Algorithmic game theory

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Applications of

regret minimization

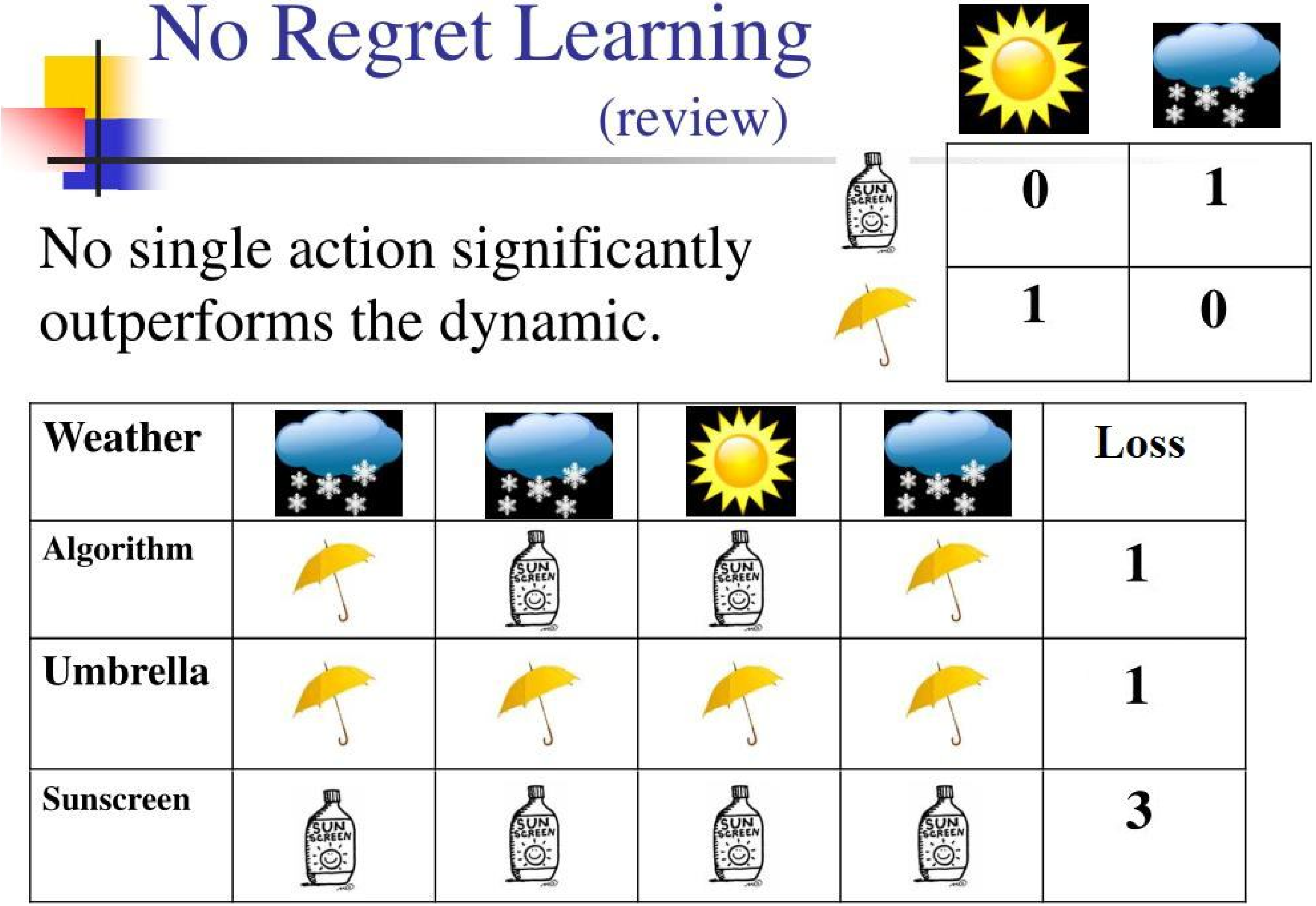
Our notation

- We have an agent A in an adversary environment.
- There are N available actions for A in the set $X = \{1, \ldots, N\}$.

- At each step $t = 1, \ldots, T$:
  - Our agent A selects a probability distribution $p^t = (p_1^t, \ldots, p_N^t)$ over X, where p_i^t is the probability that A selects i in step t.
  - Then, the adversary chooses a loss vector $\ell^t = (\ell_1^t, \ldots, \ell_N^t)$, where $\ell_i^t \in [-1, 1]$ is the loss of action i in step t.
  - The agent A then experiences loss $\ell_A^t = \sum_{i=1}^N p_i^t \ell_i^t$. This is the expected loss of A in step t.

