Algorithmic game theory

Martin Balko

1st lecture

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Basic info

- Webpage: *https://kam.mff.cuni.cz/~balko/ath2425/ATH.html*
  - lecture info, topics covered, presentations, lecture notes ...
- Recommended literature:
  - M. Balko: Algorithmic game theory: lecture notes.
  - The notes are still under construction. Comments are welcome.

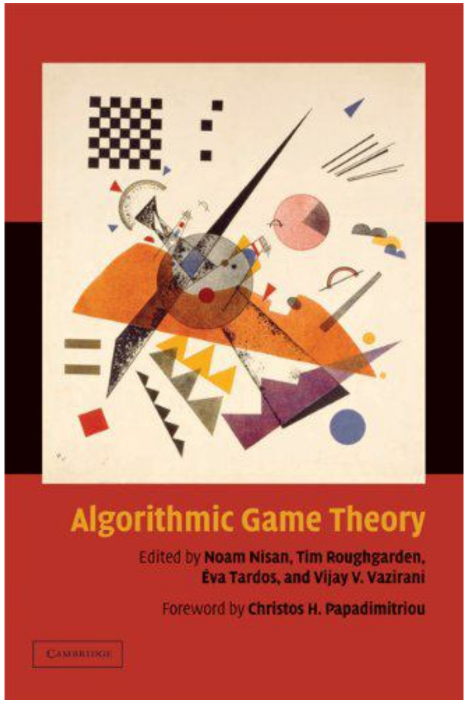


Figure: Algorithmic game theory by Nisan et al.

Source: https://amazon.com

Game theory

- study of mathematical models of strategic interaction among rational decision-makers.

Zdroj: https://quantamagazine.org

- We focus on the algorithmic side of the game theory.
- Several real-word applications.
- More than ten game theorists have won the Nobel Prize in economics.

Sylabus

- Preliminary plan:
  - Finding Nash equilibria
    - Nash equilibria and Nash's Theorem,
    - zero-sum games,
    - bimatrix games and the Lemke–Howson algorithm,
    - other notions of equilibria,
    - regret minimization.
  - Mechanism design,
    - auctions (Vickrey),
    - Myerson's lemma and its applications,
    - revenue maximization.

Finding Nash equilibria

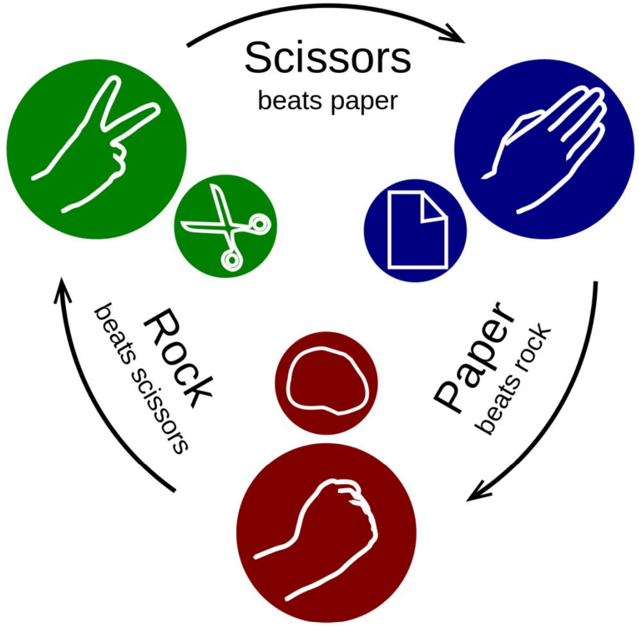
Normal-form games

- We use the following most fundamental representation of games.
- A normal-form game is a triple (P, A, u), where
  - P is a finite set of n players,
  - $A = A_1 \times \cdots \times A_n$ is a set of action profiles, where A_i is a set of actions available to player i,
  - and $u = (u_1, \ldots, u_n)$ is an n-tuple, where each $u_i \colon A \to \mathbb{R}$ is the utility function for player i.

- Knowing the utility function, all players i simultaneously choose an action a_i from A_i. The resulting action profile $a = (a_1, \ldots, a_n)$ is then evaluated using the utility function.
- The ith coordinate $u_i(a)$ of $u(a)$ is the gain of player i on a.

