# Algorithmic game theory

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## 8th lecture

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# Applications of regret minimization

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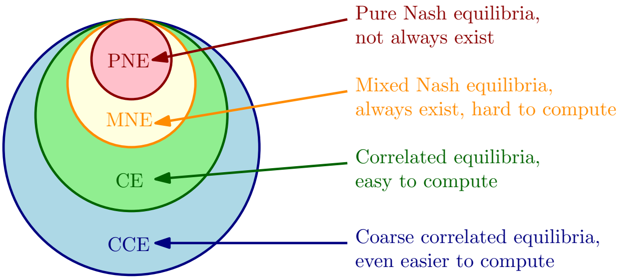

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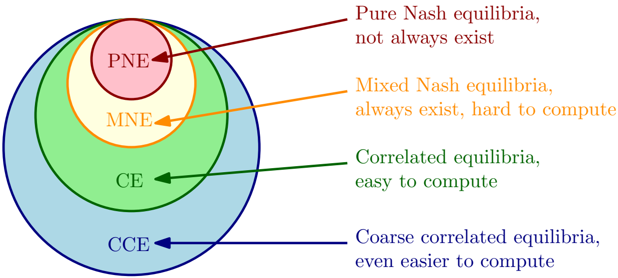


- Today, we show that the No-swap-regret dynamics converges to CE.

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- The cumulative loss of $A$ modified by $F$ is $L_{A,F}^T = \sum_{t=1}^T \sum_{i=1}^N f_i^t \ell_i^t$.

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That is, the swap regret of $M$ is at most $NR$.

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# Proof of the reduction I

- Assume that $A_1, \ldots, A_N$ are copies of the algorithm $A$. In every time step $t$, each $A_i$ outputs a probability distribution $q_i^t = (q_{i,1}^t, \ldots, q_{i,N}^t)$, where $q_{i,j}^t$ is the fraction $A_i$ assigns to an action $j \in X$.
- We construct the master algorithm $M$ by combining these copies of $A$.
- We construct a single probability distribution $p^t = (p_1^t, \ldots, p_N^t)$ by letting $p_j^t = \sum_{i=1}^N p_i^t q_{i,j}^t$ for every $j \in X$. That is, $(p^t)^\top = (p^t)^\top Q^t$, where $Q^t$ is an $N \times N$ matrix with $Q_{i,j}^t = q_{i,j}^t$.
- It can be shown that $p^t$ exists and is efficiently computable.
  - It is a "stationary distribution of the transition matrix of a Markov chain".
- This choice of $p^t$ guarantees that we can consider action selection in two equivalent ways. An action $j \in X$ is either selected with a probability $p_j^t$ or we first select an algorithm $A_i$ with probability $p_i^t$ and then use the algorithm $A_i$ to select $j$ with probability $q_{i,j}^t$.

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- That is, its average swap regret goes to 0 with $T \to \infty$.

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**Algorithm 0.3:** NO-SWAP-REGRET DYNAMICS$(G, T, \varepsilon)$

*Input* : A normal-form game $G = (P, A, C)$ of $n$ players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
*Output* : A prob. distribution $p_i^t$ on $A_i$ for each $i \in P$ and $t \in \{1, \ldots, T\}$.
**for** every step $t = 1, \ldots, T$
**do** Each player $i \in P$ independently chooses a mixed strategy $p_i^t$ using an algorithm with average swap regret at most $\varepsilon$, with actions corresponding to pure strategies.
Each player $i \in P$ receives a loss vector $\ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}$, where $\ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)]$ for the product distribution $p_{-i}^t = \prod_{j \neq i} p_j^t$.
Output $\{p^t : t \in \{1, \ldots, T\}\}$.

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**Algorithm 0.4:** NO-SWAP-REGRET DYNAMICS$(G, T, \varepsilon)$

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- No-swap-regret dynamics then converges to a correlated equilibrium.

