^t)^\top Q^t$, where Q^t is an $N \times N$ matrix with $Q_{i,j}^t = q_{i,j}^t$.
- It can be shown that p^t exists and is efficiently computable.
  - It is a "stationary distribution of the transition matrix of a Markov chain".
- This choice of p^t guarantees that we can consider action selection in two equivalent ways. An action $j \in X$ is either selected with a probability p_j^t or we first select an algorithm A_i with probability p_i^t and then use the algorithm A_i to select j with probability $q_{i,j}^t$.

Proof of the reduction II

- We show that the total loss of all algorithms A_i at step t equals $p^t \cdot \ell^t$, the actual loss of M.
- After receiving a loss vector ℓ^t, we give, for each $i \in X$, a loss vector $p_i^t \ell^t$ to A_i. Then, A_i experiences loss $(p_i^t \ell^t) \cdot q_i^t = p_i^t (q_i^t \cdot \ell^t)$.
- Since A_i is an R-external regret algorithm, we have, for each $j \in X$,

$$\sum_{t=1}^{T} p_i^t (q_i^t \cdot \ell^t) \leq \sum_{t=1}^{T} p_i^t \ell_j^t + R.$$

- Summing the losses of all algorithms A_i over $i \in X$, we get the total loss $\sum_{i=1}^{N} p_i^t (q_i^t \cdot \ell^t) = (p^t)^\top Q^t \ell^t$ of all algorithms A_i at time step t.
- By the choice of p^t, we have $(p^t)^\top = (p^t)^\top Q^t$. Thus we get what we wanted.

Proof of the reduction III

- Thus, summing
$$\sum_{t=1}^{T} p_i^t (q_i^t \cdot \ell^t) \leq \sum_{t=1}^{T} p_i^t \ell_j^t + R.$$
over all actions $i \in X$, the left-hand side equals L_M^T.
- The right-hand side of is true for every action $j \in X$, so we obtain, for every function $F\colon X \to X$,
$$L_M^T \leq \sum_{i=1}^{N} \sum_{t=1}^{T} p_i^t \ell_{F(i)}^t + NR = L_{M,F}^T + NR.$$

$\square$

- Using the PW algorithm as A, we get an algorithm with swap regret at most $O(N\sqrt{T \log N})$.
- That is, its average swap regret goes to 0 with $T \to \infty$.

No-swap-regret dynamics

- Using swap regret instead of external regret, we get:

Algorithm 0.4: NO-SWAP-REGRET DYNAMICS(G, T, ε)

Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \ldots, T\}$.
for every step $t = 1, \ldots, T$
do
- Each player $i \in P$ independently chooses a mixed strategy p_i^t using an algorithm with average swap regret at most ε, with actions corresponding to pure strategies.
- Each player $i \in P$ receives a loss vector $\ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}$, where $\ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)]$ for the product distribution $p_{-i}^t = \prod_{j \neq i} p_j^t$.

Output $\{p^t : t \in \{1, \ldots, T\}\}$.

- No-swap-regret dynamics then converges to a correlated equilibrium.

Converging to CE

Theorem 2.57

For every $G = (P, A, C)$, $\varepsilon > 0$, and $T = T(\varepsilon) \in \mathbb{N}$, if after T steps of the No-swap-regret dynamics, each player $i \in P$ has time-averaged expected regret at most ε, then p is ε-CE where $p^t = \prod_{i=1}^n p_i^t$ and $p = \frac{1}{T}\sum_{t=1}^T p^t$.

- Proof: We want to prove $\mathbb{E}_{a\sim p}[C_i(a)] \leq \mathbb{E}_{a\sim p}[C_i(F(a_i); a_{-i})] + \varepsilon$.
- By the definition of p, we have, for every player $i \in P$ and $F\colon X \to X$,

$$\mathop{\mathbb{E}}_{a\sim p}[C_i(a)] = \frac{1}{T}\sum_{t=1}^T \mathop{\mathbb{E}}_{a\sim p^t}[C_i(a)],\ \mathop{\mathbb{E}}_{a\sim p}[C_i(F(a_i); a_{-i})] = \frac{1}{T}\sum_{t=1}^T \mathop{\mathbb{E}}_{a\sim p^t}[C_i(F(a_i); a_{-i})]$$

