Algorithmic game theory

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VCG mechanism (Theorem 3.18)

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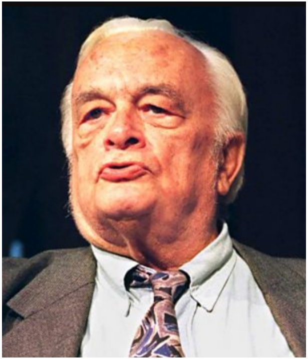

Figure: William Vickrey, Edward H. Clarke, and Theodore Groves.

Sources: : https://en.wikipedia.org, https://www.demandrevelation.com/, and https://www.researchate.net/

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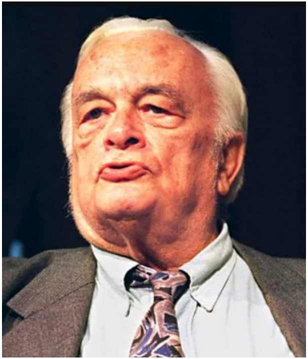

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- We now present the proof.

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together with this payment formula

$$p_i(b) = \max_{\omega\in\Omega} \left\{ \sum_{\substack{j=1\\ j\neq i}}^{n} b_j(\omega) \right\} - \sum_{\substack{j=1\\ j\neq i}}^{n} b_j(\omega^*),$$

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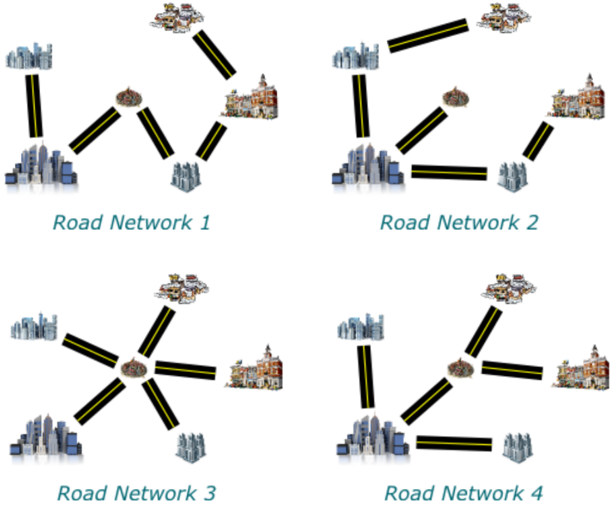

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	Road Network 1	Road Network 2	Road Network 3	Road Network 4
	6 M$	14 M$	2 M$	**16 M$**
	5 M$	8 M$	4 M$	**12 M$**
	2 M$	1 M$	**20 M$**	4 M$
	4 M$	**6 M$**	3 M$	5 M$
	1 M$	1 M$	**6 M$**	2 M$
	1 M$	2 M$	2 M$	**3 M$**
Total (social welfare)	19 M$	32 M$	37 M$	**42 M$**

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- Cities pay their negative externalities on the collectivity. Other cities would be happier without the biggest city (NYC, say). How much happier they would be is exactly what NYC must pay.

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- If NYC was not there, then road network number 3 (RN3) would have been chosen, as opposed to RN4. The value of RN3 for the other cities would be 35 M$, as opposed to the 26 M$ of RN4. Therefore, the negative externality of NYC is $35 - 26 = 9$ M$.

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- One can show that we have $0 \leq p_i(b) \leq b_i(\omega^*)$ (Exercise), hence truthtelling agents are guaranteed non-negative utility.

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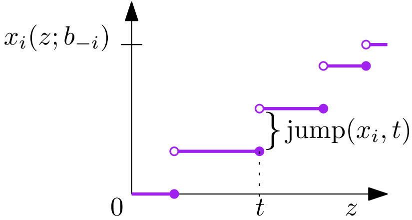

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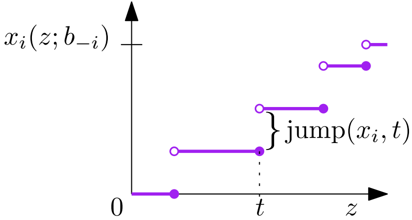

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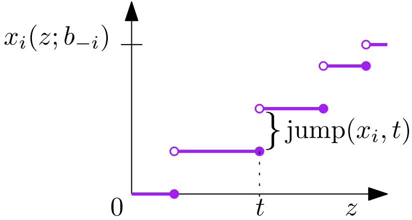

- For a piecewise constant function f, we use $\mathrm{jump}(f, t)$ to denote the magnitude of the jump of f at point t.
- If we fix z in the payment difference sandwich and let y approach z from below, then both sides become 0 if there is no jump of x_i at z. If $\mathrm{jump}(x_i, z) = h > 0$, then both sides tend to $z \cdot h$.

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- It will follow from a picture that it is optimal for bidder i to bid $b_i = v_i$.

