Algorithmic game theory

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4th lecture

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Nash equilibria in bimatrix games

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- The best response condition: If x and y are mixed strategy vectors of players 1 and 2, respectively, then x is a best response to y iff

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Analogously, y is a best response to x iff

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- Today, we reveal a geometric structure behind finding NE in bimatrix games and show one of the fastest known algorithms for this task.

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- Thus, points of $\overline{P}$ and $\overline{Q}$ are the mixed strategies with the "upper envelope" of expected payoffs of the other player.

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- Similarly, a point $(y, u) \in \overline{Q}$ has a label $i \in A_1 \cup A_2$ if either $i \in A_1$ and $M_{i,*} y = u$, or if $i \in A_2$ and $y_i = 0$.

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- Each point from $\overline{P}$ or $\overline{Q}$ may have more labels.

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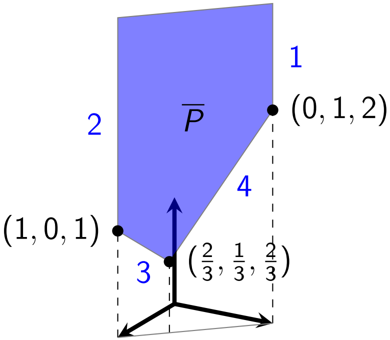

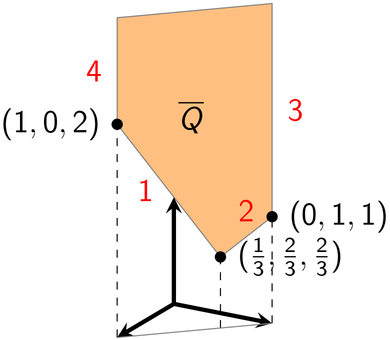

$$\overline{P} = \{(x_1, x_2, v) \in \mathbb{R}^2 \times \mathbb{R} \colon x_1, x_2 \geq 0, x_1 + x_2 = 1, x_1 \leq v, 2x_2 \leq v\}$$

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Proposition 2.27

A strategy profile s is NE of G iff the pair $((x, v), (y, u)) \in \overline{P} \times \overline{Q}$ is completely labeled, that is, every label $i \in A_1 \cup A_2$ appears as a label of either (x, v) or (y, u).

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- Proof: By the Best response condition, for every player $i \in P$, a mixed strategy s_i is a best response to s_{-i} if and only if all pure strategies in the support of s_i are best responses to s_{-i}.
- A missing label $i \in A_1$ means that a pure strategy $i \in Supp(s_1)$ $(x_i > 0)$ was not a best response $(M_{i,*}y < u)$. Analogously for $i \in A_2$. Then, s is not NE.
- If all labels appear, then s_1 and s_2 are mutually best responses, as each pure strategy is a best response or is not in the support. Then, s is NE. □

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Similarly, the best response polytope for player 2 in G is a polytope

$$Q = \{y \in \mathbb{R}^n \colon y \geq \mathbf{0}, My \leq \mathbf{1}\}.$$

Best response polytopes P and Q for the Battle of sexes

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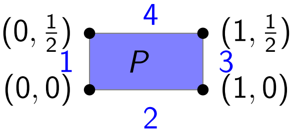

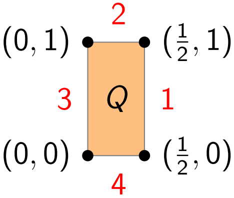

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1, x_2 \geq 0, x_1 \leq 1, 2x_2 \leq 1\}$$

$$Q = \{(y_3, y_4) \in \mathbb{R}^2 \colon y_3, y_4 \geq 0, 2y_3 \leq 1, y_4 \leq 1\}.$$

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  - Projective transformations preserve incidences, so the labels stay the same.

Corollary 2.30

A strategy profile (s_1, s_2) with mixed strategy vectors x and y is NE of G if and only if the point $(x/u_2(s), y/u_1(s)) \in P \times Q \setminus \{(\mathbf{0}, \mathbf{0})\}$ is completely labeled. □

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  - Since $dim(P) = m$ and $dim(Q) = n$, only vertices of P and Q can have m and n labels.

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- $\Rightarrow$ Algorithm for finding NE: check all pairs of vertices and their labels!