Normal-form games: Rock-Paper-Scissors

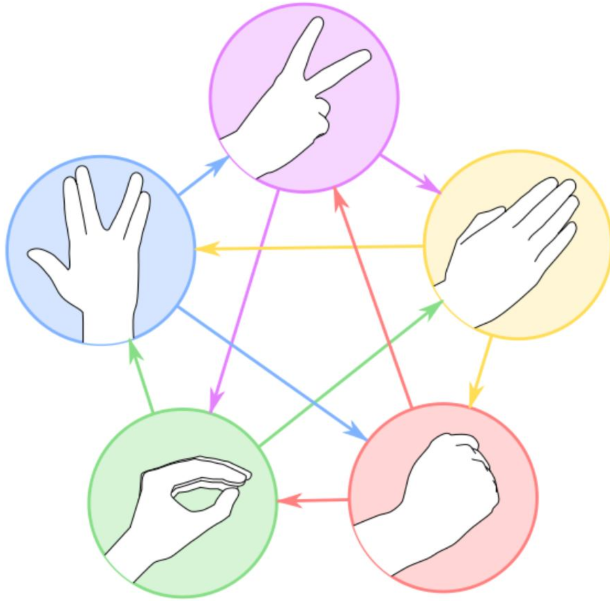
	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)



Sources: https://en.wikipedia.org/

Normal-form games: Rock-Paper-Scissors-Lizard-Spock

	Rock	Paper	Scissors	Lizard	Spock
Rock	(0,0)	(-1,1)	(1,-1)	(1,-1)	(-1,1)
Paper	(1,-1)	(0,0)	(-1,1)	(-1,1)	(1,-1)
Scissors	(-1,1)	(1,-1)	(0,0)	(1,-1)	(-1,1)
Lizard	(-1,1)	(1,-1)	(-1,1)	(0,0)	(1,-1)
Spock	(1,-1)	(-1,1)	(1,-1)	(-1,1)	(0,0)



Sources: https://bigbangtheory.fandom.com/

- "Scissors cuts Paper, Paper covers Rock, Rock crushes Lizard, Lizard poisons Spock, Spock smashes Scissors, Scissors decapitates Lizard, Lizard eats Paper, Paper disproves Spock, Spock vaporizes Rock (and as it always has) Rock crushes Scissors."

Normal-form games: Rock-Paper-Scissors-Lizard-Spock

Source: https://www.naturphilosophie.co.uk/

Normal-form games: Chess

Source: https://edition.cnn.com/

- Chess as a normal-form game: Each action of player $i \in \{\text{black}, \text{white}\}$ is a list of all possible situations that can happen on the board together with the move player i would make in that situation. Then we can simulate the whole game of chess in one round.

Strategies

- Each player i follows a certain strategy (a prescription how he chooses his actions from A_i).
- A pure strategy s_i of player i is an action from A_i.
  - "select a single action and play it",
  - a pure-strategy profile is an n-tuple $(s_1, \ldots, s_n)$, where $s_i \in A_i$ for each player i.
- A mixed strategy s_i of player i is a probability distribution over A_i.
  - that is, s_i assigns a value $s_i(a_i) \in [0, 1]$ to each $a_i \in A_i$ so that

$$\sum_{a_i \in A_i} s_i(a_i) = 1.$$

  - $s_i(a_i) =$ "probability that i chooses a_i as his action".
  - We let S_i be the set of all mixed strategies of player i.
  - a mixed-strategy profile is an n-tuple $(s_1, \ldots, s_n)$, where $s_i \in S_i$ for each player i.
- Every pure strategy is a mixed strategy.

Expected payoff

- The goal of each player is to maximize his expected payoff.
- In $G = (P, A, u)$, the expected payoff for player i of the mixed-strategy profile $s = (s_1, \dots, s_n)$ is

$$u_i(s) = \sum_{a=(a_1,\dots,a_n)\in A} u_i(a) \cdot \prod_{j=1}^{n} s_j(a_j) = \sum_{\omega\in\Omega} X(\omega) \cdot \Pr[\{\omega\}].$$

  - that is, $u_i(s)$ is the expected value of u_i under the product distribution $\prod_{j=1}^{n} s_j$.
- It satisfies the linearity of the expected payoff (Exercise):

$$u_i(s) = \sum_{a_i\in A_i} s_i(a_i) \cdot u_i(a_i; s_{-i}),$$

where $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ and $(s_i'; s_{-i}) = (s_1, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n)$ for any $s_i' \in S_i$.

Example: expected payoff in Rock-Paper-Scissors

- Consider the Rock-Paper-Scissors game where each player i uses a strategy s_i that assigns each action the probability $1/3$.