- The right-hand sides are time-averaged expected costs of i when playing according to the algorithm with small swap regret and when playing $F(a_i)$ instead of a_i. Since every player has regret at most ε, we obtain

$$\frac{1}{T}\sum_{t=1}^T \mathbb{E}_{a\sim p^t}[C_i(a)] \leq \frac{1}{T}\sum_{t=1}^T \mathbb{E}_{a\sim p^t}[C_i(F(a_i); a_{-i})] + \varepsilon.$$

- This verifies the ε-CE condition for $p = \frac{1}{T}\sum_{t=1}^T p_t$. □

Games in extensive form

Games in extensive form

- In normal-form games, players act simultaneously resulting in a static description of a game.
- Today, we describe a different representation of games which provides a dynamic description where players act sequentially.
- Instead of tables, we describe games using trees.

Zdroj: https://cz.pinterest.com

- For some of these games, we show how to compute NE.

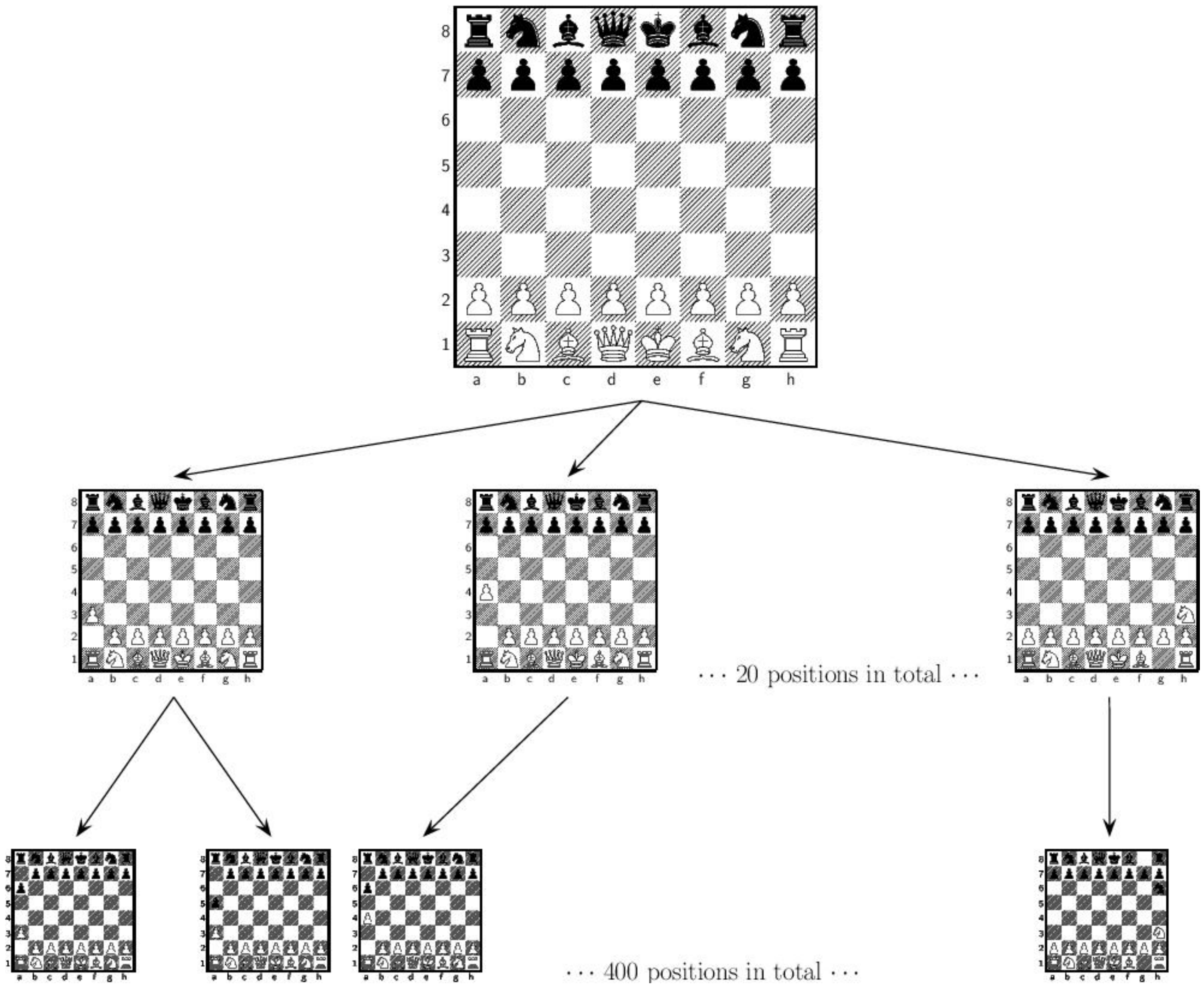
Example: normal-form of chess

Source: https://edition.cnn.com/

- Chess as a normal-form game: Each action of player $i \in \{\text{black}, \text{white}\}$ is a list of all possible situations that can happen on the board together with the move player i would make in that situation. Then we can simulate the whole game of chess in one round.

Example: extensive form of chess

