Algorithmic game theory

Martin Balko

10th lecture

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Mechanism design basics

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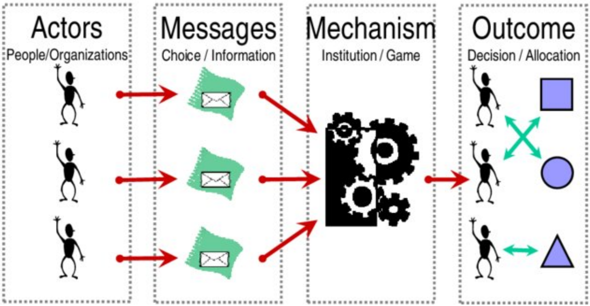

Source: Innovations in Defense Acquisition: Asymmetric Information and Incentive Contract Design

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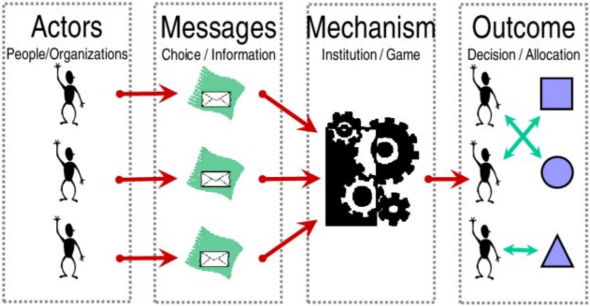

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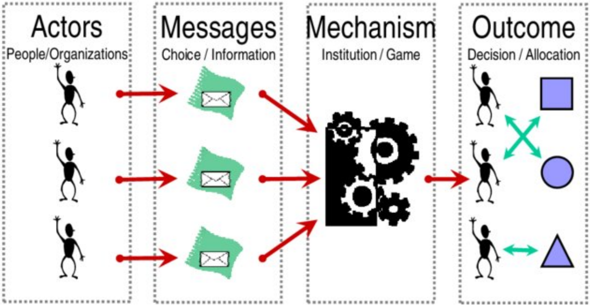

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- We start with single item auctions.
- We then extend these desired properties to a more general setting of single-parameter environments using so-called Myerson's lemma.

Single item auctions

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Source: https://www.widewalls.ch

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- To do so, we need to appropriately implement the rules for the seller how to decide the winner and the selling price.

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  - Computational efficiency: The auction can be implemented in polynomial time.

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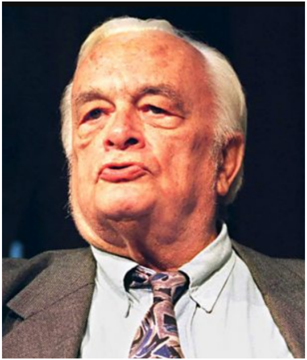

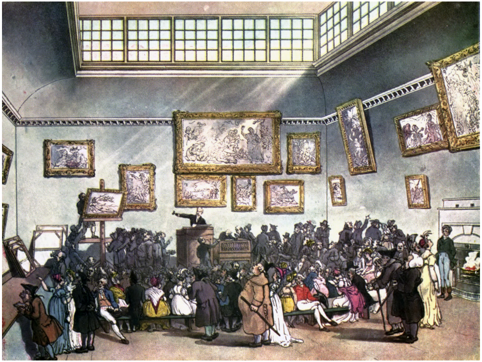

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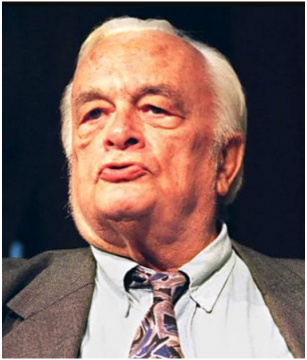

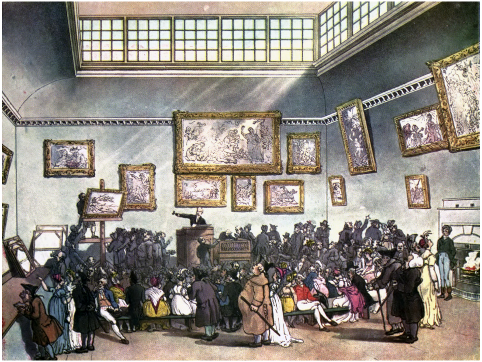

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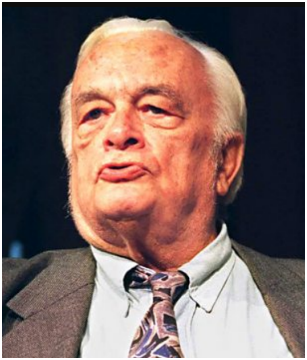