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For every $G = (P, A, C)$, $\varepsilon > 0$, and $T = T(\varepsilon) \in \mathbb{N}$, if after $T$ steps of the No-swap-regret dynamics, each player $i \in P$ has time-averaged expected regret at most $\varepsilon$, then $p$ is $\varepsilon$-CE where $p^t = \prod_{i=1}^{n} p_i^t$ and $p = \frac{1}{T} \sum_{t=1}^{T} p^t$.

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- For some of these games, we show how to compute NE.

# Example: normal-form of chess

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Source: https://edition.cnn.com/

- Chess as a normal-form game: Each action of player $i \in \{\text{black}, \text{white}\}$ is a list of all possible situations that can happen on the board together with the move player $i$ would make in that situation. Then we can simulate the whole game of chess in one round.

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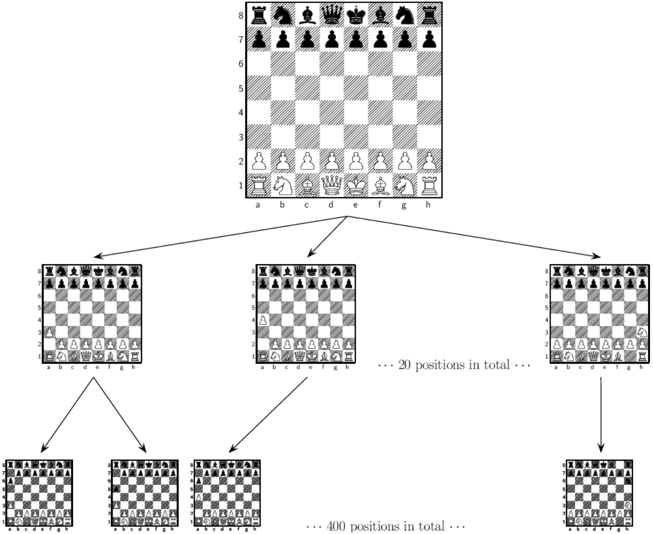
- Root corresponds to the initial position of the chessboard. Each decision node represents a position on the chessboard and its outgoing edges correspond to possible moves in such a position.



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# Example: imperfect-information game

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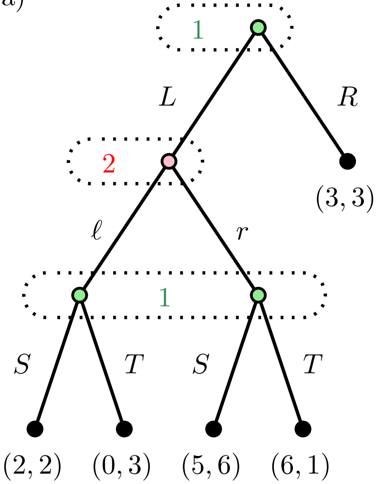
- An example of an imperfect-information game in extensive form (part (a)) and its normal-form (part (b)).

(a)




(b)

| | $(\ell)$ | $(r)$ |
|---|---|---|
| $(L, S)$ | (2,2) | (5,6) |
| $(L, T)$ | (0,3) | (6,1) |
| $(R, S)$ | (3,3) | (3,3) |
| $(R, T)$ | (3,3) | (3,3) |

# Example: Prisoner's dilemma

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- Prisoner's dilemma in extensive form (part (a)) and its normal-form (part (b)).

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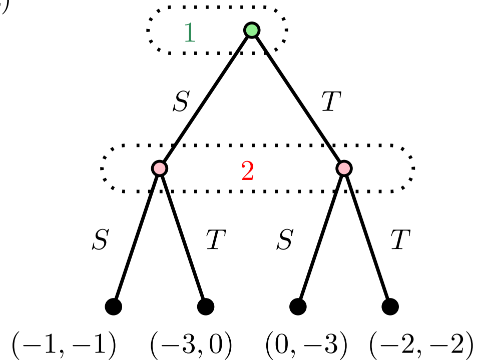


(b)

| | T | S |
|---|---|---|
| T | (-2,-2) | (0,-3) |
| S | (-3,0) | (-1,-1) |

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  - Unlike in mixed strategy, here a player might play different moves in different encounters of $h$.