- Root corresponds to the initial position of the chessboard. Each decision node represents a position on the chessboard and its outgoing edges correspond to possible moves in such a position.



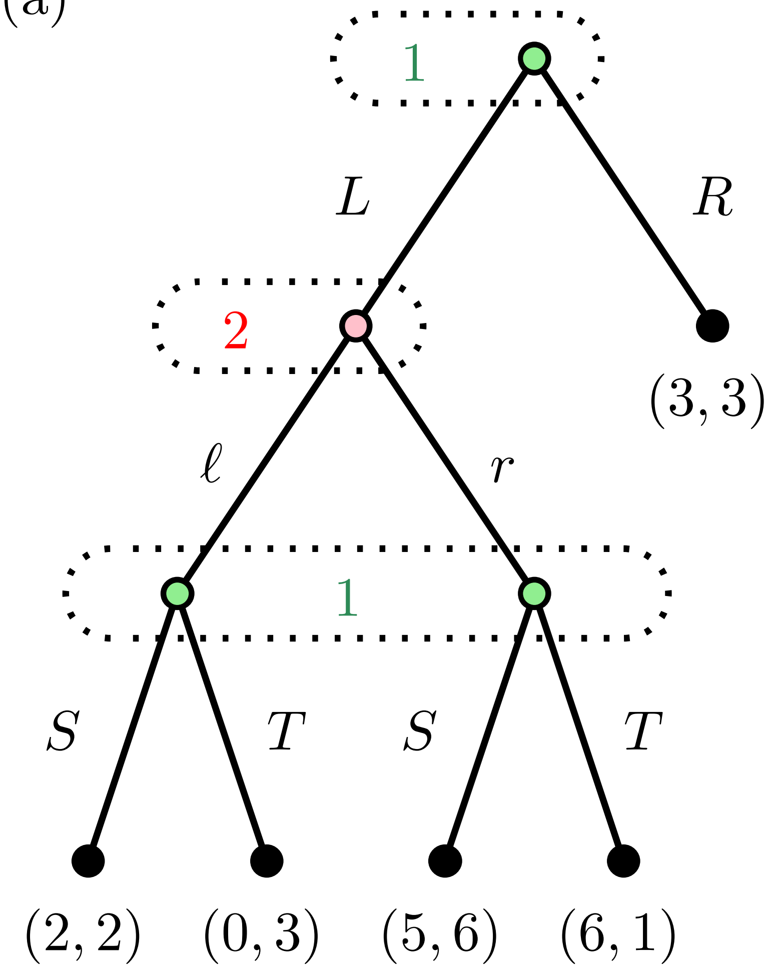
Basic definitions

- Extensive game consists of a directed tree where nodes represent game states. The tree encodes the full history of play.
- The game starts at the root of the tree and ends at a leaf, where each player receives a payoff. A tree edge corresponds to one player moving from one state to a different state of the game.
- Each node that is not a leaf is called a decision node.
- Moves a player can make in a given state are assigned to the outgoing edges of the corresponding decision node.
- In perfect-information game all players know the node they are in (that is, they know the history of the play that led them there).
  - For example, Chess.
- In imperfect-information games players have only partial knowledge.
  - For example, Poker.
  - We partition decision nodes into information sets where all nodes belong to the same player, and have the same moves.
  - For player i, let H_i be the set of information sets of i and, for $h \in H_i$, let C_h be the set of moves at h.

Example: imperfect-information game

- An example of an imperfect-information game in extensive form (part (a)) and its normal-form (part (b)).

