Algorithmic game theory – Tutorial 9*

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1 Motivation

The main goal of mechanism design is to design rules of the game so that strategic behavior by participants leads to a desirable outcome. This is not always easy and bad mechanisms can lead to undesirable situations.

Exercise 1. *Consider the following auction. The auctioneer offers 50 CZK to the highest bidder. Bidding starts at 10 CZK and increases incrementally by 5 CZK each time. The highest bidder wins the 50 CZK, but both the highest and the second-highest bidders must pay their bids.*

Exercise 2. *The mayor wants to motivate bus drivers and decides to give them a bonus, which equals 10% of the cost of the bus tickets they sell. How does this change the bus traffic?*

Exercise 3. *During the occupation of India, the British soldiers were having problems with over-population of cobras in Delhi. To get the situation under control, they offered a small reward to locals for each killed cobra snake. Can you guess what happened?*

2 Mechanism design basics

In a *single-parameter environment*, there are n bidders, each bidding for a certain goods. Each bidder i has a private *valuation* v_i and there is a *feasible set* $X \subseteq \mathbb{R}^n$ corresponding to feasible outcomes. The sealed-bid auction in this environment then proceeds in three steps.

(a) Collect bids $b = (b_1, \dots, b_n)$, where b_i is the bid of bidder i.

(b) *Allocation rule*: Choose a feasible outcome (allocation) $x = x(b)$ from the feasible set X as a function of the bids b.

(c) *Payment rule*: Choose payments $p(b) = (p_1(b), \dots, p_n(b)) \in \mathbb{R}^n$ as a function of the bids b.

The pair (x, p) then forms a *(direct) mechanism*. The *utility* $u_i(b)$ of bidder i is defined as $u_i(b) = v_i \cdot x_i(b) - p_i(b)$.

An auction is *dominant-strategy incentive-compatible (DSIC)* if it satisfies the following two properties. Every bidder has a dominant strategy: *bid truthfully*, that is, set his bid b_i to his private valuation v_i. Moreover, the utility of every truth-telling bidder is guaranteed to be non-negative. An example of DSIC auction is *Vickrey's auction* where the highest bidder is the winner and he has to pay the second largest bid.

An allocation rule x is *implementable* if there is a payment rule p such that the mechanism (x, p) is DSIC. An allocation rule x is *monotone* if, for every bidder i and all bids b_{-i} of the other bidders, the allocation $x_i(z; b_{-i})$ to i is nondecreasing in his bid z.

Theorem 1 (Myerson's lemma). *In a single-parameter environment, the following three claims hold.*

(a) *An allocation rule is implementable if and only if it is monotone.*

(b) *If an allocation rule x is monotone, then there exists a unique payment rule p such that the mechanism (x, p) is DSIC (assuming that $b_i = 0$ implies $p_i(b) = 0$).*

(c) *The payment rule p is given by the following explicit formula*

$$p_i(b_i; b_{-i}) = \int_0^{b_i} z \cdot \frac{\mathrm{d}}{\mathrm{d}z} x_i(z; b_{-i}) \, \mathrm{d}z$$

for every $i \in \{1, \dots, n\}$.

*Information about the course can be found at http://kam.mff.cuni.cz/~balko/

Exercise 4. *Consider a single-item auction with at least three bidders. Prove that selling the item to the highest bidder at a price equal to the third-highest bid, yields an auction that is not DSIC.*

Exercise 5. *Consider single-item auction, where the winner is the bidder with the highest bid and pays the price corresponding to the second highest bid with 10% discount. For example, given bids* $b = (11, 7, 10)$*, the winner is bidder 1 and pays* $10 \cdot 0.9 = 9$*. Can you find a dominant strategy in this auction? Is it truthtelling?*

Exercise 6. *Consider the following single-item auction with prioritized eligibility. The seller assigns a publicly known priority level* p_i *to each bidder* i *(for example, based on loyalty or seniority). Only bidders with priority* $p = \max_j p_j$ *are eligible to win and the item is allocated to the highest eligible bidder. The winner pays the maximum of the second-highest bid among all eligible bidders. Is this auction DSIC? Is it awesome?*

Exercise 7. *Use Myerson's Lemma to prove that the Vickrey auction is the unique single-item auction that is DSIC, always awards the good to the highest bidder, and charges the other bidders 0.*

Exercise 8. *Assume there are* k *identical items and* $n > k$ *bidders. Also assume that each bidder can receive at most one item. What is the analog of the second-price auction? Is it the* j*th highest bidder paying the* $(j + 1)$*st highest bid for* $j \leq k$*? Prove that your auction is DSIC.*