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

- By definition, the expected payoff of player 1 on $s = (s_1, s_2)$ is

$$u_1(s) = 1 \cdot \left(\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}\right) + (-1) \cdot \left(\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}\right) + 0 \cdot \left(\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}\right) = 0.$$

- By linearity, $u_1(s) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 0 = 0.$

Examples of normal-form games

- We now give four more examples of normal-form games.
- Several of these are used later in the lecture and the tutorials.

- We focus here only on two-player games, that is, $P = \{1, 2\}$.
- These games are called bimatrix games, as they can be represented with two real matrices.
- Player 1 will be the "row player" while player 2 will be the "column player".

Prisoner's dilemma

- Two prisoners, are being held in solitary confinement and cannot communicate with the other. Each can either betray the other one by testifying or cooperate with the other one by remaining silent.

	Testify	Remain silent
Testify	(-2,-2)	(0,-3)
Remain silent	(-3,0)	(-1,-1)

Sources: Serena Maylon (MtG)

- Paradoxically, the only stable solution is when both testify.

Matching pennies

- We introduce an easier variant of the Rock-Paper-Scissors game.
- Each of the two players has a penny and chooses either Heads of Tails. If the pennies match, then player 1 wins and keeps both pennies. Otherwise, player 2 keeps both pennies.

	Heads	Tails
Heads	(1,-1)	(-1,1)
Tails	(-1,1)	(1,-1)

Sources: https://www.fourstateshomepage.com/

- Like Rock-Papers-Scissors, this is a zero-sum game (whatever one player gets, the other one loses). Prisoner's dilemma is not.

Battle of sexes

- A husband and wife wish to spend an evening together rather than separately, but cannot decide which event to attend. The husband wishes to go to a football match while the wife wants to go to opera.

	Football	Opera
Football	(2,1)	(0,0)
Opera	(0,0)	(1,2)

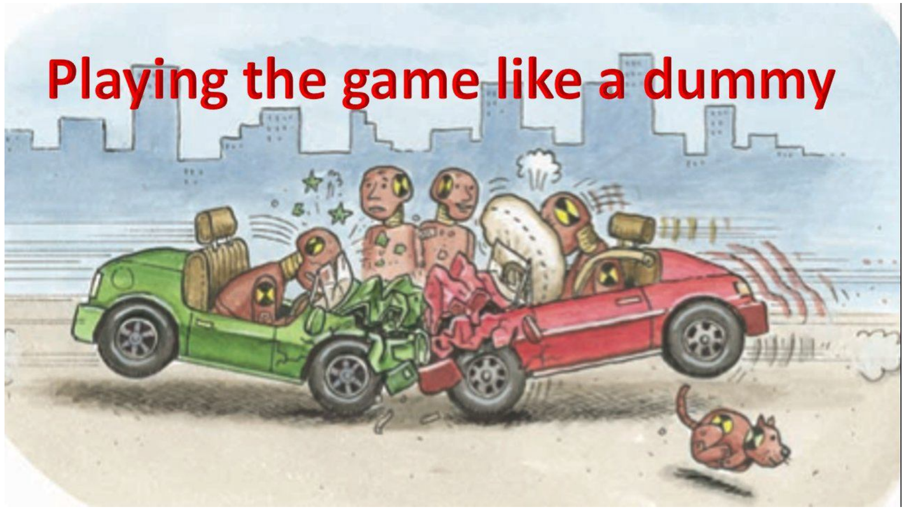
Sources: https://media.istockphoto.com/

- This game displays both cooperation and competition.

Game of chicken

- Two drivers drive towards each other on a collision course: one must swerve, or both die in the crash. However, if one driver swerves and the other does not, the one who swerved will be called a "chicken"

	Turn	Go straight
Turn	(0,0)	(-1,1)
Go straight	(1,-1)	(-10,-10)



Sources: https://peakd.com/

- What is the best strategy for the players?

Nash equilibrium

- In game theory, we typically study rules for predicting how a game will be played, called solution concepts.
- We now introduce perhaps the most influential solution concept, which captures a notion of stability.