- After T steps, the cumulative loss of action i is $L_i^T = \sum_{t=1}^T \ell_i^t$.
- The cumulative loss of A is $L_A^T = \sum_{t=1}^T \ell_A^t$.
- Given a comparison class $\mathcal{A}_X$ of agents A_i that select a single action i in all steps, we let $L_{min}^T = \min_{i \in X}\{L_{A_i}^T\}$ be the minimum cumulative loss of an agent from $\mathcal{A}_X$.
- Our goal is to minimize the external regret $R_A^T = L_A^T - L_{min}^T$.

Example



Source: No regret algorithms in games (Georgios Piliouras)

The Polynomial weights algorithm (PW algorithm)

Algorithm 0.4: POLYNOMIAL WEIGHTS ALGORITHM(X, T, η)

Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$w_i^1 \leftarrow 1$ for every $i \in X$,
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$

$$\textbf{do} \begin{cases} w_i^t \leftarrow w_i^{t-1}(1 - \eta \ell_i^{t-1}), \\ W^t \leftarrow \sum_{i \in X} w_i^t, \\ p_i^t \leftarrow w_i^t / W^t \text{ for every } i \in X. \end{cases}$$

Output $\{p^t : t \in \{1, \ldots, T\}\}$.

- For any sequence of loss vectors, we have $R^T_{\mathrm{PW}} \leq 2\sqrt{T \ln N}$.
- So the average regret $\frac{1}{T} \cdot R^T_{\mathrm{PW}}$ goes to 0 with $T \to \infty$.

Applications of regret minimization

- Today, we will see how to apply regret minimization in the theory of normal-form games.
- Let $G = (P, A, C)$ be a normal-form game of n players with a cost function $C = (C_1, \ldots, C_n)$, where $C_i \colon A \to [-1, 1]$. Cost $=$ $-$utility.
- This will be done via the so-called No-regret dynamics:
  - "Players play against each other by selecting actions according to an algorithm with small external regret."
  - Each player $i \in P$ chooses a mixed strategy $p_i^t = (p_i^t(a_i))_{a_i \in A_i}$ using some algorithm with small external regret such that actions correspond to pure strategies.
  - Then, i receives a loss vector $\ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}$, where

$$\ell_i^t(a_i) = \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)]$$

    for the product distribution $p_{-i}^t = \prod_{j \neq i} p_j^t$.
  - That is, $\ell_i^t(a_i)$ is the expected cost of the pure strategy a_i given the mixed strategies chosen by the other players.

The No-regret dynamics

- "Players play against each other by selecting actions according to an algorithm with small external regret."

Algorithm 0.11: NO-REGRET DYNAMICS(G, T, ε)

Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \ldots, T\}$.
for every step $t = 1, \ldots, T$

do Each player $i \in P$ independently chooses a mixed strategy p_i^t using an algorithm with average regret at most ε, with actions corresponding to pure strategies.
Each player $i \in P$ receives a loss vector $\ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}$, where $\ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)]$ for the product distribution $p_{-i}^t = \prod_{j \neq i} p_j^t$.

Output $\{p^t : t \in \{1, \ldots, T\}\}$.

Application: Modern proof of the Minimax Theorem

- A new proof of the Minimax theorem.
- A zero-sum game $G = (\{1,2\}, A, C)$ with $A_1 = \{a_1, \ldots, a_m\}$, $A_2 = \{b_1, \ldots, b_n\}$ is represented with an $m \times n$ matrix M where $M_{i,j} = -C_1(a_i, b_j) = C_2(a_i, b_j) \in [-1, 1]$.
- The expected cost $C_2(s)$ for player 2 equals $x^\top My$, where x and y are the mixed strategy vectors.
- The Minimax theorem then states

$$\max_{x \in S_1} \min_{y \in S_2} x^\top My = \min_{y \in S_2} \max_{x \in S_1} x^\top My.$$

Source: https://www.privatdozent.co/

- We can prove it without LP!

