Algorithmic game theory

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11th lecture

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- Is there awesome mechanism (x, p) for single-parameter environments?
- We started by looking for DSIC mechanisms and saw a powerful tool for designing them.

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Myerson's lemma

In a single-parameter environment, the following three claims hold.

(a) An allocation rule is implementable if and only if it is monotone.

(b) If an allocation rule x is monotone, then there exists a unique payment rule p such that (x, p) is DSIC (assuming $b_i = 0$ implies $p_i(b) = 0$).

(c) For every i, the payment rule p is given by the following explicit formula

$$p_i(b_i; b_{-i}) = \int_0^{b_i} z \cdot \frac{\mathrm{d}}{\mathrm{d}z} x_i(z; b_{-i}) \, \mathrm{d}z.$$

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- We saw two applications: single-item and sponsored-search auctions.

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- Today, we try to maximize the revenue.

Revenue maximizing auctions

Source: Reprofoto

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- How to maximize the revenue $\sum_{i=1}^{n} p_i(b)$?

Source: https://merger.com/recurring-revenue/

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- The situation then becomes more complicated, but we will see some nice results today.

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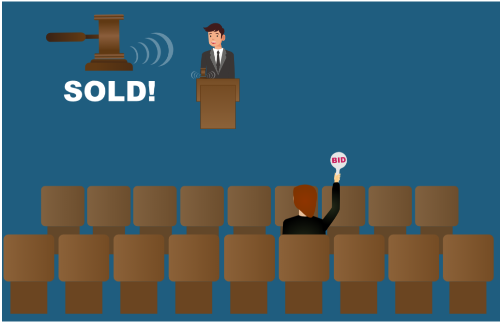
- Consider single-item auction with 1 bidder with valuation v.



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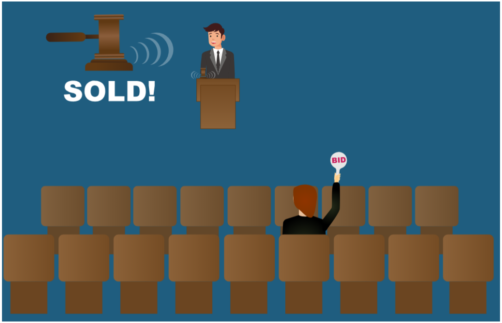

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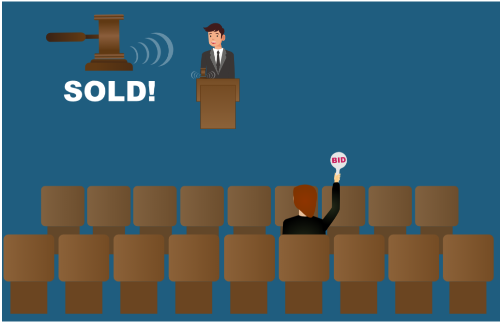

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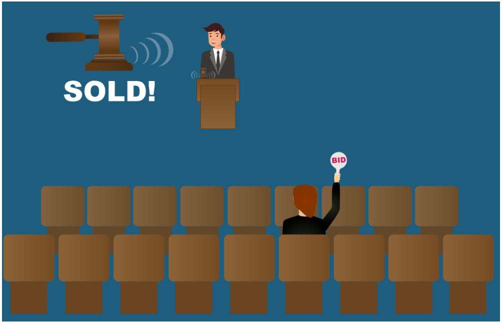

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- Maximizing the social surplus is trivial by putting $r = 0$.
- However, when maximizing the revenue, it is not clear how we should set r, since we do not know the valuation v.

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  - If F is a probability distribution drawing some value v, then $F(z)$ is the probability that v has value at most z.
  - If F is a probability distribution with density f and with support $[0, v_{max}]$, then

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) \text{ and } F(x) = \int_0^x f(z)\,\mathrm{d}z.$$

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  - For a random variable X, the expected value of X is

$$\mathbb{E}_{z \sim F}[X(z)] = \int_0^{v_{max}} X(z) \cdot f(z)\,\mathrm{d}z.$$

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where the expectation is taken with respect to the given distribution $F_1 \times \cdots \times F_n$ over the valuations $(v_1, \ldots, v_n)$.