Algorithm for finding NE with vertex enumeration

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Algorithm 0.2: VERTEX ENUMERATION(G)

Input : A nondegenerate bimatrix game G.
Output : All Nash equilibria of G.
for each pair (x, y) of vertices from $(P \setminus \{\mathbf{0}\}) \times (Q \setminus \{\mathbf{0}\})$
$\begin{cases} \text{if } (x, y) \text{ is completely labeled,} \\ \text{then return } (x/(\mathbf{1}^\top x), y/(\mathbf{1}^\top y)) \text{ as a Nash equilibrium} \end{cases}$

Algorithm for finding NE with vertex enumeration

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- All vertices of a simple polytope in $\mathbb{R}^d$ with v vertices and N defining inequalities can be found in time $O(dNv)$ (Avis and Fukuda).

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Algorithm 0.4: VERTEX ENUMERATION(G)

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- All vertices of a simple polytope in $\mathbb{R}^d$ with v vertices and N defining inequalities can be found in time $O(dNv)$ (Avis and Fukuda).
- However, if $m = n$, the best response polytopes can have c^n vertices for some constant c with $1 < c < 2.9$.

Polytopes can be weird and complex!

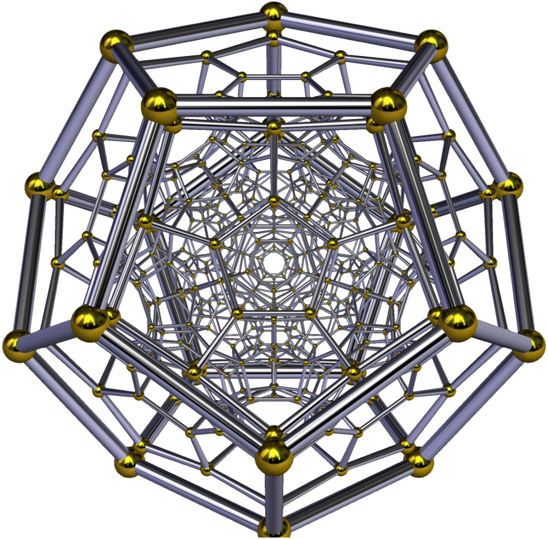

Figure: Schlegel diagram for the 120-cell.

Source: https://en.wikipedia.org/

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Algorithm for finding NE with vertex enumeration

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- We can speed up the search by performing a walk on $(P \setminus \{\mathbf{0}\}) \times (Q \setminus \{\mathbf{0}\})$ guided by the labels.

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Figure: Carlton E. Lemke (1920–2004) and J. T. Howson (1937–2022).

Source: https://oldurls.inf.ethz.ch

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- Since P is simple and $dim(P) = m$, its vertices are incident to exactly m facets and m edges. So each vertex of P has m labels and each edge of P has $m - 1$ labels. Similarly for Q and n.
- Dropping a label $l \in A_1 \cup A_2$ in a vertex x of P means traversing the unique edge of P that is incident to x and has all m labels that x has besides l. The other endpoint of this edge has the same labels as x, only l is replaced with a new label, which is said to be picked up. Dropping and picking up a label in vertices of Q is defined analogously.
- The algorithm starts at $(\mathbf{0}, \mathbf{0})$ and alternately follows edges of P and Q.
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- Duplicate label is either a new best response, which gets a positive probability, or a pure strategy whose probability became 0 and we move away from its best response facet.

The Lemke–Howson algorithm: pseudocode

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Algorithm 0.8: LEMKE–HOWSON(G)

Input : A nondegenerate bimatrix game G.
Output : One Nash equilibrium of G.
$(x, y) \leftarrow (\mathbf{0}, \mathbf{0}) \in \mathbb{R}^m \times \mathbb{R}^n$,
$k \leftarrow$ arbitrary label from $A_1 \cup A_2$, $l \leftarrow k$,
while (*true*)
do {
In P, drop l from x and redefine x as the new vertex, redefine l as the newly picked up label. Switch to Q.
If $l = k$, stop looping.

In Q, drop l from y and redefine y as the new vertex, redefine l as the newly picked up label. Switch to P.
If $l = k$, stop looping.
}
Output $(x/(\mathbf{1}^\top x), y/(\mathbf{1}^\top y))$.