Modern proof of the Minimax Theorem I

- First, the inequality $\max_x \min_y x^\top My \le \min_y \max_x x^\top My$ follows easily, since it is only worse to go first.
- Second, we prove the inequality $\max_x \min_y x^\top My \ge \min_y \max_x x^\top My$.
- We choose a parameter $\varepsilon \in (0, 1]$ and run the No-regret dynamics for a sufficient number T of steps so that both players have average expected external regret at most ε.
- With the PW algorithm, we can set $T = 4 \ln(\max\{m, n\})/\varepsilon^2$.
- Let $p^1, \dots, p^T$ and $q^1, \dots, q^T$ be strategies played by players 1 and 2.
- We let $\overline{x} = \frac{1}{T} \sum_{t=1}^{T} p^t$ and $\overline{y} = \frac{1}{T} \sum_{t=1}^{T} q^t$ be the time-averaged strategies of players 1 and 2.
- The payoff vector revealed to each no-regret algorithm after step t is the expected payoff of each strategy, given the mixed strategy played by the other player.
- Thus, players 1 and 2 get the payoff vectors Mq^t and $-(p^t)^\top M$.
- The time-averaged expected payoff of 1 is then $v = \frac{1}{T} \sum_{t=1}^{T} (p^t)^\top Mq^t$.

Modern proof of the Minimax Theorem II

- For $i = 1, \ldots, m$, let $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$ be the mixed strategy vector for the pure strategy a_i. Since the external regret of player 1 is at most ε, we have

$$e_i^\top M\overline{y} = \frac{1}{T}\sum_{t=1}^{T} e_i^\top Mq^t \leq \frac{1}{T}\sum_{t=1}^{T}(p^t)^\top Mq^t + \varepsilon = v + \varepsilon.$$

- Since every strategy $x \in S_1$ is a convex combination of the vectors e_i, the linearity of expectation gives $x^\top M\overline{y} \leq v + \varepsilon$. Analogously, $(\overline{x})^\top My \geq v - \varepsilon$ for every $y \in S_2$.
- Putting everything together, we get

$$\begin{aligned}\max_{x\in S_1}\min_{y\in S_2} x^\top My &\geq \min_{y\in S_2}(\overline{x})^\top My \geq v - \varepsilon \\ &\geq \max_{x\in S_1} x^\top M\overline{y} - 2\varepsilon \geq \min_{y\in S_2}\max_{x\in S_1} x^\top My - 2\varepsilon.\end{aligned}$$

- For $T \to \infty$, we get $\varepsilon \to 0$ and we obtain the desired inequality. $\square$

Application: Coarse correlated equilibria

- Recall: a prob. distribution p on A is a correlated equilibrium (CE) if

$$\sum_{a_{-i} \in A_{-i}} C_i(a_i; a_{-i}) p(a_i; a_{-i}) \leq \sum_{a_{-i} \in A_{-i}} C_i(a_i'; a_{-i}) p(a_i; a_{-i})$$

  for every player $i \in P$ and all pure strategies $a_i, a_i' \in A_i$.

  - In other words,

$$\mathbb{E}_{a \sim p}[C_i(a) \mid a_i] \leq \mathbb{E}_{a \sim p}[C_i(a_i'; a_{-i}) \mid a_i].$$

  - We define an even more tractable concept and use no-regret dynamics to converge to it.

Coarse correlated equilibrium

- For a normal-form game $G = (P, A, C)$ of n players, a probability distribution p on A is a coarse correlated equilibrium (CCE) in G if

$$\sum_{a \in A} C_i(a)p(a) \leq \sum_{a \in A} C_i(a'_i; a_{-i})p(a)$$

  for every player $i \in P$ and every $a'_i \in A_i$.
- CCE can be expressed as

$$\mathbb{E}_{a \sim p}[C_i(a)] \leq \mathbb{E}_{a \sim p}[C_i(a'_i; a_{-i})]$$

  for every $i \in P$ and each $a'_i \in A_i$.
- The difference between CCE and CE is that CCE only requires that following your suggested action a_i when a is drawn from p is only a best response in expectation before you see a_i. This makes sense if you have to commit to following your suggested action or not upfront, and do not have the opportunity to deviate after seeing it.