- The best response of player i to a strategy profile s_{-i} is a mixed strategy s_i^* such that $u_i(s_i^*; s_{-i}) \geq u_i(s_i'; s_{-i})$ for each $s_i' \in S_i$.
  - If i knew what strategies the others follow, he would choose this one. It maximizes his expected payoff if others play s_{-i}.
- For a normal-form game $G = (P, A, u)$ of n players, a Nash equilibrium (NE) in G is a strategy profile $(s_1, \ldots, s_n)$ such that s_i is a best response of player i to s_{-i} for every $i \in P$.
  - A stable solution concept: no player would like to change his strategy if he knew the strategies of the other players.
  - Introduced by Nash and by Von Neumann and Morgenstern.

Nash equilibria: remarks

- Neither best responses nor Nash equilibria are determined uniquely.
- Example 1: In the Rock-Paper-Scissors game, there is a unique mixed Nash equilibrium (both players play everything with probability $1/3$).
- Example 2: In the Battle of sexes game, there are three Nash equilibria, two pure and one mixed.

- Do Nash equilibria always exist in every game? Is there always a stable solution concept?
- Yes, they do! Shown by Nash in 1950.
- Maybe the most influential result in game theory. Later, Nash received a Nobel prize for economics.

Nash’s Theorem

Nash’s Theorem (Theorem 2.16)

Every normal-form game has a Nash equilibrium.

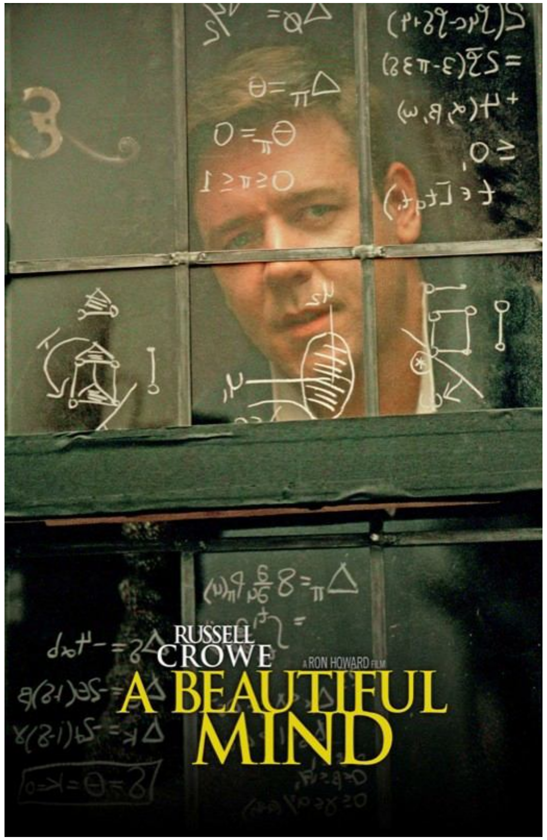


Figure: John Forbes Nash Jr. (1928–2015) and his depiction in the movie A Beautiful mind.

Sources: https://britannica.com and https://medium.com

Preparations for the proof of Nash's theorem

- The proof is essentially topological, as its main ingredient is a fixed-point theorem. We use a theorem due to Brouwer.

- For $d \in \mathbb{N}$, a subset X of $\mathbb{R}^d$ is compact if X is closed and bounded.
- We say that a subset Y of $\mathbb{R}^d$ is convex if every line segment containing two points from Y is fully contained in Y. Formally: for all x, y from Y, $tx + (1-t)y \in Y$ for every $t \in [0,1]$.
- For n affinely independent points $x_1, \ldots, x_n \in \mathbb{R}^d$, an $(n-1)$-simplex Δ_n on $x_1, \ldots, x_n$ is the set of convex combinations of the points $x_1, \ldots, x_n$. Each simplex is a compact convex set in $\mathbb{R}^d$.

Lemma (Lemma 2.18)

For $n, d_1, \ldots, d_n \in \mathbb{N}$, let $K_1, \ldots, K_n$ be compact sets, each K_i lying in $\mathbb{R}^{d_i}$. Then, $K_1 \times \cdots \times K_n$ is a compact set in $\mathbb{R}^{d_1 + \cdots + d_n}$.

Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem (Thoerem 2.17)

For each $d \in \mathbb{N}$, let K be a non-empty compact convex set in $\mathbb{R}^d$ and $f : K \to K$ be a continuous mapping. Then, there exists a fixed point $x_0 \in K$ for f, that is, $f(x_0) = x_0$.

Figure: L. E. J. Brouwer (1881–1966).

Source: https://arxiv.org/pdf/1612.06820.pdf

Figure: John Forbes Nash Jr. receiving a Nobel prize for economics.

Source: https://pbs.org

Thank you for your attention.