Algorithmic game theory

Martin Balko

2nd lecture

October 11th 2024

Proof of Nash's Theorem

Nash equilibria in normal-form games

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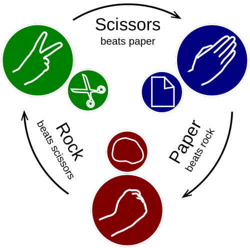

Sources: https://en.wikipedia.org/

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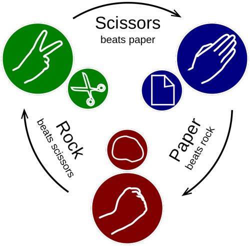

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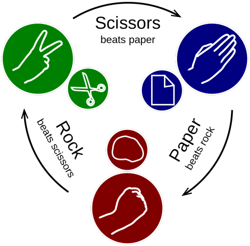

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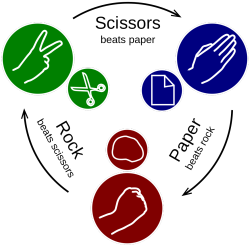

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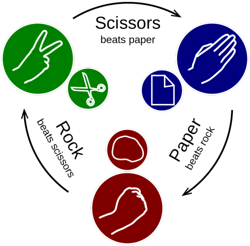

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- Amazingly, every normal-form game has a Nash equilibrium.

Nash's Theorem

Nash's Theorem (Theorem 2.16)

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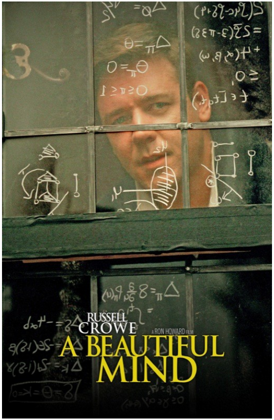

Figure: John Forbes Nash Jr. (1928–2015) and his depiction in the movie A Beautiful mind.

Sources: https://britannica.com and https://medium.com

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- For $d \in \mathbb{N}$, a subset X of $\mathbb{R}^d$ is compact if X is closed and bounded.
- We say that a subset Y of $\mathbb{R}^d$ is convex if every line segment containing two points from Y is fully contained in Y. Formally: for all x, y from Y, $tx + (1-t)y \in Y$ for every $t \in [0,1]$.
- For n affinely independent points $x_1, \ldots, x_n \in \mathbb{R}^d$, an $(n-1)$-simplex Δ_n on $x_1, \ldots, x_n$ is the set of convex combinations of the points $x_1, \ldots, x_n$. Each simplex is a compact convex set in $\mathbb{R}^d$.

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Lemma (Lemma 2.18)

For $n, d_1, \ldots, d_n \in \mathbb{N}$, let $K_1, \ldots, K_n$ be compact sets, each K_i lying in $\mathbb{R}^{d_i}$. Then, $K_1 \times \cdots \times K_n$ is a compact set in $\mathbb{R}^{d_1+\cdots+d_n}$.

Brouwer's Fixed Point Theorem

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- For each $d \in \mathbb{N}$, let K be a non-empty compact convex set in $\mathbb{R}^d$ and $f : K \to K$ be a continuous mapping. Then, there exists a fixed point $x_0 \in K$ for f, that is, $f(x_0) = x_0$.

Figure: L. E. J. Brouwer (1881–1966).

Source: https://arxiv.org/pdf/1612.06820.pdf

- https://www.youtube.com/watch?v=csInNn6pfT4&t=268s&ab_

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  - By Lemma 2.18, the set $K = S_1 \times \cdots \times S_n$ is compact.
  - For any strategy profiles $s = (s_1, \ldots, s_n), s' = (s'_1, \ldots, s'_n) \in K$ and a number $t \in [0, 1]$, the point

$$ts + (1 - t)s' = (ts_1 + (1 - t)s'_1, \ldots, ts_n + (1 - t)s'_n)$$

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- For every player $i \in P$ and action $a_i \in A_i$, we define a mapping $\varphi_{i,a_i}\colon K \to \mathbb{R}$ by setting

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- We then define f by setting $f(s) = s'$.

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- Then, f is continuous, since the mappings φ_{i,a_i} are.
- It remains to show that fixed points of f are exactly NE in G. Then, Brouwer's theorem gives us a fixed point of f, which is NE in G.
- First, if s is NE, then all functions φ_{i,a_i} are constant zero functions and thus $f(s) = s$.

