Algorithmic game theory – Tutorial 2*

October 7th 2024

1 Nash equilibria

A *normal-form game* is a triple (P, A, u), where P is a finite set of n *players*, $A = A_1 \times \cdots \times A_n$ is a set of *action profiles*, where A_i is a set of *actions* available to player i, and $u = (u_1, \ldots, u_n)$ is an n-tuple, where each $u_i \colon A \to \mathbb{R}$ is the *utility function* for player i.

The set of *pure strategies* of player i is the set A_i of available actions for i. The set S_i of *mixed strategies* of player i is the set of all probability distributions on A_i. The *expected payoff* for player i of the mixed-strategy profile $s = (s_1, \ldots, s_n)$ is

$$u_i(s) = \sum_{a=(a_1,\ldots,a_n)\in A} u_i(a) \prod_{j=1}^{n} s_j(a_j).$$

We use the notation $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ and, for a strategy $s'_i \in S_i$ of player i, we use $u_i(s'_i; s_{-i})$ to denote the number $u_i(s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$.

The *best response* of player i to the strategy profile s_{-i} is a mixed strategy s^*_i such that $u_i(s^*_i; s_{-i}) \geq u_i(s'_i; s_{-i})$ for each strategy $s'_i \in S_i$ of i. A *Nash equilibrium* in G is a strategy profile $(s_1, \ldots, s_n)$ such that s_i is a best response of player i to s_{-i} for every $i \in P$.

Observation 1 (Best response condition)**.** *In a normal-form game $G = (P, A, u)$ of n players, for every player $i \in P$, a mixed strategy s_i is a best response to s_{-i} if and only if all pure strategies in the support of s_i are best responses to s_{-i}.*

Exercise 1. *Verify that the expected payoff of a mixed strategy in a normal-form game $G = (P, A, u)$ of n players is linear. That is, prove that $u_i(s) = \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$ for every player $i \in P$ and every mixed-strategy profile $s = (s_1, \ldots, s_n)$.*

Exercise 2. *Compute mixed Nash equilibria in the following games:*

(a) *Prisoner's dilemma,*

	Testify	Remain silent
Testify	(-2,-2)	(0,-3)
Remain silent	(-3,0)	(-1,-1)

Table 1: A normal form of the game Prisoner's dilemma.

(b) *Rock-Paper-Scissors.*

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Table 2: A normal form of the game Rock-paper-scissors.

*Information about the course can be found at http://kam.mff.cuni.cz/~balko/

and formally show that no other Nash equilibria exist in these games.

Exercise 3 (Iterated dominance equilibrium)**.** *Let $G = (P, A, u)$ be a normal-form game of n players. For player i, we say that a strategy $s_i \in S_i$ is* strictly dominated *by a strategy $s'_i \in S_i$ if, for every $s_{-i} \in S_{-i}$, we have $u_i(s_i; s_{-i}) < u_i(s'_i; s_{-i})$. Consider the following iterated process that will help us find Nash equilibria in some games.*

Set $A_i^0 = A_i$ and $S_i^0 = S_i$ for every player $i \in P$. For $t \geq 1$ and $i \in P$, let A_i^t be the set of pure strategies from A_i^{t-1} that are not strictly dominated by a strategy from S_i^{t-1} and let S_i^t be the set of mixed strategies with support contained in A_i^t. Let T be the first step, when the sets A_i^T and S_i^T are no longer shrinking for any $i \in P$. If each player $i \in P$ is left with one strategy $a_i \in A_i^T$, we call $a_1 \times \cdots \times a_n$ an iterated dominance equilibrium *of G.*

(a) *Show that every iterated dominance equilibrium is a Nash equilibrium.*

(b) *Find an example of a two-person game in normal form game with a pure Nash equilibrium that is not iterated dominance equilibrium.*

Exercise 4. *Use iterated elimination of strictly dominated strategies (introduced in Exercise 3) to find the unique Nash equilibrium in the following normal-form game of 2 players (see Table 3) by first reducing the game to a 2×2 game.*

	c_1	c_2	c_3	c_4
r_1	(5, 2)	(22, 4)	(4, 9)	(7, 6)
r_2	(16, 4)	(18, 5)	(1, 10)	(10, 2)
r_3	(15, 12)	(16, 9)	(18, 10)	(11, 3)
r_4	(9, 15)	(23, 9)	(11, 5)	(5, 13)

Table 3: A game from Exercise 4.