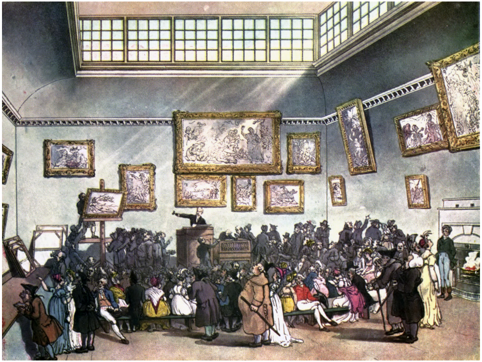

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- The utility $u_i(b)$ of bidder i is $u_i(b) = v_i \cdot x_i(b) - p_i(b)$.

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- We now illustrate single-parameter environments with a few specific examples.

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  We have two assumptions: first, the more the slot is on the top, the higher the probability α_j that the slot is clicked on, and, second, the click-through rates do not depend on the occupant of the slot.

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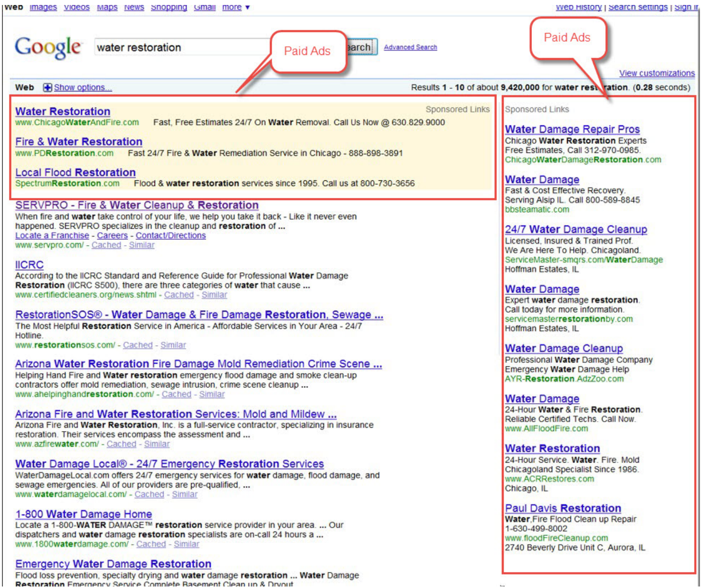

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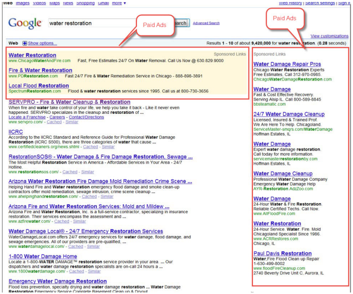

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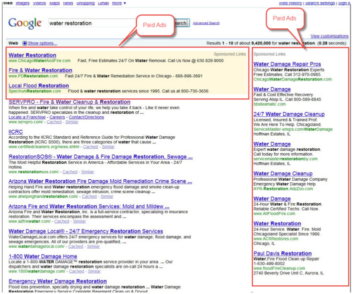

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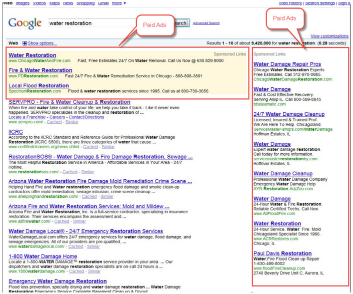

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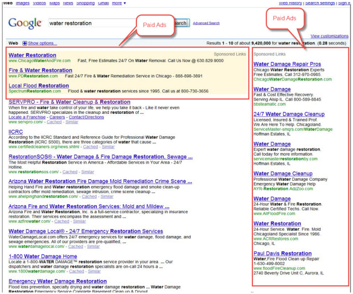

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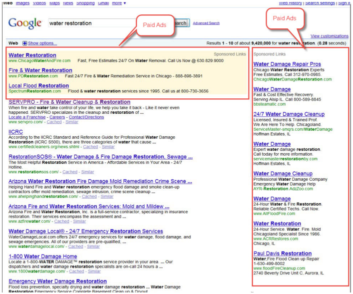

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Figure: Roger Myerson (born 1951) receiving a Nobel prize in economics.

Sources: https://en.wikipedia.org and https://twitter.com

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- We will see the proof next week, now we show some applications.

