Algorithmic game theory – Tutorial 3*

October 21st 2024

1 Bimatrix games

A *bimatrix game* is a normal-form game of 2 players. A bimatrix game is *non-degenerate* of every player has at most k pure best responses to every strategy with support of size k. A *zero-sum bimatrix game* is a game where the utility of one player equals the loss of the other one. For a bimatrix game $G = (\{1,2\}, A, u)$ with $A_1 = \{1,\dots,m\}$ and $A_2 = \{1,\dots,n\}$, we use the payoff matrices M and N where $(M)_{i,j} = u_1(i,j)$ and $(N)_{i,j} = u_2(i,j)$ for all $i \in A_1$ and $j \in A_2$.

The following algorithm for computing NE in non-degenerate games was shown at the lecture.

Algorithm 1.1: SUPPORT ENUMERATION(G)

Input: A non-degenerate game G.
Output: All Nash equilibria of G.
for every $k \in \{1,\dots,\min\{m,n\}\}$ and a pair of supports (I,J) of size k
- solve the system of equations $\sum_{i\in I}(N^\top)_{j,i}x_i = v, \sum_{j\in J}(M)_{i,j}y_j = u$,
- for all $i \in I, j \in J$ and $\sum_{i\in I} x_i = 1, \sum_{j\in J} y_j = 1$
- if $x, y \geq \mathbf{0}$ and $u = \max\{(M)_i y : i \in A_1\}, v = \max\{(N^\top)_j x : j \in A_2\}$,
- return (x,y) as Nash equilibrium

Exercise 1. *Use the Support enumeration algorithm to find a Nash equilibrium of the Game of chicken with supports of size 2.*

	Turn (1)	Go straight (2)
Turn (1)	$(0,0)$	$(-1,1)$
Go straigth (2)	$(1,-1)$	$(-10,-10)$

Table 1: The Game of chicken.

Exercise 2. *Decide whether the Game for Gotham's soul is degenerate and find all Nash equilibria of this game. How is the set of equilibria different from previously computed examples?*

	Cooperate (1)	Detonate (2)
Cooperate (1)	$(0,0)$	$(0,1)$
Detonate (2)	$(1,0)$	$(0,0)$

Table 2: The Game for Gotham's soul.

Exercise 3. *Decide which of these two payoff matrices determines a degenerate game.*

$$(a)\ M = \begin{pmatrix} 0 & 4 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 2 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix}.$$

*Informace o cvičení naleznete na http://kam.mff.cuni.cz/~balko/

(b) $M = \begin{pmatrix} 0 & 4 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 2 \end{pmatrix}$ *and* $N = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix}$.

Exercise 4. *Prove that the following linear programs from the proof of the Minimax theorem are dual to each other.*

(a) *For a matrix* $M \in \mathbb{R}^{m\times n}$,

	Program P	Program D
Variables	$y_1, \ldots, y_n$	x_0
Objective function	$\min x^\top M y$	$\max x_0$
Constraints	$\sum_{j=1}^{n} y_j = 1$,	$\mathbf{1}x_0 \le M^\top x$.
	$y_1, \ldots, y_n \ge 0$.	

(b) *For a matrix* $M \in \mathbb{R}^{m\times n}$,

	Program P'	Program D'
Variables	$y_0, y_1, \ldots, y_n$	$x_0, x_1, \ldots, x_m$
Objective function	$\min y_0$	$\max x_0$
Constraints	$\mathbf{1}y_0 - My \ge \mathbf{0}$,	$\mathbf{1}x_0 - M^\top x \le \mathbf{0}$,
	$\sum_{j=1}^{n} y_j = 1$,	$\sum_{i=1}^{m} x_i = 1$,
	$y_1, \ldots, y_n \ge 0$.	$x_1, \ldots, x_m \ge 0$.

You can use the following recipe for duality.

	Primal	Dual
Variables	$\mathbf{x} = (x_1, \ldots, x_m)$	$\mathbf{y} = (y_1, \ldots, y_n)$
Constraint matrix	$A \in \mathbb{R}^{n\times m}$	$A^\top \in \mathbb{R}^{m\times n}$
Right-hand side	$\mathbf{b} \in \mathbb{R}^n$	$\mathbf{c} \in \mathbb{R}^m$
Objective function	$\max \mathbf{c}^\top \mathbf{x}$	$\min \mathbf{b}^\top \mathbf{y}$
Constraints	ith constraint has $\le$	$y_i \ge 0$
	$\ge$	$y_i \le 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \ge 0$	jth contraints has $\ge$
	$x_j \le 0$	$\le$
	$x_j \in \mathbb{R}$	$=$

Exercise 5 (*)**.** *Prove that if a bimatrix game is non-degenerate, then the system of equations in the Support enumeration algorithm has a unique solution with* $\mathbf{x}, \mathbf{y} > \mathbf{0}$, $u = \max\{(M)_i y : i \in A_1\}$ *and* $v = \max\{(N^\top)_j x : j \in A_2\}$.

Hint: Prove that if there are more solutions, then we can reduce the support.

Exercise 6. *Prove that if* (s_1, s_2) *and* (s_1', s_2') *are mixed Nash equilibria of a two-player zero-sum game, then so are* (s_1, s_2') *and* (s_1', s_2).