Algorithmic game theory

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- What if the seller does not know the distributions $F_1, \ldots, F_n$?

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Let $F = F_1 = \cdots = F_n$ be a regular probability distribution. Then,

$$\mathbb{E}_{v_1,\ldots,v_{n+1}\sim F}\left[\mathrm{Rev}(VA_{n+1})\right] \geq \mathbb{E}_{v_1,\ldots,v_n\sim F}\left[\mathrm{Rev}(OPT_{F,n})\right],$$

where $\mathrm{Rev}(VA_{n+1})$ is the revenue of Vickrey auction VA_{n+1} with $n+1$ bidders (and no reserve) and $\mathrm{Rev}(OPT_{F,n})$ denotes the revenue of the optimal auction $OPT_{F,n}$ for F with n bidders.

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Figure: Jeremy Bulow and Paul Klemperer.
Sources: https://economics.stanford.edu/ and https://www.economics.ox.ac.uk/

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- More competition is better than finding the right auction format.

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- Informally: extra competition is more important than getting the auction format just right.

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□

Knapsack auctions

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  - computational efficiency: the mechanism runs in polynomial time.

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- Can we design an awesome mechanism that assigns the slots?

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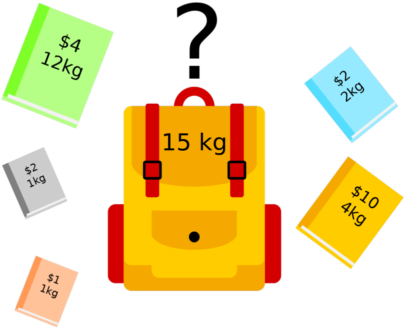

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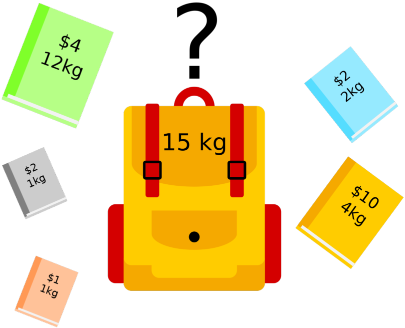

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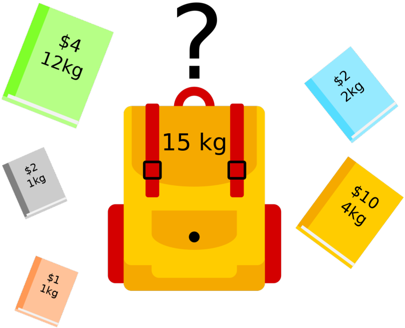

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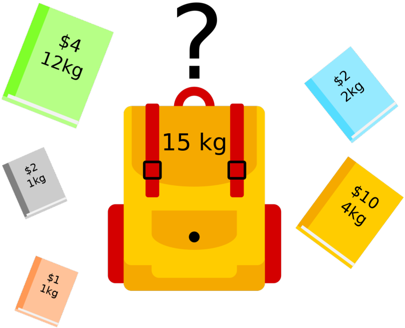

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- The dominant paradigm is to relax the second constraint (optimal surplus) as little as possible, subject to the first (DSIC) and the third (polynomial-time) constraints.
- Myerson's Lemma implies that the following goal is equivalent: design a polynomial-time and monotone allocation rule that comes as close as possible to maximizing the social surplus.

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- We now illustrate this approach by designing an allocation rule that gives at least half of the optimum social surplus in knapsack auctions.

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- The rule x_G is monotone (Exercise).

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- We show that this algorithm maximizes the surplus over all feasible solutions to the fractional knapsack problem.
    - Let $1, \ldots, k$ be the winners selected by the greedy algorithm and suppose for contradiction that there is another feasible solution that gives higher social surplus.

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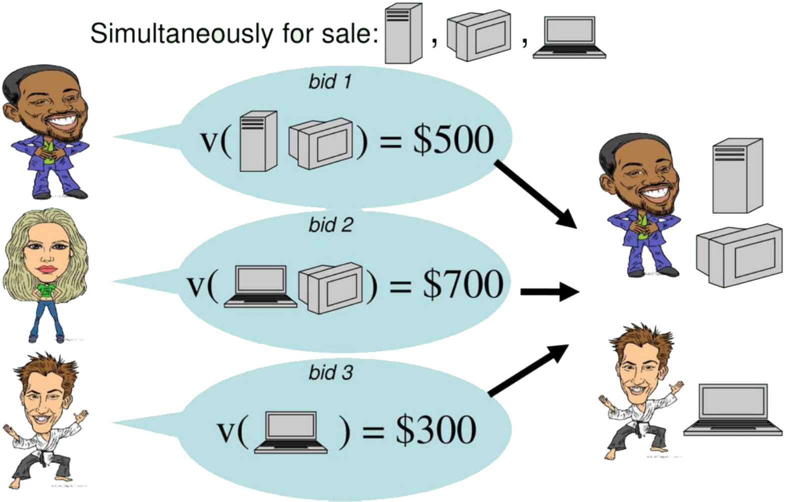

Sources: Auctions & Combinatorial auctions (Vincent Conitzer)

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VCG mechanism (Theorem 3.18)

In every multi-parameter mechanism design environment, there is a DSIC social-surplus-maximizing mechanism.

