

# Algorithmic game theory

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5th lecture

November 1st 2024



# Nash equilibria in bimatrix games

# What have we learned so far

- We have seen **three algorithms** to find NE in bimatrix games:
  - the brute-force algorithm with support enumeration,
  - the algorithm with vertex enumeration,
  - the Lemke–Howson algorithm.
- All these algorithms have **exponential running time** in the worst case.



Source: <https://www.shutterstock.com/>

- **Is there a chance to get an efficient algorithm?**
- **NASH** = the problem of finding NE in bimatrix games.
- Today, we discuss the **computational complexity of NASH**.

# Where does NASH belong to?

- Is NASH NP-complete?
  - No. NP is a class of decision problems (yes/no answers) while NE always exist (so the answer is always yes).
- Another candidate is the complexity class FNP (“functional NP”).
  - The input of FNP problem is an instance of a problem from NP. The algorithm outputs a solution if one exists. If there is no solution, the algorithm outputs ‘no’.
  - That is, we demand a solution for ‘yes’ instances.
  - NASH belongs to FNP, as checking whether a strategy profile is NE can be done using the Best Response Condition.
  - Is NASH FNP-complete? Unlikely, because of the following result.

**Theorem 2.34** (Megiddo and Papadimitriou, 1991)

If the problem NASH is FNP-complete, then  $NP = coNP$ .

- Without proof (but you can find it in the lecture notes).

# New complexity class

- The proof of the correctness of the Lemke–Howson algorithm reveals the **structure of NASH** (finding another endpoint of a path in graph of maximum degree 2).
- Let us capture this abstract structure.
- The **END-OF-THE-LINE problem**: for a directed graph  $G$  with every vertex having at most one predecessor and one successor, given a vertex  $s$  of  $G$  with no predecessor, find a vertex  $t \neq s$  with no predecessor or no successor. The graph  $G$  is not given on the input, but it is specified by some polynomial-time computable function  $f(v)$  that returns the predecessor and successor (if they exist) of  $v$ .
  - Thus,  $G$  can be exponentially large with respect to the input.
- Let **PPAD** be a complexity class consisting of problems that admit a polynomial-time reduction to END-OF-THE-LINE.

# The class PPAD

- The class PPAD was introduced in 1994 by Papadimitrou.



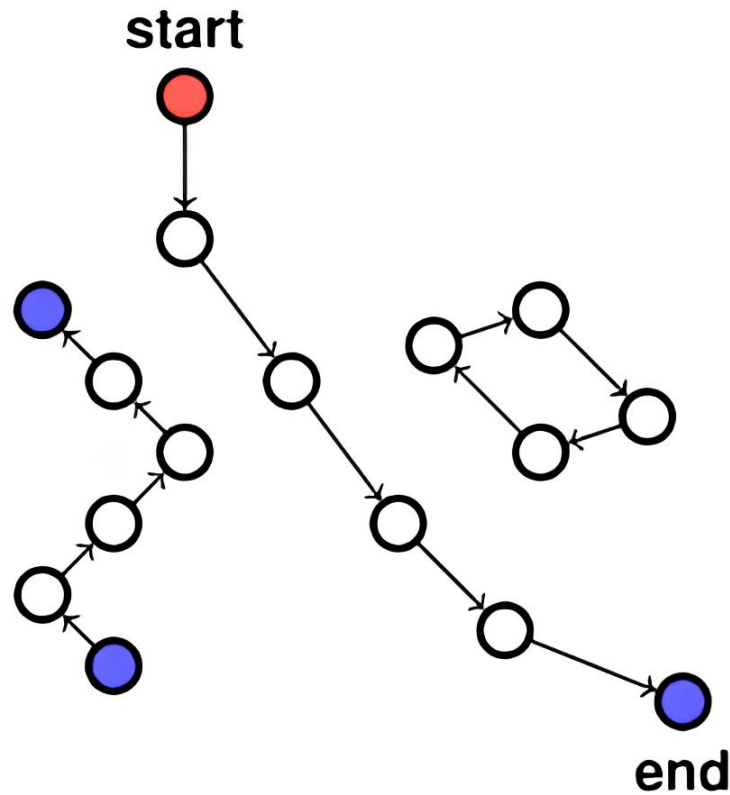
Figure: Christos Papadimitriou (born 1949).

