Algorithmic game theory – Tutorial 11*

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1 Bulow–Klemperer theorem

Theorem 1 (The Bulow–Klemperer Theorem). *Let $F = F_1 = \cdots = F_n$ be a regular probability distribution and let n be a positive integer. Then the following inequality holds*

$$\mathbb{E}_{v_1,\dots,v_{n+1}\sim F}\left[\mathrm{Rev}(VA_{n+1})\right] \geq \mathbb{E}_{v_1,\dots,v_n\sim F}\left[\mathrm{Rev}(OPT_{F,n})\right], \tag{1}$$

where $\mathrm{Rev}(VA_{n+1})$ denotes the revenue of Vickrey auction VA_{n+1} with $n+1$ bidders (and no reserve) and $\mathrm{Rev}(OPT_{F,n})$ denotes the revenue of the optimal auction $OPT_{F,n}$ for F with n bidders.

Exercise 1. *Consider a single-item auction with $n \geq 2$ bidders that draw their valuations from a regular probability distribution F. Prove that the expected revenue of the Vickrey auction with no reserve is at least $\frac{n-1}{n}$-fraction of the expected revenue of the optimal auction with the same number n of bidders.*

Hint: deduce this statement from the Bulow–Klemperer theorem. When one new bidder is added, how much can the maximum-possible expected revenue increase?

2 Multi-parameter mechanism design

In *multi-parameter mechanism design*, we have the following setting:

(*a*) n strategic participants (or bidders),

(*b*) a finite set Ω of outcomes,

(*c*) each bidder i has a private valuation $v_i(\omega) \geq 0$ for every outcome $\omega \in \Omega$.

Each bidder i submits his bids $b_i(\omega)$ for each $\omega \in \Omega$ and our goal is to design a mechanism that selects an outcome $\omega \in \Omega$ so that it maximizes the *social surplus* $\sum_{i=1}^{n} v_i(\omega)$.

Theorem 2 (The Vickrey–Clarke–Groves (VCG) mechanism). *In every multi-parameter mechanism design environment, there is a DSIC social-surplus-maximizing mechanism.*

Exercise 2. *Prove that the payment rule from the proof of the VCG mechanism is always non-negative and bounded from above by $b_i(\omega^*)$. That is, show that $0 \leq p_i(b) \leq b_i(\omega^*)$ for every vector b of bids, where*

$$p_i(b) = \max_{\omega\in\Omega}\left\{\sum_{\substack{j=1\\ j\neq i}}^{n} b_j(\omega)\right\} - \sum_{\substack{j=1\\ j\neq i}}^{n} b_j(\omega^*)$$

and

$$\omega^* = \mathrm{argmax}_{\omega\in\Omega}\sum_{i=1}^{n} b_i(\omega).$$

Exercise 3. *Consider a three-item auction with two bidders 1 and 2. The three items A, B, and C are being auctioned simultaneously, and each bidder can bid on any possible subset of the items. The valuations of the bidders for each subset of the items are shown in Table 1. What are the outcomes of this VCG auction? In other words, which of the two bidders will get which item(s) and what payments will they each pay?*

*Information about the course can be found at http://kam.mff.cuni.cz/~balko/

bidder i	$v_i(\emptyset)$	$v_i(A)$	$v_i(B)$	$v_i(C)$	$v_i(AB)$	$v_i(AC)$	$v_i(BC)$	$v_i(ABC)$
$i = 1$	0	24	4	9	29	38	20	50
$i = 2$	0	15	18	11	30	34	32	47

Table 1: Valuations of the bidders from Exercise 3.

3 Knapsack auctions

In a single-parameter environment, we assume that the bidders $1, \dots, n$ are sorted in the order $<$ so that $\frac{b_1}{w_1} \geq \cdots \geq \frac{b_n}{w_n}$. Consider the following *greedy allocation rule* $x^G = (x_1^G, \dots, x_n^G) \in X$, which for given bids $b = (b_1, \dots, b_n)$ selects a subset of bidders so that $\sum_{i=1}^n x_i^G w_i \leq W$ using the following procedure.

1. Pick winners in the order $<$ until one does not fit and then halt.
2. Return either the solution from the first step or the highest bidder, whichever creates more social surplus.

Exercise 4. *(*a*) Prove that the Knapsack auction allocation rule x^G induced by the greedy* (1/2)*-approximation algorithm is monotone.*

*(*b*) Prove that it suffices to change only two coefficients α_i and β_j in the proof of Theorem 3.10 (the correctness of the* (1/2)*-approximation algorithm that is based on x^G).*