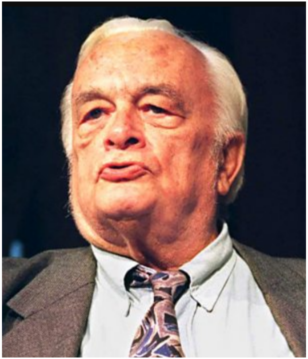

Figure: William Vickrey, Edward H. Clarke, and Theodore Groves.

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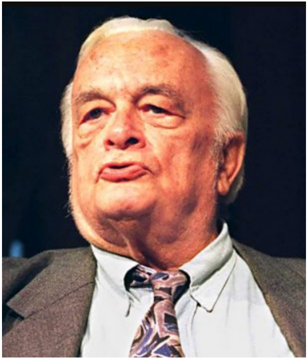

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- We postpone the proof of this result to the last lecture.

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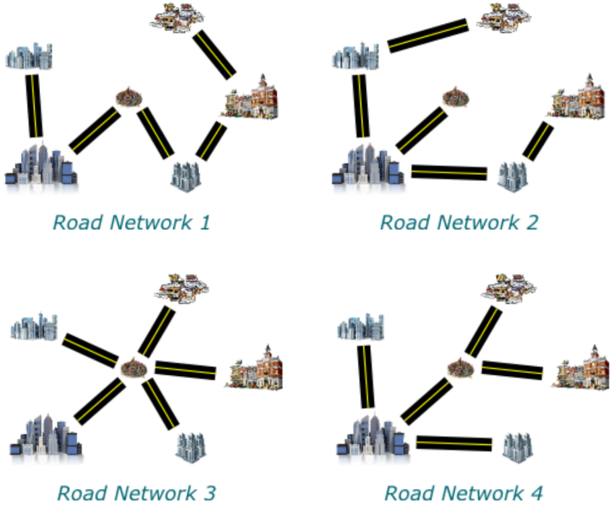

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	6 M$	14 M$	2 M$	**16 M$**
	5 M$	8 M$	4 M$	**12 M$**
	2 M$	1 M$	**20 M$**	4 M$
	4 M$	**6 M$**	3 M$	5 M$
	1 M$	1 M$	**6 M$**	2 M$
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Total (social welfare)	19 M$	32 M$	37 M$	**42 M$**

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Total (social welfare)	19 M$	32 M$	37 M$	**42 M$**

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- Cities pay their negative externalities on the collectivity.

VCG auction example

	Road Network 1	Road Network 2	Road Network 3	Road Network 4
	6 M$	14 M$	2 M$	16 M$
	5 M$	8 M$	4 M$	12 M$
	2 M$	1 M$	20 M$	4 M$
	4 M$	6 M$	3 M$	5 M$
	1 M$	1 M$	6 M$	2 M$
	1 M$	2 M$	2 M$	3 M$
Total (social welfare)	19 M$	32 M$	37 M$	42 M$

Sources: https://www.science4all.org/article/auction-design/

- Cities pay their negative externalities on the collectivity. Other cities would be happier without the biggest city (NYC, say). How much happier they would be is exactly what NYC must pay.

VCG auction example

	Road Network 1	Road Network 2	Road Network 3	Road Network 4
	6 M$	14 M$	2 M$	**16 M$**
	5 M$	8 M$	4 M$	**12 M$**
	2 M$	1 M$	**20 M$**	4 M$
	4 M$	**6 M$**	3 M$	5 M$
	1 M$	1 M$	**6 M$**	2 M$
	1 M$	2 M$	2 M$	**3 M$**
Total (social welfare)	19 M$	32 M$	37 M$	**42 M$**

Sources: https://www.science4all.org/article/auction-design/

VCG auction example

	Road Network 1	Road Network 2	Road Network 3	Road Network 4
	6 M$	14 M$	2 M$	**16 M$**
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	2 M$	1 M$	**20 M$**	4 M$
	4 M$	**6 M$**	3 M$	5 M$
	1 M$	1 M$	**6 M$**	2 M$
	1 M$	2 M$	2 M$	**3 M$**
Total (social welfare)	19 M$	32 M$	37 M$	**42 M$**

Sources: https://www.science4all.org/article/auction-design/

- If NYC was not there, then road network number 3 (RN3) would have been chosen, as opposed to RN4. The value of RN3 for the other cities would be 35 M$, as opposed to the 26 M$ of RN4. Therefore, the negative externality of NYC is $35 - 26 = 9$ M$.

Source: https://www.kindpng.com/

Thank you for your attention and merry Christmas!