Source: <https://cs.columbia.edu>

- Abbreviation for “Polynomial Parity Arguments on Directed graphs”.
- This complexity class contains a lot of well-known problems.

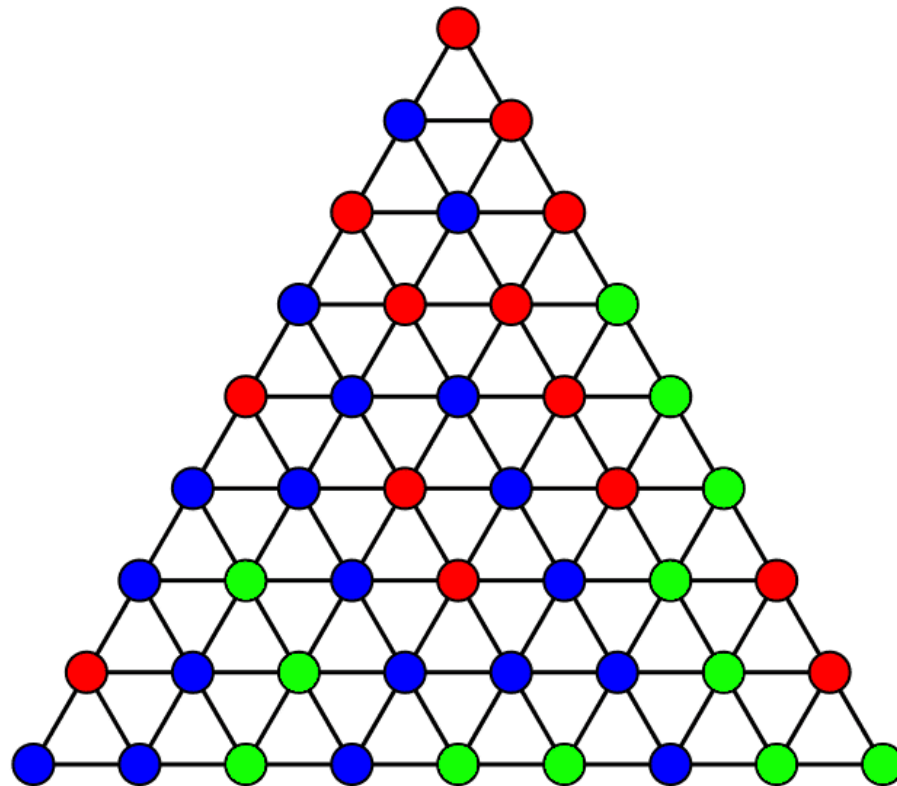
# Problems from PPAD: End-of-the-line

- For an oriented graph  $G$  with max. indegree and outdegree 1 and a source in  $G$ , find a target in  $G$ . The graph is given by a polynomial-time computable function  $f(v)$  that returns predecessor and successor of  $v$ .



# Problems from PPAD: Sperner's lemma

- Given a **legal** 3-coloring of a triangulated triangle, find a triangle with vertices colored by all 3 colors.



Source: <https://lesswrong.com>

- Discrete version of the **Brouwer's fixed point theorem**.