Proof of Myerson's lemma by picture

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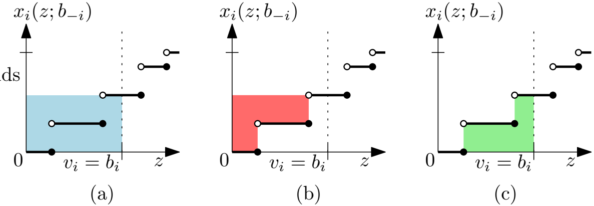
Bidder i bids truthfully:



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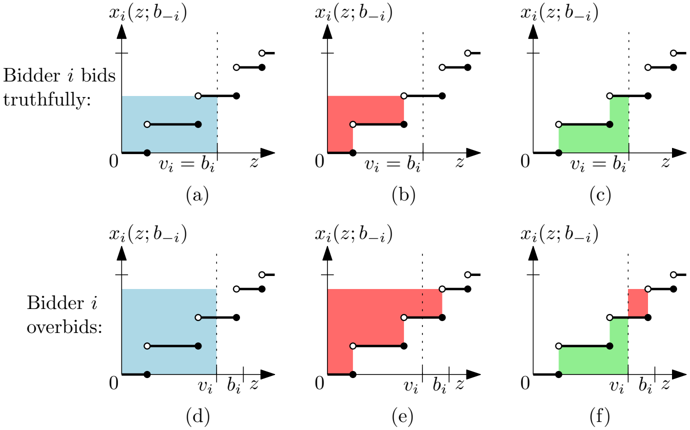

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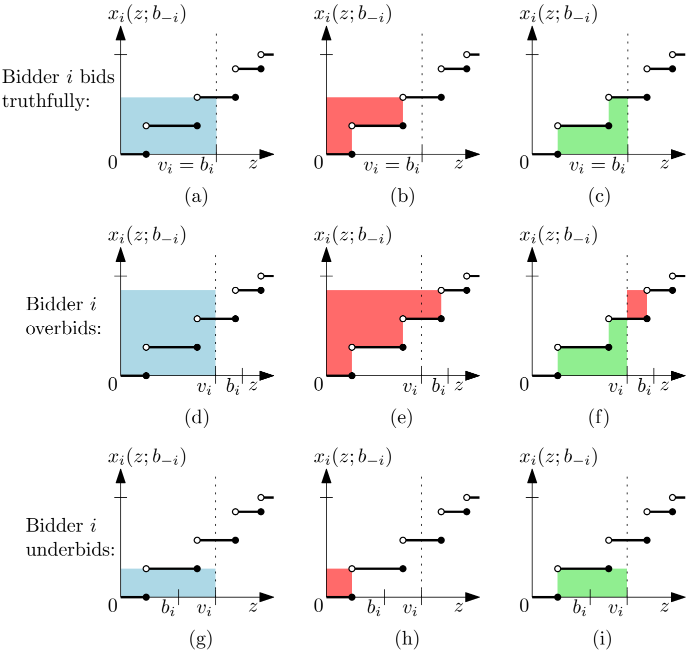

Revelation principle

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- So is DSIC too restrictive?

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Revelation principle (Theorem 3.19)

For every multi-parameter mechanism M in which every bidder has a dominant strategy, no matter what his private valuation is, there is an equivalent mechanism M' in which each bidder has a dominant strategy that is a direct revelation.

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Exams

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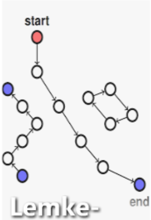
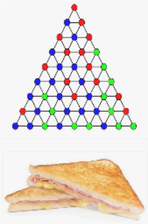
- What you should know: everything that we covered (everything is included in the lecture notes).

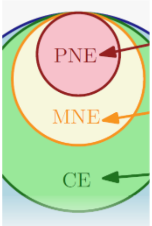

Nash equilibria


Minimax theorem

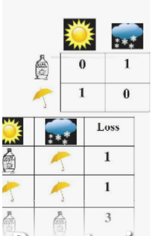


Lemke-Howson algorithm

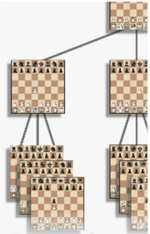

Complexity of NASH



Variants of NE



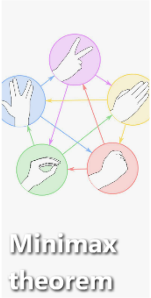
Regret minimization

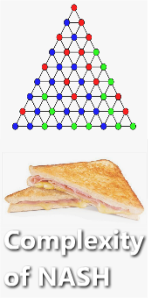

Extensive games

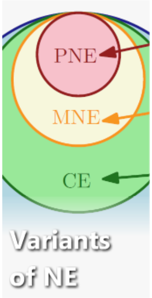

Mechanism design

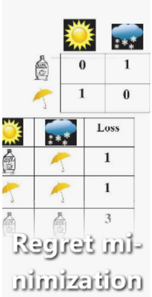
Revenue maximization

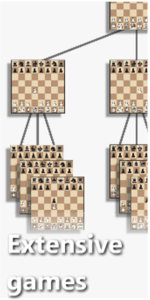

VCG mechanism











Thank you for your attention.