Algorithmic game theory

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4th lecture

October 31th 2022

Nash equilibria in bimatrix games

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- The best response condition: If x and y are mixed strategy vectors of players 1 and 2, respectively, then x is a best response to y if and only if for all $i \in A_1$,

$$x_i > 0 \Longrightarrow (M)_i y = \max\{(M)_k y \colon k \in A_1\}.$$

  Analogously, y is the best response to x if and only if for all $j \in A_2$,

$$y_j > 0 \Longrightarrow (N^\top)_j x = \max\{(N^\top)_k x \colon k \in A_2\}.$$

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- Today, we notice a geometric structure behind this task and show the fastest known algorithm for computing NE in bimatrix games.

Examples of polytopes

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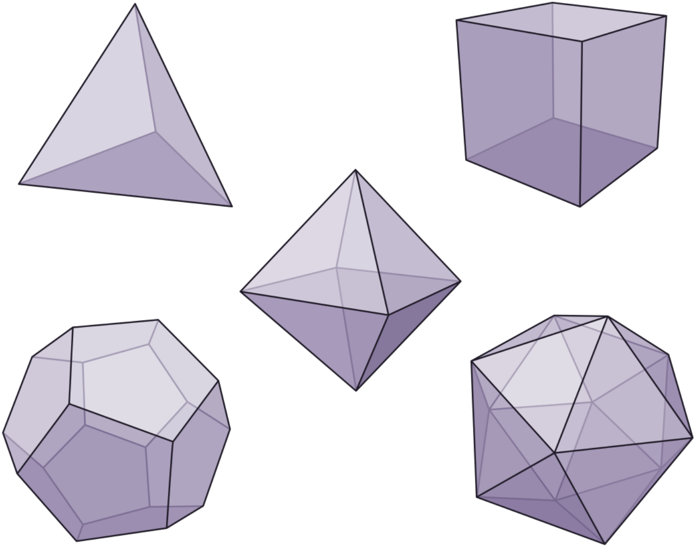

Best response polyhedra $\overline{P}$ and $\overline{Q}$ for the Battle of sexes

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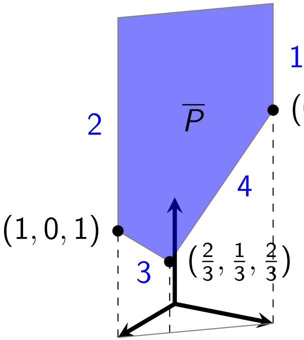

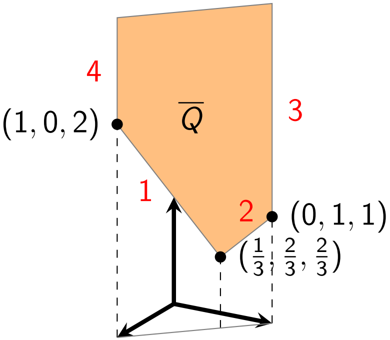

$$\overline{P} = \{(x_1, x_2, v) \in \mathbb{R}^2 \times \mathbb{R} \colon x_1, x_2 \geq \mathbf{0}, x_1 + x_2 = 1, x_1 \leq v, 2x_2 \leq v\}$$

$$\overline{Q} = \{(y_3, y_4, u) \in \mathbb{R}^2 \times \mathbb{R} \colon y_3, y_4 \geq \mathbf{0}, y_3 + y_4 = 1, 2y_3 \leq u, y_4 \leq u\}.$$

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Lemke–Howson on the Battle of sexes ($k = 3$)

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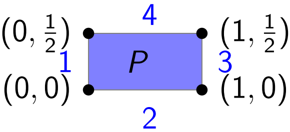

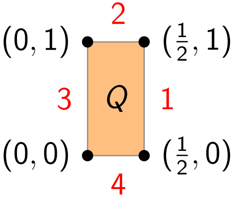

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Figure: Carlton E. Lemke (1920–2004) and J. T. Howson (?).

Source: https://oldurls.inf.ethz.ch

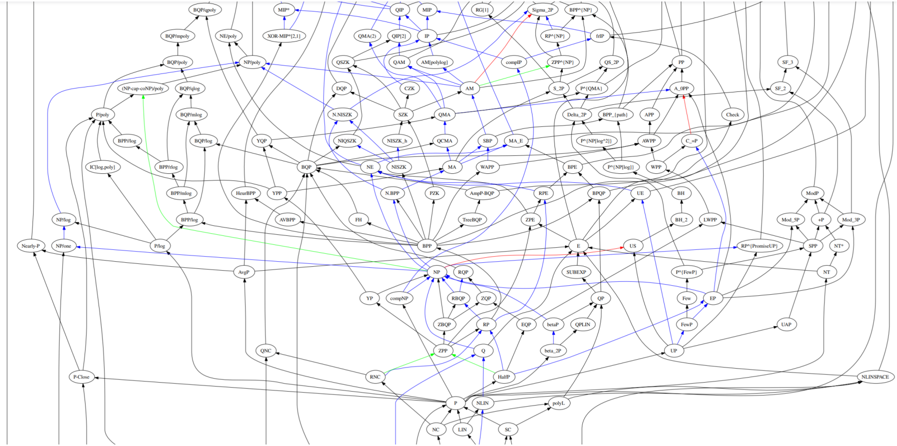
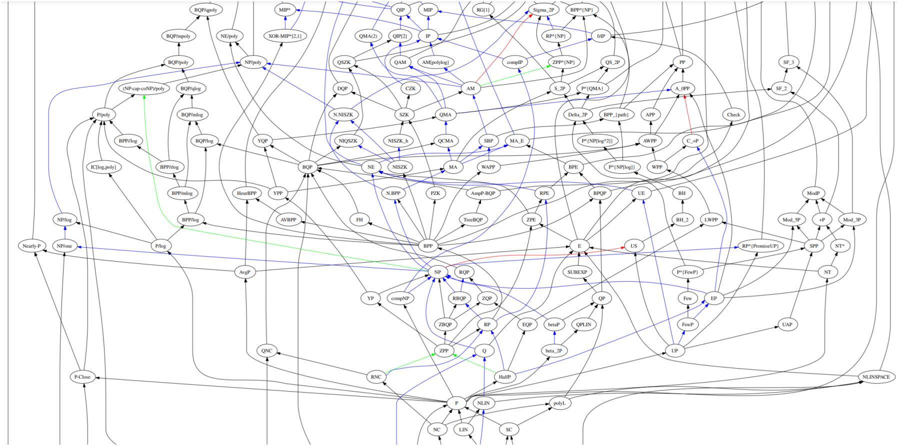


Figure: A view on the complexity classes classification.

Source: https://complexityzoo.uwaterloo.ca/Complexity_Zoo



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Thank you for your attention.