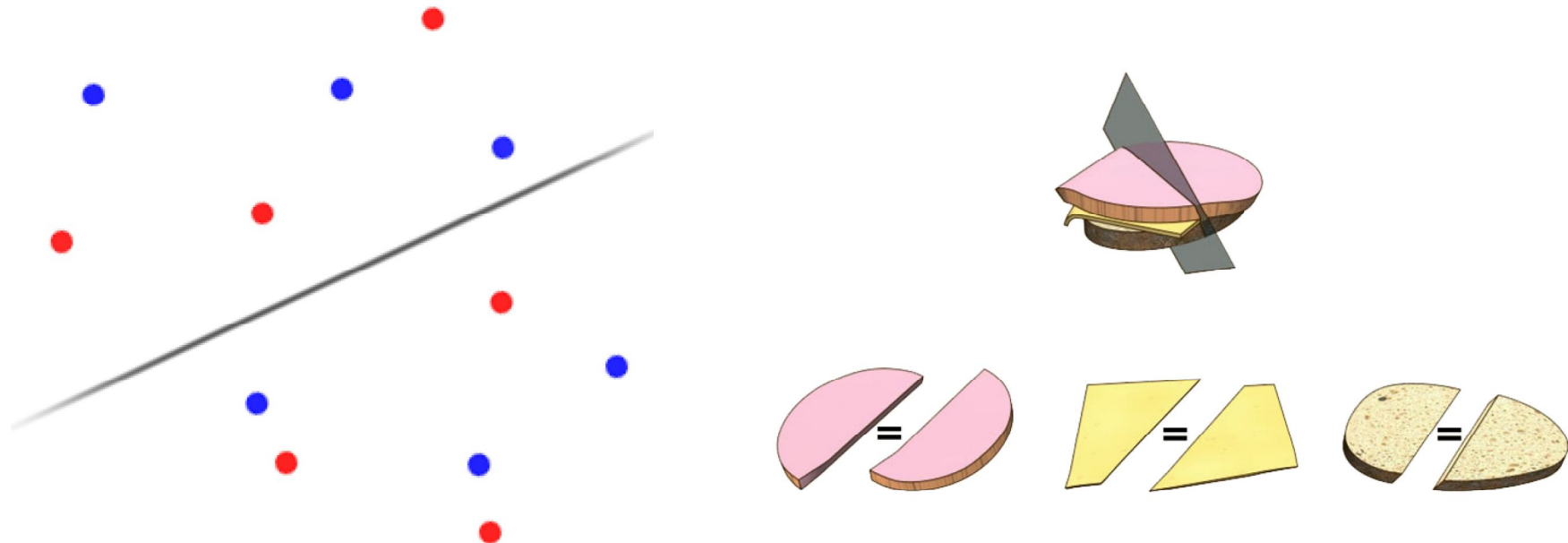
# Problems from PPAD: Ham sandwich theorem