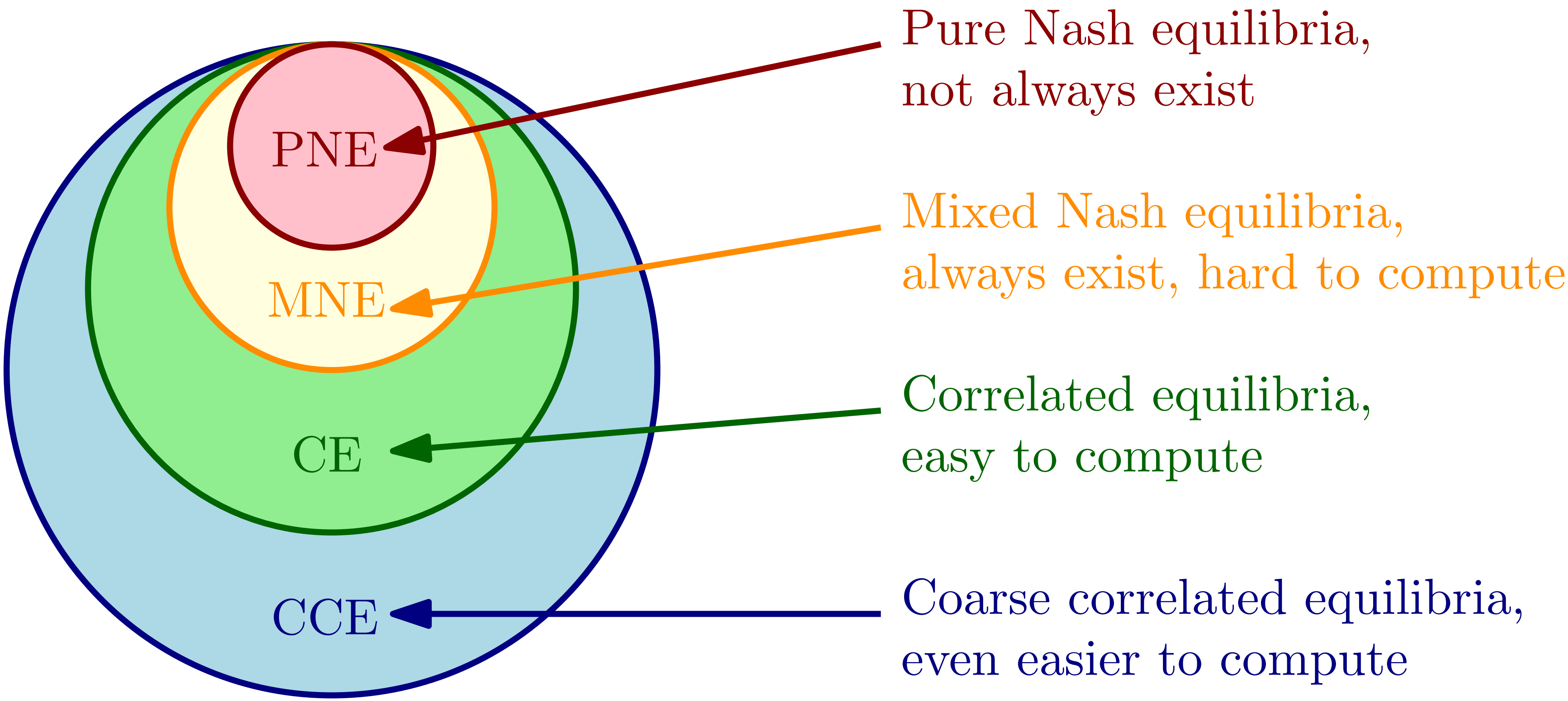
Example: Coarse correlated equilibrium

- Giving probability $1/6$ to each red outcome gives coarse correlated equilibrium in the Rock-Paper-Scissors game.

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

- Then, the expected payoff of each player is 0 and deviating to any pure strategy gives the expected payoff 0.
- It is not a correlated equilibrium though.

Hierarchy of Nash equilibria



- In general normal-form game, no-regret dynamics converges to a coarse correlated equilibrium.
- For $\varepsilon > 0$, a probability distribution p on A is an ε-coarse correlated equilibrium (ε-CCE) if $\mathbb{E}_{a\sim p}[C_i(a)] \leq \mathbb{E}_{a\sim p}[C_i(a'_i; a_{-i})] + \varepsilon$.

Converging to CCE

Theorem 2.54

For every $G = (P, A, C)$, $\varepsilon > 0$, and $T = T(\varepsilon) \in \mathbb{N}$, if after T steps of the No-regret dynamics, each player $i \in P$ has time-averaged expected regret at most ε, then p is ε-CCE where $p^t = \prod_{i=1}^n p_i^t$ and $p = \frac{1}{T}\sum_{t=1}^T p^t$.