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- For example, if F_1 is the uniform distribution on $[0, 1]$, where $F_1(x) = x$ for every $x \in [0, 1]$, then we achieve the maximum revenue by choosing $r = 1/2$.
- The posted price r that makes the expected revenue as high as possible is called the monopoly price.

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- Besides Vickrey's auctions, there are other DSIC auctions that perform better with respect to the expected revenue (Exercise).
- Today, we describe such optimal auctions in more general settings using Myerson's lemma.

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- Example: consider a bidder i with valuation v_i drawn from the uniform distribution on $[0, 1]$.

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- In other words, for every DSIC auction, the expected revenue is equal to the expected virtual social surplus $\sum_{i=1}^{n} \varphi_i(v_i) \cdot x_i(v)$. So when maximizing the expected revenue, we end up with maximizing a similar value as before.
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- A coarse intuition behind the virtual valuations: the first term v_i in this expression can be thought of as the maximum revenue obtainable from bidder i and the second term as the inevitable revenue loss caused by not knowing v_i in advance.

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- The first term is zero, as it equals to 0 for $z = 0$ and also for $z = v_{max}$, as v_{max} is a finite number.

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$$\mathop{\mathbb{E}}_{v \sim F} \left[\sum_{i=1}^n p_i(v) \right] = \sum_{i=1}^n \mathop{\mathbb{E}}_{v \sim F} [p_i(v)] = \sum_{i=1}^n \mathop{\mathbb{E}}_{v \sim F} [\varphi_i(v_i) x_i(v)] = \mathop{\mathbb{E}}_{v \sim F} \left[\sum_{i=1}^n \varphi_i(v_i) x_i(v) \right]$$

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- In a Vickrey auction with reserve price r, the item is given to the highest bidder, unless all bids are less than r, in which case no one gets the item. The winner (if any) is charged the second-highest bid or r, whichever is larger.

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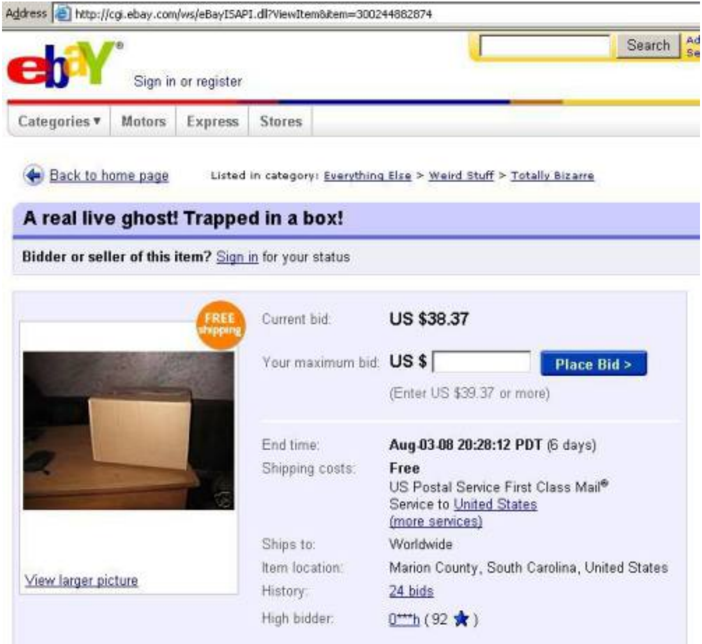

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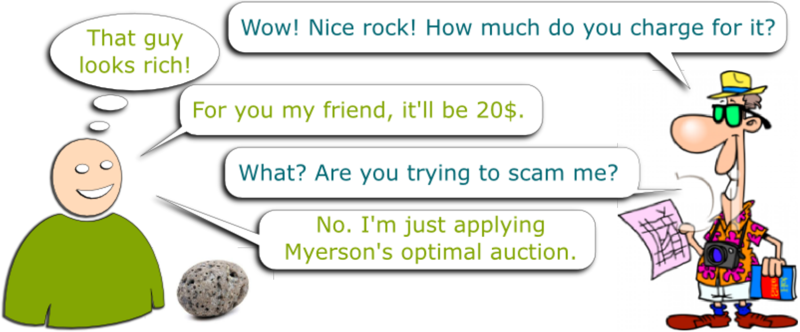
- Very roughly, if the seller believes that bidders have high valuations, he should set a high reserve price accordingly.