Source: <https://www.seekpng.com/>

# Problems from PPAD: Ham sandwich theorem

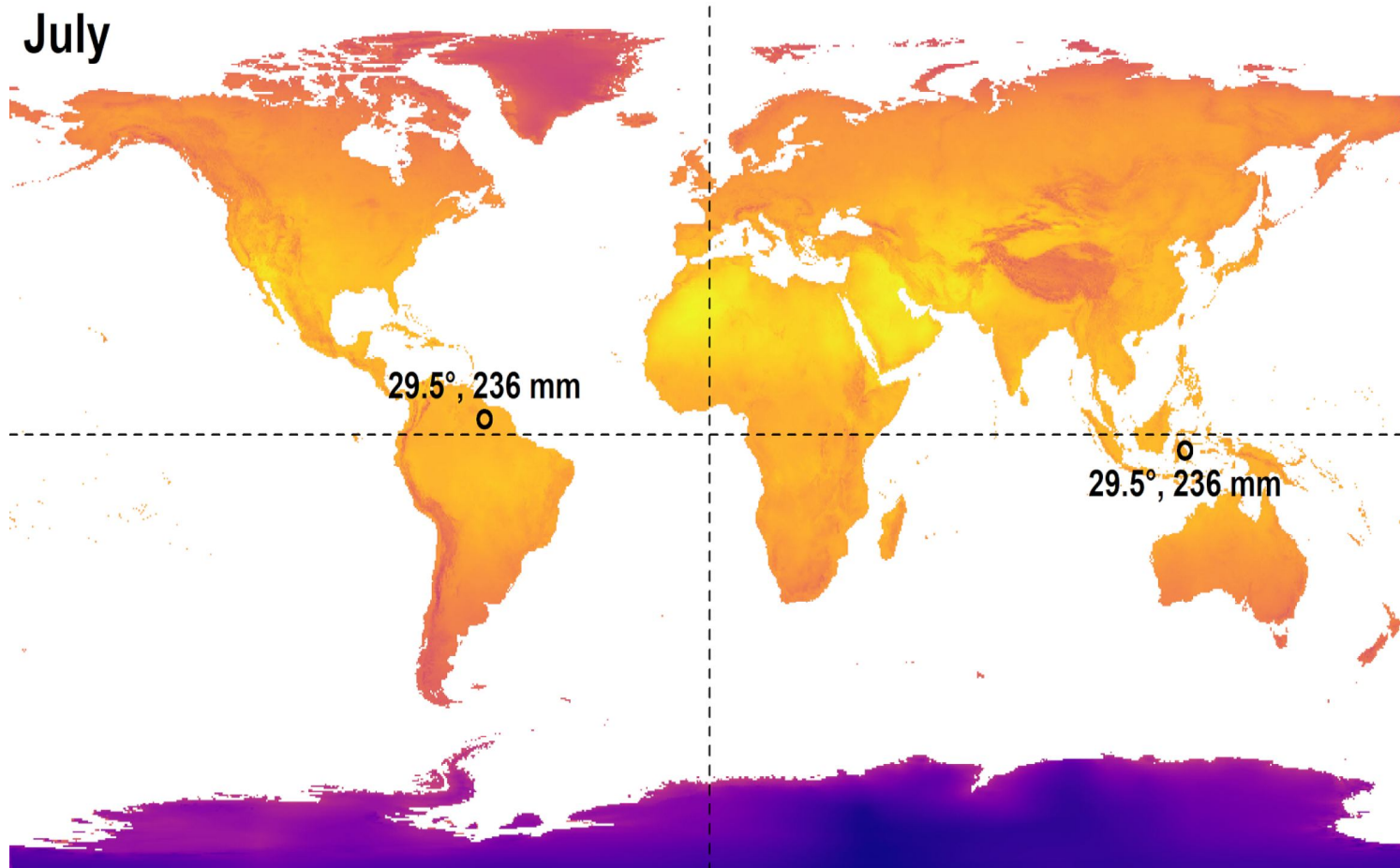
- Given  $n$  sets of  $2n$  points in  $\mathbb{R}^n$ , find a hyperplane  $H$  that contains exactly  $n$  points from each of the sets in each open halfspace determined by  $H$ .



Sources: <https://ejarzo.github.io> and <https://curiosamathematica.tumblr.com>

# Problems from PPAD: The Borsuk–Ulam theorem

- An approximate version of the following theorem is in PPAD: For every continuous  $f: S^n \rightarrow \mathbb{R}^n$  there is  $x \in S^n$  with  $f(x) = f(-x)$ .



Source: <https://scientificgems.wordpress.com/>

# NASH and PPAD

- The proof of the correctness of the Lemke–Howson algorithm shows that **NASH belongs to PPAD** (for nondegenerate games).
- Is NASH **PPAD-complete**?
  - That is, is it among the most difficult problems in this class?
  - PPAD-completeness gives some evidence of computational intractability, although somehow weaker than NP-completeness.
  - Open for a long time.

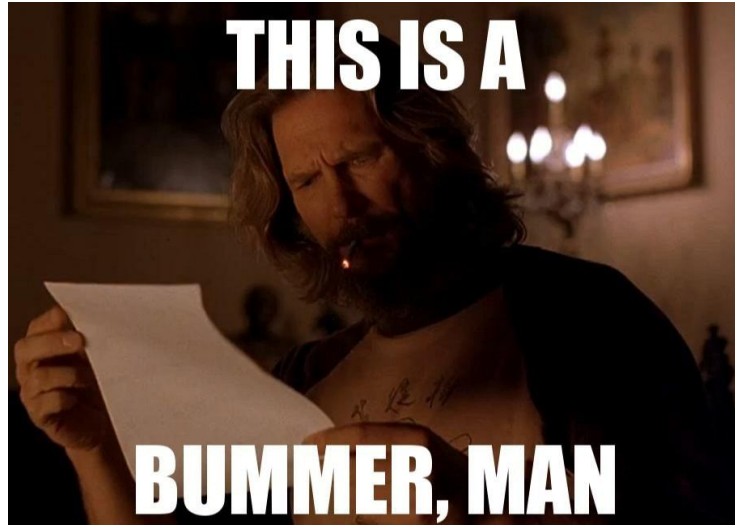
**Theorem 2.35** (Chen, Deng, and Teng and Daskalakis, Goldberg, and Papadimitriou, 2009)