- Proof: We want to prove $\mathbb{E}_{a\sim p}[C_i(a)] \leq \mathbb{E}_{a\sim p}[C_i(a_i'; a_{-i})] + \varepsilon$.
- By the definition of p, we have, for every player $i \in P$ and $a_i' \in A_i$,

$$\mathop{\mathbb{E}}_{a\sim p}[C_i(a)] = \frac{1}{T}\sum_{t=1}^T \mathop{\mathbb{E}}_{a\sim p^t}[C_i(a)] \text{ and } \mathop{\mathbb{E}}_{a\sim p}[C_i(a_i'; a_{-i})] = \frac{1}{T}\sum_{t=1}^T \mathop{\mathbb{E}}_{a\sim p^t}[C_i(a_i'; a_{-i})].$$

- The right-hand sides are time-averaged expected costs of i when playing according to the algorithm with small external regret and when playing a_i' every iteration. Since every player has regret at most ε, we obtain

$$\frac{1}{T}\sum_{t=1}^T \mathbb{E}_{a\sim p^t}[C_i(a)] \leq \frac{1}{T}\sum_{t=1}^T \mathbb{E}_{a\sim p^t}[C_i(a_i'; a_{-i})] + \varepsilon.$$

- This verifies the ε-CCE condition for $p = \frac{1}{T}\sum_{t=1}^T p_t$. □

Other notions of regret

- Converging to CCE is nice, but how about converging to CE? We can do that with a different notion of regret!
- We consider an "internal setting" when we compare our agent to its modifications.

- A modification rule is a function $F\colon X \to X$.
- We modify a sequence $(p^t)_{t=1}^T$ with F by replacing it with a sequence $(f^t)_{t=1}^T$, where $f^t = (f_1^t, \ldots, f_N^t)$ and $f_i^t = \sum_{j\colon F(j)=i} p_j^t$.
  - "The modified agent plays $F(i)$ whenever A plays i."
- The cumulative loss of A modified by F is $L_{A,F}^T = \sum_{t=1}^T \sum_{i=1}^N f_i^t \ell_i^t$.

- Given a set of modification rules $\mathcal{F}$, we can compare our agent to his modifications by rules from $\mathcal{F}$, obtaining different notions of regret.

Internal and swap regret

- For a set $\mathcal{F}^{ex} = \{F_i : i \in X\}$ of rules where F_i always outputs action i, we obtain exactly the external regret:

$$R^T_{A,\mathcal{F}^{ex}} = \max_{F \in \mathcal{F}^{ex}} \left\{L^T_A - L^T_{A,F}\right\} = \max_{j \in X} \left\{ \sum_{t=1}^{T} \left(\left(\sum_{i \in X} p_i^t \ell_i^t \right) - \ell_j^t \right) \right\}.$$

- For $\mathcal{F}^{in} = \{F_{i,j} : (i,j) \in X \times X, i \neq j\}$ where $F_{i,j}$ is defined by $F_{i,j}(i) = j$ and $F_{i,j}(i') = i'$ for each $i' \neq i$, we get the internal regret:

$$R^T_{A,\mathcal{F}^{in}} = \max_{F \in \mathcal{F}^{in}} \left\{L^T_A - L^T_{A,F}\right\} = \max_{i,j \in X} \left\{ \sum_{t=1}^{T} p_i^t (\ell_i^t - \ell_j^t) \right\}.$$

- For the set $\mathcal{F}^{sw}$ of all modification rules, we get the swap regret:

$$R^T_{A,\mathcal{F}^{sw}} = \max_{F \in \mathcal{F}^{sw}} \left\{L^T_A - L^T_{A,F}\right\} = \sum_{i=1}^{N} \max_{j \in X} \left\{ \sum_{t=1}^{T} p_i^t (\ell_i^t - \ell_j^t) \right\}.$$

- Since $\mathcal{F}^{ex}, \mathcal{F}^{in} \subseteq \mathcal{F}^{sw}$, we immediately have $R^T_{A,\mathcal{F}^{ex}}, R^T_{A,\mathcal{F}^{in}} \leq R^T_{A,\mathcal{F}^{sw}}$.

The No-swap-regret dynamics

- Using swap regret instead of external regret, we will get:

Algorithm 0.15: NO-SWAP-REGRET DYNAMICS(G, T, ε)

Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \ldots, T\}$.
for every step $t = 1, \ldots, T$

do $\begin{cases} \text{Each player } i \in P \text{ independently chooses a mixed strategy } p_i^t \\ \text{using an algorithm with average swap regret at most } \varepsilon, \text{ with} \\ \text{actions corresponding to pure strategies.} \\ \text{Each player } i \in P \text{ receives a loss vector } \ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}, \text{ where} \\ \ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)] \text{ for the product distribution} \\ p_{-i}^t = \prod_{j \neq i} p_j^t. \end{cases}$

Output $\{p^t : t \in \{1, \ldots, T\}\}$.

- No-swap-regret dynamics then converges to a correlated equilibrium.

Thank you for your attention.