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  - Since s is a fixed point, we get $s'_i(a'_i) = s_i(a'_i)$

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distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \to Q$, $P_n \to P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

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REMARK ON WEYL'S NOTE "INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION"*

BY GEORGE POLYA

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY

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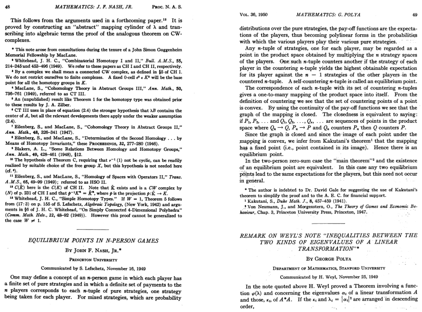

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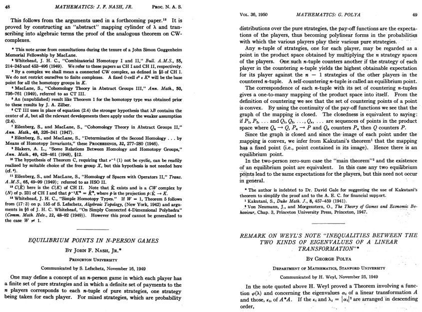

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EQUILIBRIUM POINTS IN N-PERSON GAMES

By John F. Nash, Jr.*

Princeton University

Communicated by S. Lefschetz, November 16, 1949

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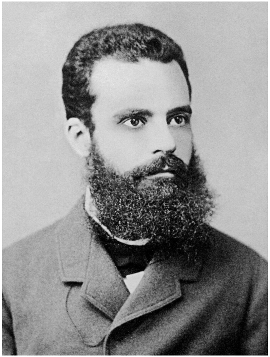

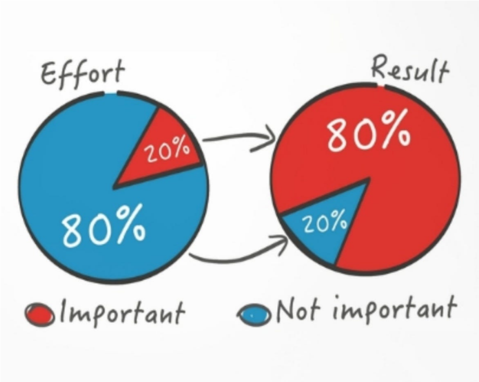

Figure: Vilfredo Pareto (1848–1923).

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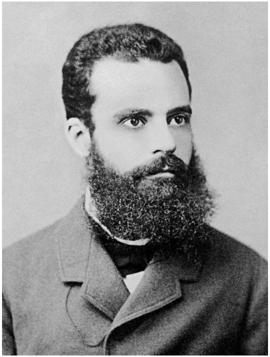

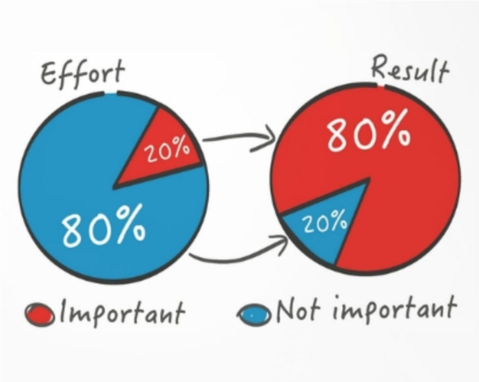

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- Pareto principle: for many outcomes roughly 80% of consequences come from 20% of the causes.

Finding Nash equilibria in zero-sum games

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- Historically, zero-sum games were considered first in game game theory (by Morgenstern and Von Neumann in the 1940s).

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	Rock	Paper	Scissors
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Paper	(1,-1)	(0,0)	(-1,1)
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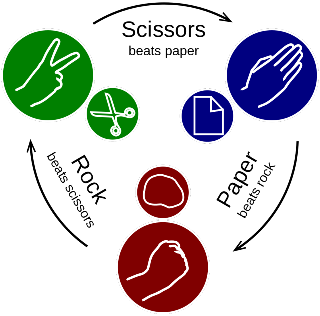

Sources: https://en.wikipedia.org/

Zero-sum games examples: chess

Source: https://edition.cnn.com/

Zero-sum games examples: table tennis

Source: https://www.reddit.com/

Zero-sum games examples: derivative trading

Source: https://www.linkedin.com/

Zero-sum games examples: elections

Source: https://news.sky.com/

Zero-sum games examples: many more

Source: https://lhongtortai.com/collection/what-is-a-non-zero-sum-game

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$$u_1(s) = \sum_{a=(i,j)\in A} u_1(a) s_1(i) s_2(j)$$

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Figure: John von Neumann (1903–1957) and Oskar Morgenstern (1902–1977).

Sources: https://en.wikiquote.org and https://austriainusa.org

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Source: https://czthomas.files.wordpress.com

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