The problem **NASH is PPAD-complete**.

- One of the main breakthroughs in algorithmic game theory.
- We omit the proof, as it is complicated (the papers have over 50 and 70 pages).

# What now?

- So it is likely that there is no polynomial-time algorithm for NASH.



- Finding approximate NE in games with **at least three players** lies in PPAD, but the problem appears to be strictly harder than PPAD.
- If we modify NASH so that the existence is not always guaranteed, then the resulting problem often becomes **NP-complete**.
- This seems to be a **problem with the concept of NE**. “How can we expect the players to find a Nash equilibrium, if our computers cannot?”
- We introduce **other solution concepts** that possess some qualities of NE and yet are easier to compute.

# Other notions of equilibria

# Two new solution concepts

- Since finding NE is computationally difficult unless  $PPAD \subseteq FP$ , we look for **different solution concepts** that are computationally tractable.
- We introduce two such solution concepts:  $\epsilon$ -Nash equilibria and correlated equilibria.
  - The first one will seem natural with an easy-to-understand definition, but we will later notice some of its drawbacks.
  - The second one will have a rather complicated definition at first sight, but we will later learn to appreciate it and see that it might be even **more natural than NE!**

# $\varepsilon$ -Nash equilibria

- For  $\varepsilon > 0$ , a strategy profile  $s = (s_1, \dots, s_n)$  in a normal-form game  $G = (P, A, u)$  is an  $\varepsilon$ -Nash equilibrium ( $\varepsilon$ -NE) if, for every player  $i \in P$  and every  $s'_i \in S_i$ , we have  $u_i(s_i; s_{-i}) \geq u_i(s'_i; s_{-i}) - \varepsilon$ .
  - That is, no other strategy can improve the payoff by more than  $\varepsilon$ .
  - If we allowed  $\varepsilon = 0$ , we would get the standard NE.
- Advantages:
  - Easy-to-understand definition
  - $\varepsilon$ -NE always exist by Nash's theorem (every NE is  $\varepsilon$ -NE).
  - Using  $\varepsilon$  as the “machine precision” we do not have to work with irrational numbers.
- Disadvantages:
  - There are  $\varepsilon$ -NE that are not close to any NE (so  $\varepsilon$ -NE are not exactly approximations of NE).
  - We will see that this concept is also somehow computationally difficult.



# Algorithmic aspects of $\varepsilon$ -Nash equilibria

- An optimization problem  $P$  with input of size  $n$  and a parameter  $\varepsilon > 0$  has a **PTAS** if there is an algorithm that computes an  $\varepsilon$ -approximate solution of  $P$  in time  $O(n^{f(1/\varepsilon)})$  for some function  $f$ .
- The problem  $P$  has **FPTAS** if there is such an algorithm that runs in time  $O((1/\varepsilon)^c n^d)$  for some constants  $c$  and  $d$ .
- Do we have **FPTAS for  $\varepsilon$ -NE**?
  - **No**, unless  $PPAD \subseteq FP$  (Chen, Deng, and Teng, 2006).
- Do we have **PTAS for  $\varepsilon$ -NE**?
  - **Open problem!**
- So what do we have? A **quasi-polynomial-time algorithm**.

## Theorem 2.37 (Lipton, Markakis, and Mehta, 2003)

Let  $G = (P, A, u)$  be a normal-form game of two players, each having  $m$  actions, such that the payoff matrices have entries in  $[0, 1]$ . For every  $\varepsilon > 0$ , there is an **algorithm for computing  $\varepsilon$ -NE of  $G$  in time  $m^{O(\log m/\varepsilon^2)}$** .