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Optimal auctions more generally

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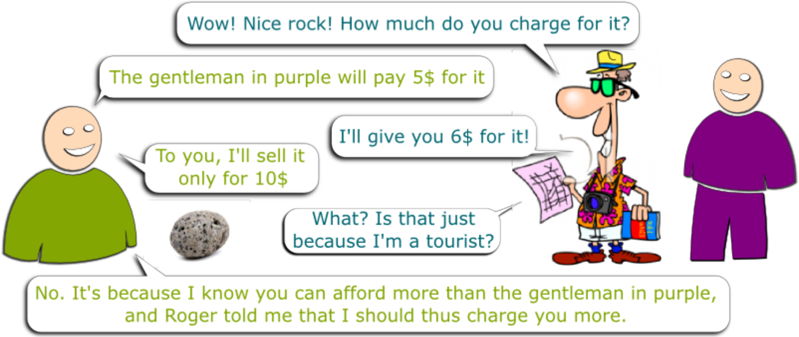
- There are also optimal DISC auctions even if we relax the conditions by not insisting on $F_1, \ldots, F_n$ being identical. However, such optimal auctions can get weird, and they does not generally resemble any auctions used in practice (Exercise).



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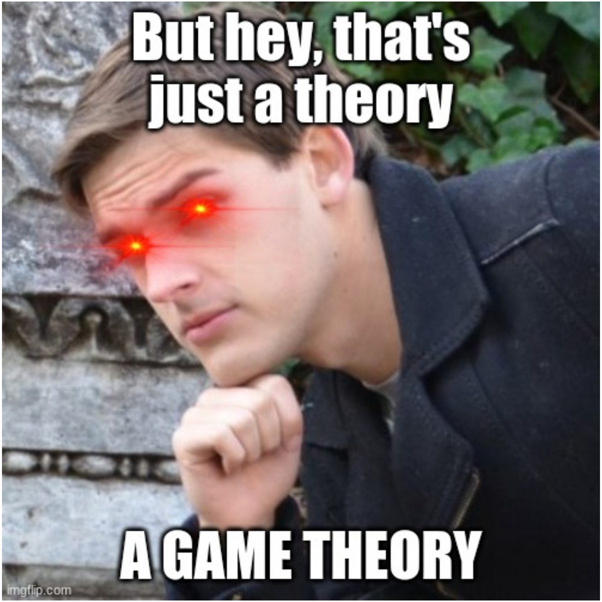

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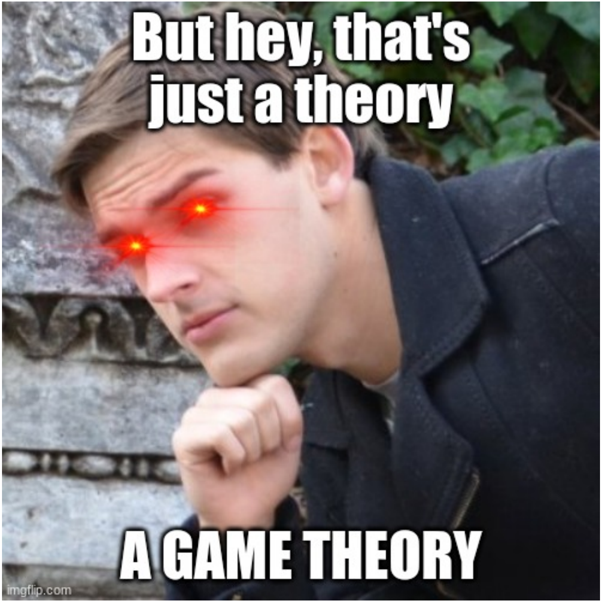

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Thank you for your attention.