Lemke–Howson on the Battle of sexes ($k = 3$)

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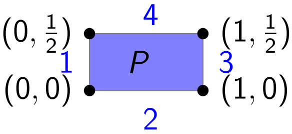

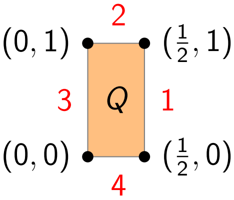

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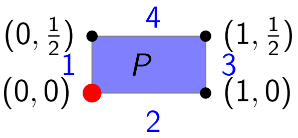

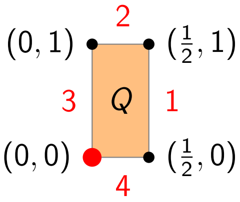

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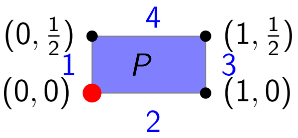

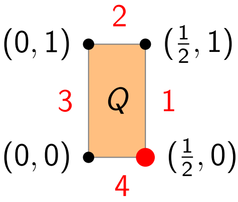

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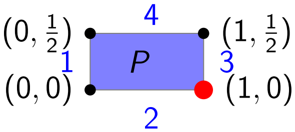

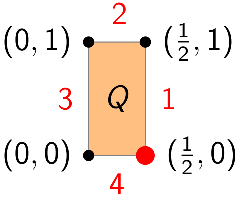

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- Is there an efficient algorithm to find NE?

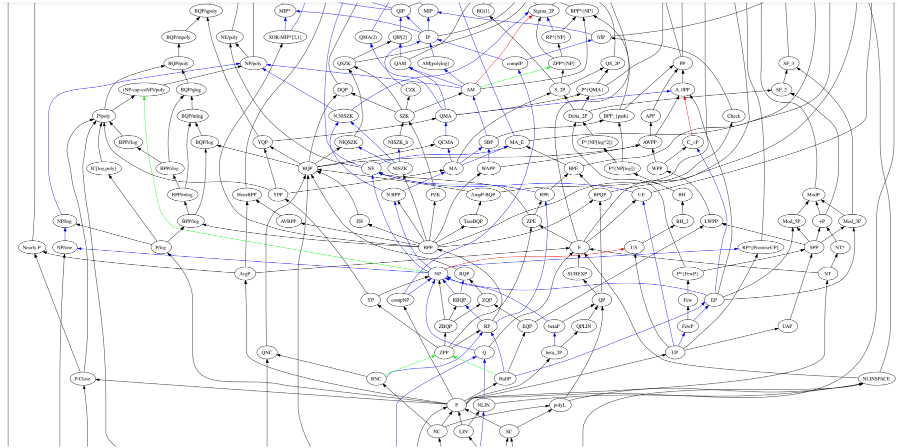

Figure: A view on the complexity classes classification.

Source: https://complexityzoo.uwaterloo.ca/Complexity_Zoo

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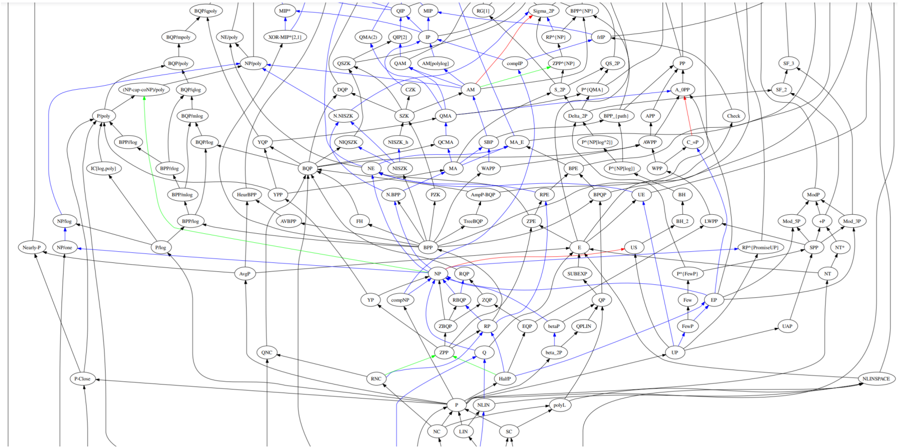

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Thank you for your attention.