- I no longer present the proof (see the lecture notes).

# Correlated equilibria

- The **most fundamental solution concept** according to several people.
- *“If there is intelligent life on other planets, in majority of them, they would have discovered correlated equilibrium before NE.”* (Myerson)
- In  $G = (P, A, u)$ , let  $p$  be a probability distribution on  $A$ , that is,  $p(a) \geq 0$  for every  $a \in A$  and  $\sum_{a \in A} p(a) = 1$ . The distribution  $p$  is a **correlated equilibrium (CE)** in  $G$  if

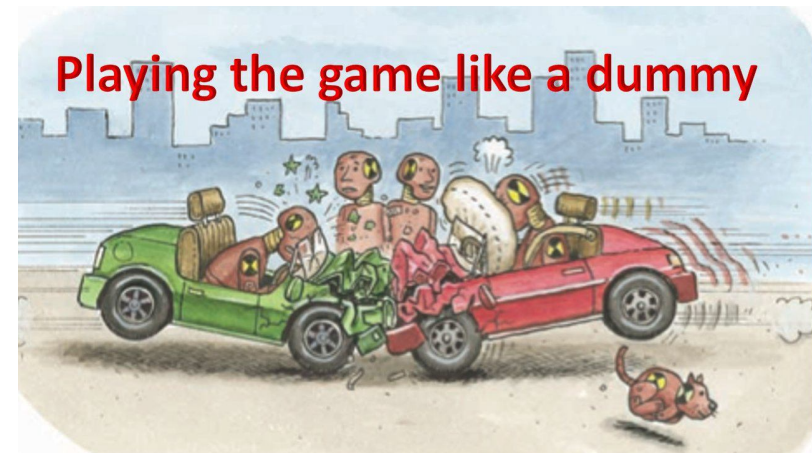
$$\sum_{a_{-i} \in A_{-i}} u_i(a_i; a_{-i}) p(a_i; a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a'_i; a_{-i}) p(a_i; a_{-i})$$

for every player  $i \in P$  and all pure strategies  $a_i, a'_i \in A_i$ .

- Imagine a trusted third party with the distribution  $p$  being publicly known. The trusted third party samples  $a \in A$  according to  $p$  and privately suggests the strategy  $a_i$  to  $i$ , but does not reveal  $a_{-i}$  to  $i$ . The player  $i$  can follow this suggestion, or not. Then,  $p$  is **CE if every player maximizes his expected utility by playing the suggested strategy  $a_i$ .**

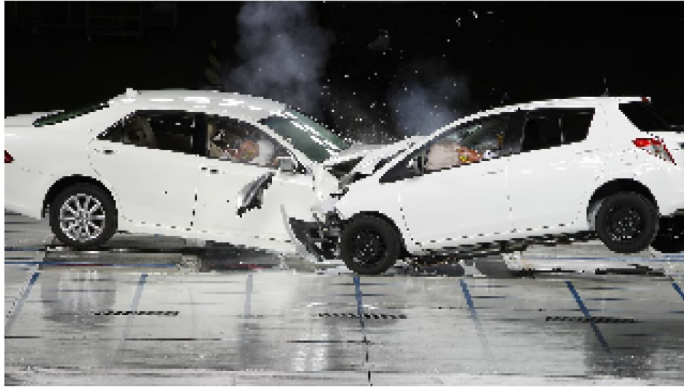
# Example of correlated equilibria: Game of Chicken

|      | Stop   | Go        |
|------|--------|-----------|
| Stop | (0,0)  | (-1,1)    |
| Go   | (1,-1) | (-10,-10) |



Sources: <https://peakd.com/>

- There are **two pure NE** with  $(s_1(S), s_2(S)) = (1, 0)$  and  $(s_1(S), s_2(S)) = (0, 1)$ , and one **mixed NE** with  $(s_1(S), s_2(S)) = (9/10, 9/10)$ .
- Consider a trusted third party, a **traffic light**. The traffic light chooses  $(S, S)$ ,  $(S, G)$ , and  $(G, S)$  independently at random with probability  $1/3$ . **The traffic light gives CE.**
  - If 1 follows the suggestion “go”, then he gets 1 while deviating gives him 0.
  - If 1 follows the suggestion “stop”, then he gets  $-1/2$  while deviating gives him  $-9/2$ .
  - By symmetry, driver 2 does not deviate as well.