(a)



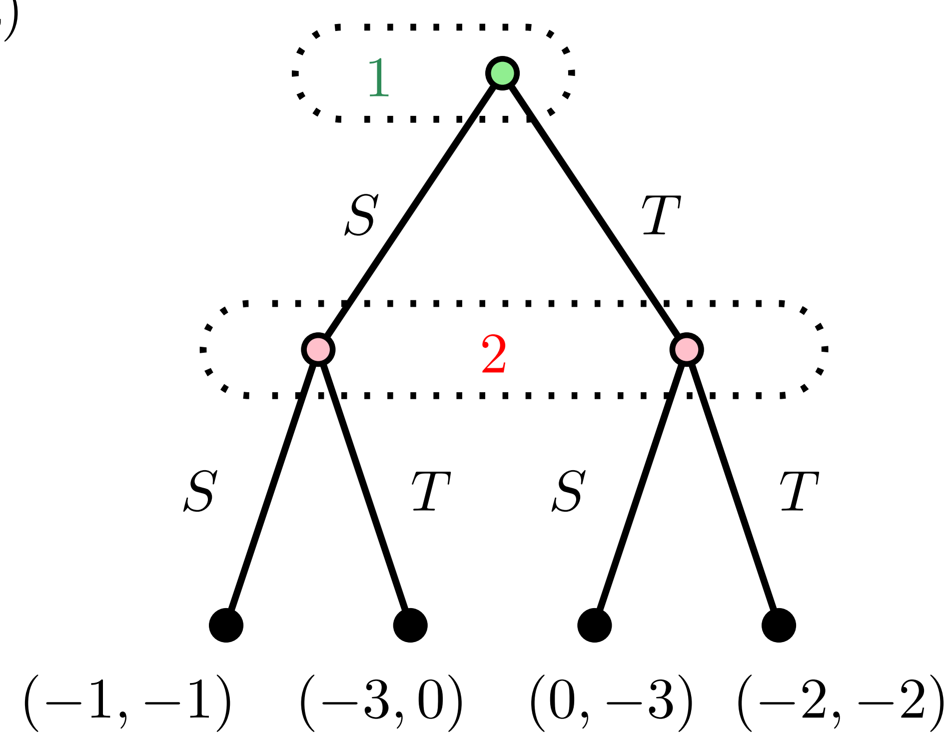
(b)

	(ℓ)	(r)
(L, S)	(2,2)	(5,6)
(L, T)	(0,3)	(6,1)
(R, S)	(3,3)	(3,3)
(R, T)	(3,3)	(3,3)

Example: Prisoner's dilemma

- Prisoner's dilemma in extensive form (part (a)) and its normal-form (part (b)).

(a)



(b)

	T	S
T	(-2,-2)	(0,-3)
S	(-3,0)	(-1,-1)