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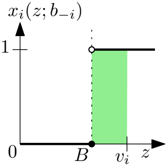

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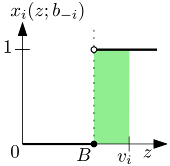

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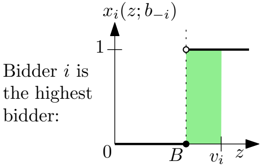

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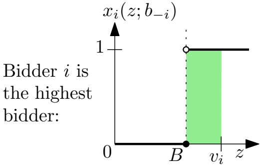

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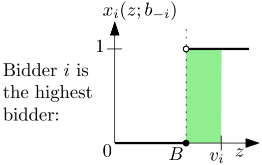

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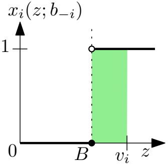

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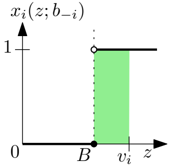

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- In general, for ith highest bidder, Myerson's lemma gives the payment formula (for $\alpha_{k+1} = 0$)

$$p_i(b) = \sum_{j=i}^{k} b_{j+1}(\alpha_j - \alpha_{j+1}).$$

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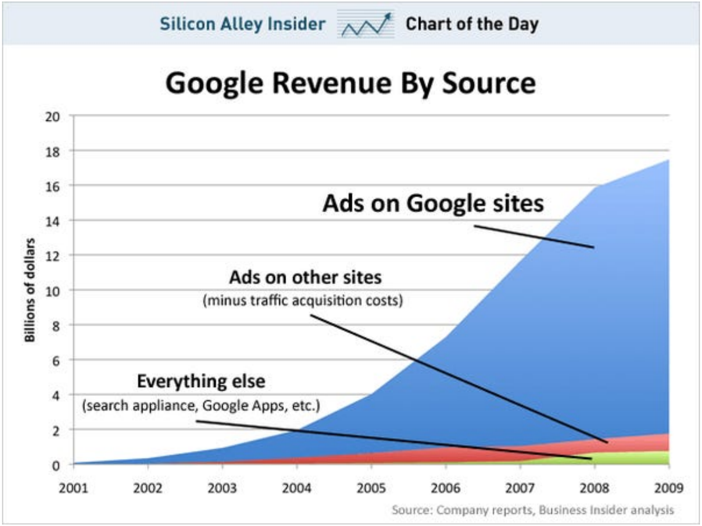

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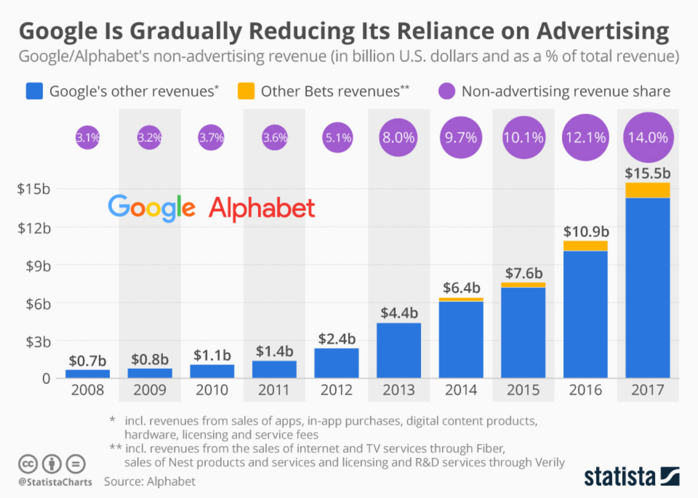

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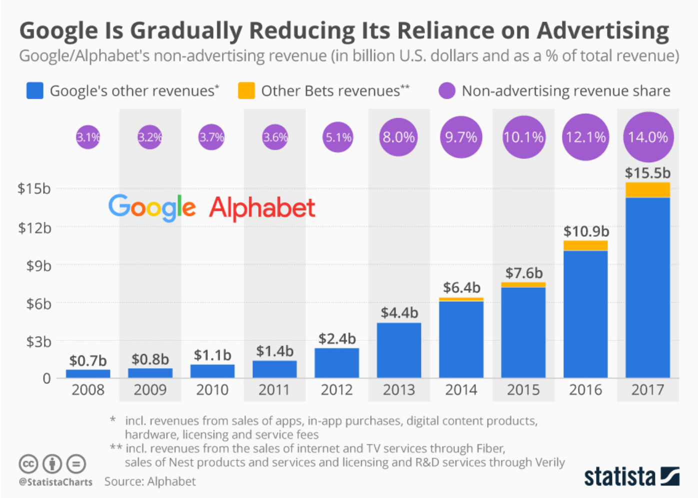

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