Source: Students of MFF UK

# Example of correlated equilibria: Battle of sexes

|          | Football | Opera |
|----------|----------|-------|
| Football | (2,1)    | (0,0) |
| Opera    | (0,0)    | (1,2) |



Sources: <https://media.istockphoto.com/>

- There are **two pure NE** with  $(s_1(F), s_2(F)) = (1, 1)$  and  $(s_1(F), s_2(O)) = (0, 0)$ , and one **mixed NE** with  $(s_1(F), s_2(O)) = (2/3, 2/3)$ .
- Consider a trusted third party, a **mother-in-law**. The mother-in-law flips a coin and chooses  $(F, F)$  or  $(O, O)$  independently at random with probability  $1/2$ . **The mother-in-law gives CE.**
  - If the husband follows the suggestion “football”, then he gets 2 while deviating gives him 0.
  - If the husband follows the suggestion “opera”, then he gets 1 while deviating gives him 0.
  - By symmetry, the wife does not deviate as well.

# Advantages and disadvantages of correlated equilibria

- **Disadvantages:**

- The definition of CE takes some getting used to.

- **Advantages:**

- Every NE is CE (**Exercise**). So CE always exist by **Nash's theorem**.
- Each NE  $s$  is CE with the product distribution  $p = \prod_{i=1}^n s_i$ . So CE can give better payoffs than NE.
- Can be computed in **polynomial time using** LP! Consider the following LP with variables  $(p(a))_{a \in A}$ :

$$\max \left\{ \sum_{i \in P} \left( \sum_{a \in A} u_i(a) p(a) \right) \right\} \text{ subject to, for all } i \in P, a_i, a'_i \in A_i,$$

$$\sum_{a_{-i} \in A_{-i}} u_i(a_i; a_{-i}) p(a_i; a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a'_i; a_{-i}) p(a_i; a_{-i})$$

$$\sum_{a \in A} p(a) = 1, p(a) \geq 0 \text{ for every } a \in A.$$

- The objective function can be arbitrary as long as it is linear.

- The concept of correlated equilibria was introduced by **Robert Aumann**, who received a **Nobel prize** in economics for his work in game theory.

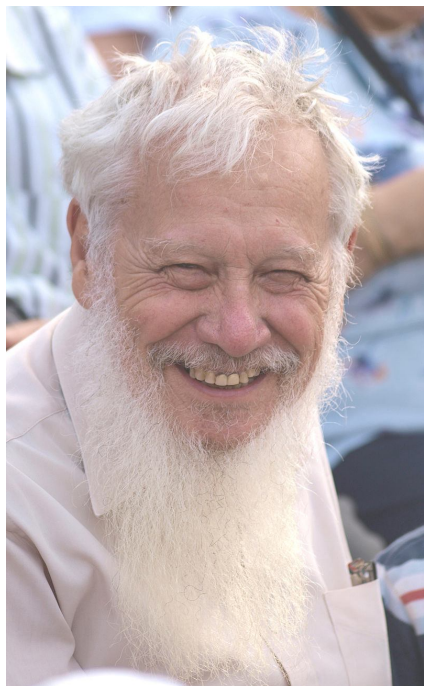


Figure: **Robert Aumann** (born 1930).

Sources: <https://en.wikipedia.org> and <https://slideslive.com/38910863/strategic-information-theory>

- In 2018, Robert Aumann visited Prague and gave a lecture at Prague mathematical colloquium. You can see the lecture here: <https://slideslive.com/38910863/strategic-information-theory>.

**Thank you for your attention.**