# Algorithmic game theory – Tutorial 4*

November 4th 2024

## 1 The Lemke–Howson algorithm

The *best response polyhedron* for player 1 in $G$ is a polyhedron $\overline{P}$ is defined as

$$\overline{P} = \{(x, v) \in \mathbb{R}^m \times \mathbb{R} \colon x \geq \mathbf{0}, \mathbf{1}^\top x = 1, N^\top x \leq \mathbf{1}v\}.$$

Similarly, we define the best response polyhedron for player 2 in $G$ as

$$\overline{Q} = \{(y, u) \in \mathbb{R}^n \times \mathbb{R} \colon y \geq \mathbf{0}, \mathbf{1}^\top y = 1, My \leq \mathbf{1}u\}.$$

A point $(x, v)$ of $\overline{P}$ has *label* $i \in A_1 \cup A_2$ if either $i \in A_1$ and $x_i = 0$ or if $i \in A_2$ and $(N^\top)_i x = v$, where $(N^\top)_i$ is the $i$th row of $N^\top$. A point $(y, u)$ has a label $i \in A_1 \cup A_2$ if either $i \in A_1$ and $(M)_i y = u$, where $(M)_i$ is the $i$th row of $M$, or if $i \in A_2$ and $y_i = 0$.

Assume that the matrices $M$ and $N^\top$ are non-negative and have no zero column. The *best response polytope* for player 1 in $G$ is a polytope $P$ defined as

$$P = \{x \in \mathbb{R}^m \colon x \geq \mathbf{0}, N^\top x \leq \mathbf{1}\}.$$

Similarly, the best response polytope for player 2 in $G$ is a polytope $Q$ defined as

$$Q = \{y \in \mathbb{R}^n \colon y \geq \mathbf{0}, My \leq \mathbf{1}\}.$$

The labels in $P$ and $Q$ are defined analogously as in $\overline{P}$ and $\overline{Q}$.

Nash equilibria in a non-degenerate game correspond to completely labeled pairs of vertices from $P \times Q \setminus \{(\mathbf{0}, \mathbf{0})\}$.

**Exercise 1.** *Draw the best response polyhedron and normalized best response polytope for the Game of Chicken. Then, find all completely labeled pairs of vertices that correspond to Nash equilibria.*

| | Turn (3) | Go straight (4) |
|---|---|---|
| Turn (1) | $(10, 10)$ | $(9, 11)$ |
| Go straight (2) | $(11, 9)$ | $(0, 0)$ |

Table 1: The Game of Chicken.

**Exercise 2.** *Use the Lemke–Howson algorithm and compute a Nash equilibrium of the following bimatrix game*

$$M = \begin{pmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{pmatrix} \quad a \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{pmatrix}.$$

*Start the algorithm by choosing the label* 2.

A *configuration graph* with the vertex set formed by those pairs $(x, y)$ of vertices from $P \times Q$ that are *$k$-almost completely labeled*, meaning that every label from $A_1 \cup A_2 \setminus \{k\}$ appears as a label of $x$ or of $y$. Two vertices $(x, y)$ and $(x', y')$ of the configuration graph are connected by an edge if either $x = x'$ and $yy'$ is an edge of $Q$ or if $xx'$ is an edge of $P$ and $y = y'$.

**Exercise 3.** *Show that the Lemke–Howson algorithm does not terminate in a vertex of the form $(x, \mathbf{0})$ or $(\mathbf{0}, y)$ in the configuration graph.*

*Information about the course can be found at http://kam.mff.cuni.cz/~balko/