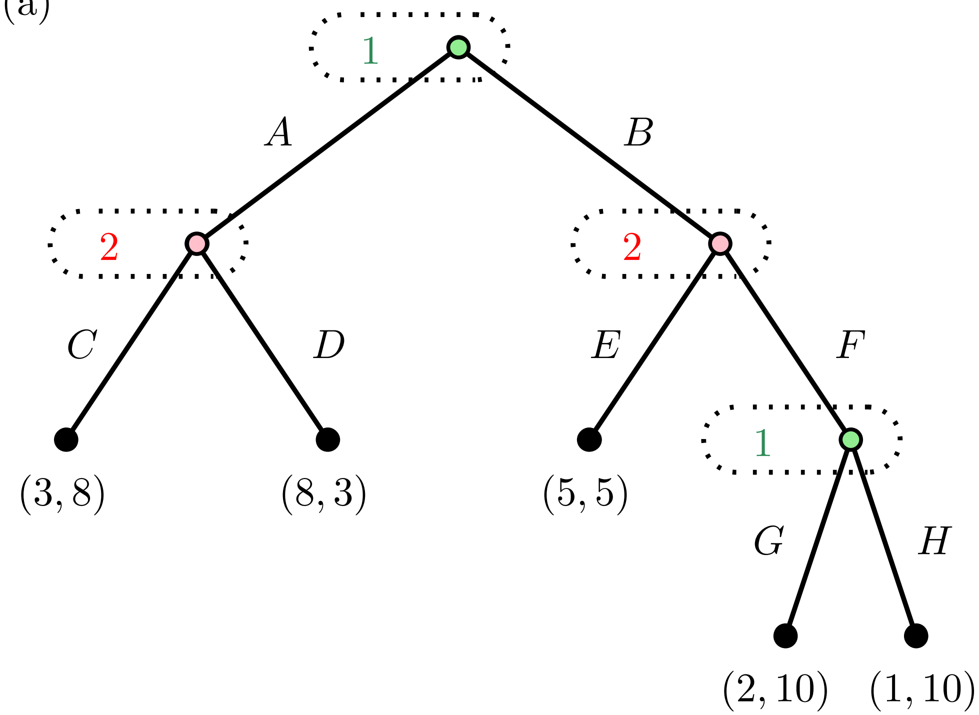
Strategies in extensive games

- A pure strategy for player i is a complete specification of which deterministic action to take at every information set belonging to i.
  - Formally, a pure strategy of player i is a vector $(c_h)_{h \in H_i}$ from the Cartesian product $\prod_{h \in H_i} C_h$.
  - Using pure strategies, we can transform an extensive game G into a normal-form game G' simply by tabulating all pure strategies of the players and recording the resulting expected payoffs.
- Mixed strategies of G are the mixed strategies of G'.
- In the same way, we also define the set of Nash equilibria of G.
- A behavioral strategy of player i is a probability distribution on C_h for each $h \in H_i$.
  - This is a strategy in which each player's choice at each information set is made independently of his choices at other information sets.
  - So a behavioral strategy is a vector of probability distributions while a mixed strategy is a probability distribution over vectors.
  - Unlike in mixed strategy, here a player might play different moves in different encounters of h.

Example: behavioral strategy

- An example of a perfect-information game in extensive form (part (a)) and its normal-form (part (b)).

(a)



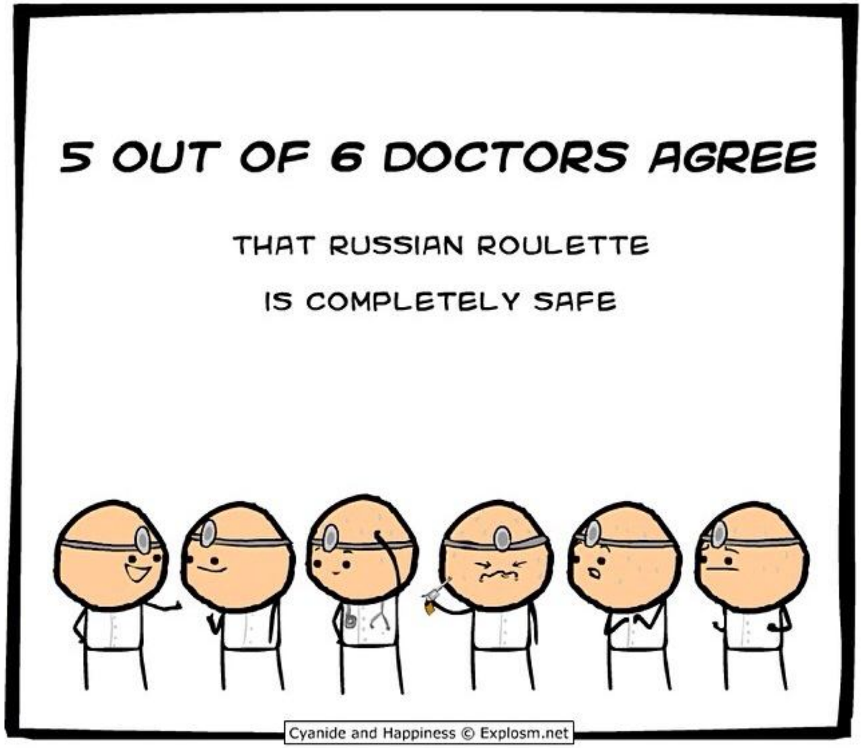
(b)

	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	(3,8)	(3,8)	(8,3)	(8,3)
(A, H)	(3,8)	(3,8)	(8,3)	(8,3)
(B, G)	(5,5)	(2,10)	(5,5)	(2,10)
(B, H)	(5,5)	(1,0)	(5,5)	(1,0)

- A strategy of player 1 that selects A with probability $\frac{1}{2}$ and G with probability $\frac{1}{3}$ is a behavioral strategy.
- The mixed strategy $(\frac{3}{5}(A, G), \frac{2}{5}(B, H))$ is not a behavioral strategy for 1 as the choices made by him at the two nodes are not independent.