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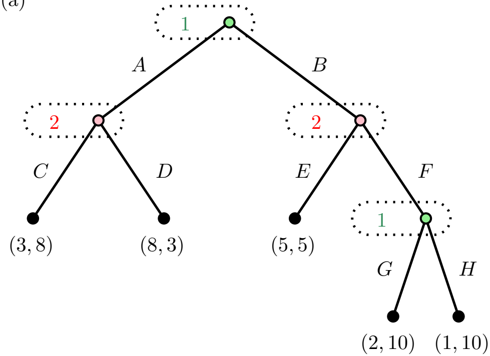
- An example of a perfect-information game in extensive form (part (a)) and its normal-form (part (b)).

(a)




(b)

| | $(C,E)$ | $(C,F)$ | $(D,E)$ | $(D,F)$ |
|---|---|---|---|---|
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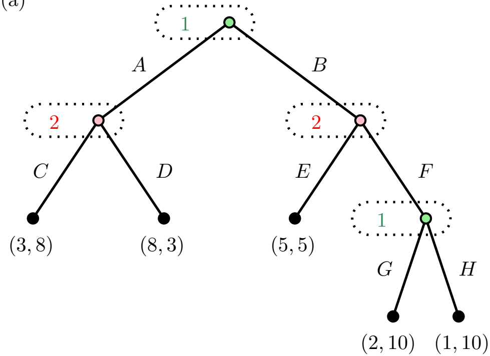


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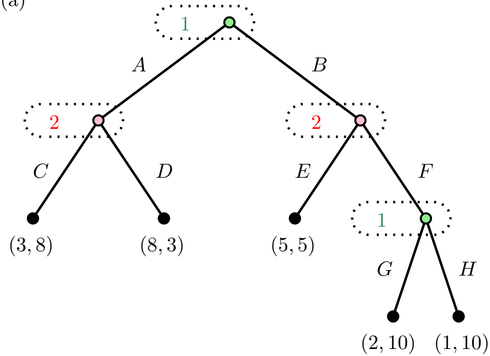


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- The mixed strategy $(\frac{3}{5}(A,G), \frac{2}{5}(B,H))$ is not a behavioral strategy for 1 as the choices made by him at the two nodes are not independent.

# Example: Russian roulette

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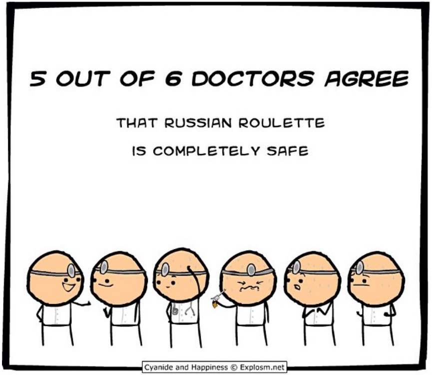
- We have two players with a six-shot revolver containing a single bullet. Each player has two moves: shoot or give up. If player gives up, he loses the game immediately. If he shoots, then he either dies or survives, in which case the other player is on turn.

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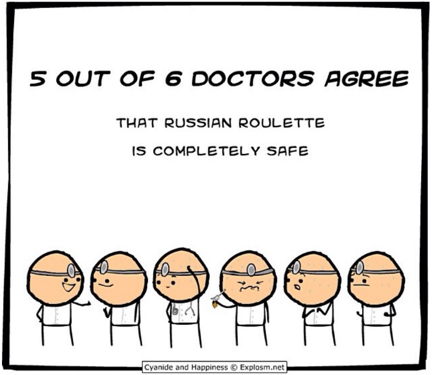


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- Consider that player 1 has payoffs $(10, 2, 1)$ for (Win, Loss, Death) and that player 2 has payoffs $(10, 0, 0)$.