Example: Russian roulette

- We have two players with a six-shot revolver containing a single bullet. Each player has two moves: shoot or give up. If player gives up, he loses the game immediately. If he shoots, then he either dies or survives, in which case the other player is on turn.

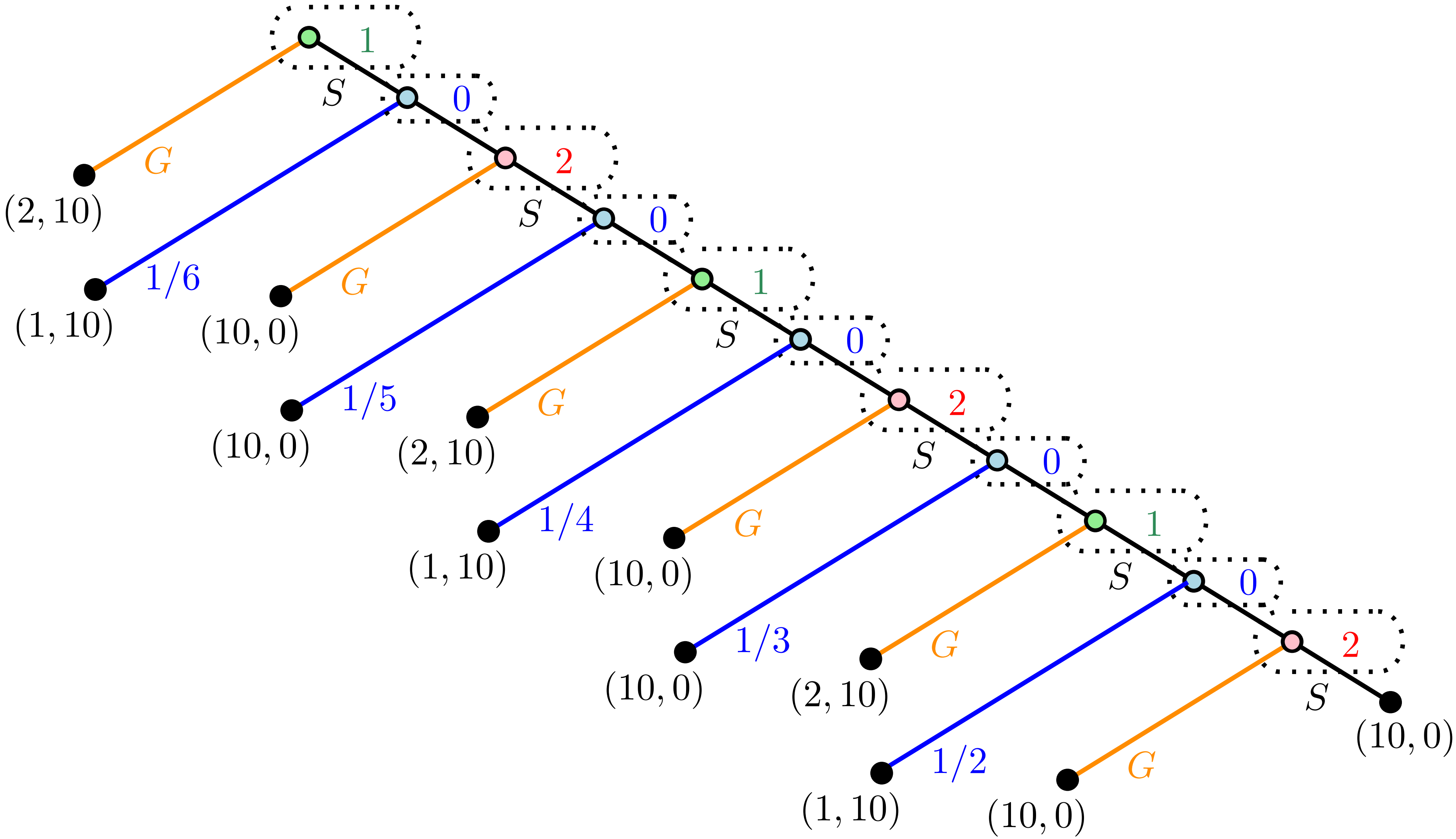


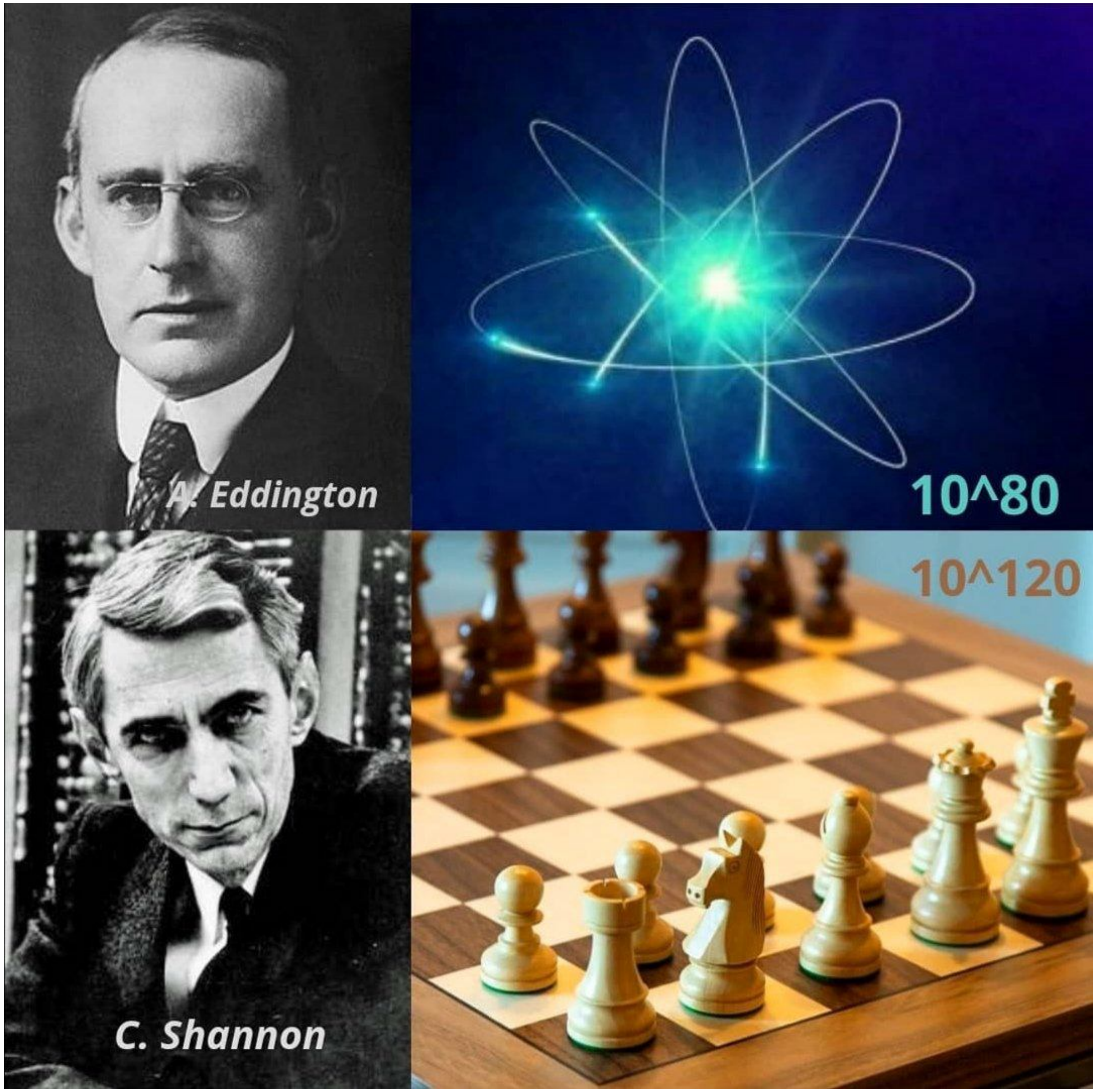
Source: https://www.memedroid.com/

- Consider that player 1 has payoffs $(10, 2, 1)$ for (Win, Loss, Death) and that player 2 has payoffs $(10, 0, 0)$.

Example: Russian roulette

- The Russian roulette in the extensive form using the random player who plays according to a known behavior strategy β_0.





Source: https://twitter.com/curiosite12

Thank you for your attention.