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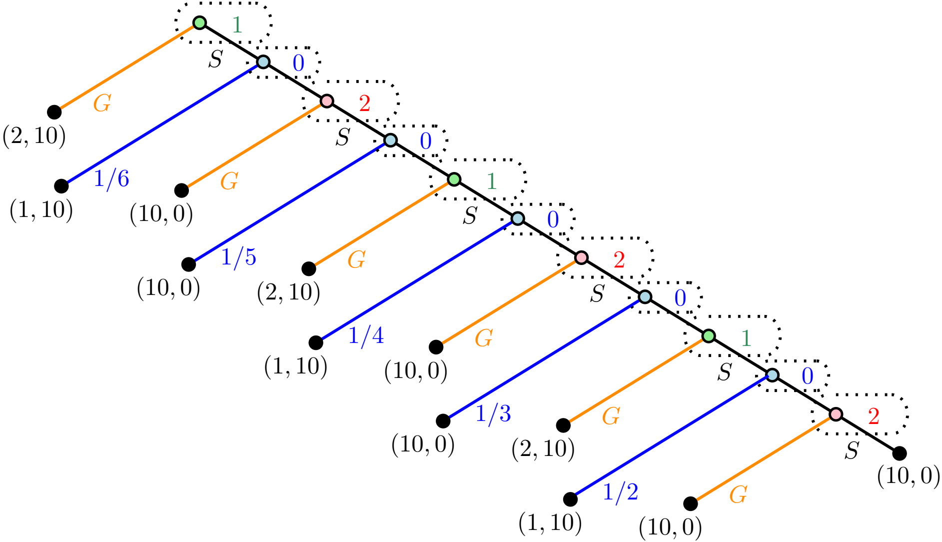
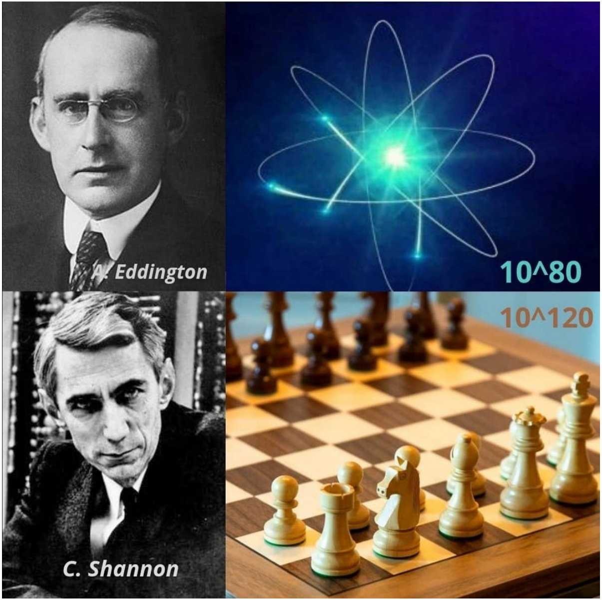
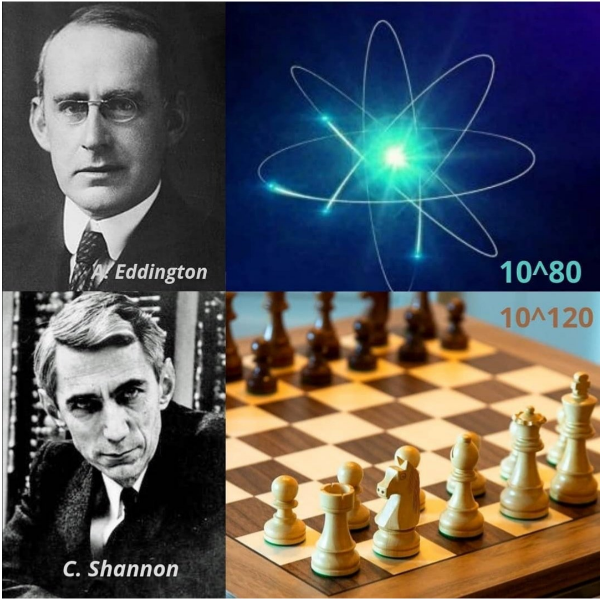
- The Russian roulette in the extensive form using the random player who plays according to a known behavior strategy $\beta_0$.






Source: https://twitter.com/curiosite12

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