

Algorithmic game theory – Tutorial 7*

January 2, 2023

1 Revenue-maximizing auctions

We consider the *Bayesian model*, which consists of a single-parameter environment (x, p) with n bidders, where, for each bidder i , the private valuation v_i of i is drawn from a probability distribution F_i with density function f_i and with support contained in $[0, v_{max}]$. The distributions F_1, \dots, F_n are independent, but not necessarily the same. We recall that if F is a probability distribution with density f and with support $[0, v_{max}]$, then $f(z) = \frac{d}{dz}F(z)$ and $F(x) = \int_0^x f(z) dz$. Also, for a random variable X , we have $\mathbb{E}_{z \sim F}[X(z)] = \int_0^{v_{max}} X(z) \cdot f(z) dz$.

The *virtual valuation* of bidder i with valuation v_i drawn from F_i is $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$. The *virtual social surplus* is $\sum_{i=1}^n \varphi_i(v_i) \cdot x_i(v)$.

We consider only DSIC auctions.

Exercise 1. Let F be the uniform probability distribution on $[0, 1]$. Consider a single-item auction with two bidders 1 and 2 that have probability distributions $F_1 = F$ and $F_2 = F$ on their valuations. Prove that the expected revenue obtained by the Vickrey auction (with no reserve) is $1/3$.

Exercise 2. Compute the virtual valuation function of the following probability distributions and show which of these distributions are regular (meaning the virtual valuation function is strictly increasing).

- (a) The uniform distribution $F(z) = z/a$ on $[0, a]$ with $a > 0$,
- (b) The exponential distribution $F(z) = 1 - e^{-\lambda z}$ with rate $\lambda > 0$ on $[0, \infty)$,

Exercise 3. Consider a single-item auction where bidder i draws his valuation from his own regular distribution F_i , that is, the probability distributions F_1, \dots, F_n can be different but all virtual valuation functions $\varphi_1, \dots, \varphi_n$ are strictly increasing.

- (a) Give a formula for the winner's payment in an optimal auction, in terms of the bidders' virtual valuation functions φ_i . Verify that if $F_1 = \dots = F_n$ are uniform probability distributions on $[0, 1]$, then your formula yields Vickrey auction with reserve price $1/2$.
- (b) Find an example of an optimal auction in which the highest bidder does not win, even if he has a positive virtual valuation.

Hint: It suffices to consider two bidders with valuations from different uniform distributions.

We assume that the bidders $1, \dots, n$ are sorted in the order $<$ so that $\frac{b_1}{w_1} \geq \dots \geq \frac{b_n}{w_n}$. Consider the following *greedy allocation rule* $x^G = (x_1^G, \dots, x_n^G) \in X$, which for given bids $b = (b_1, \dots, b_n)$ selects a subset of bidders so that $\sum_{i=1}^n x_i^G w_i \leq W$ using the following procedure.

1. Pick winners in the order $<$ until one does not fit and then halt.
2. Return either the solution from the first step or the highest bidder, whichever creates more social surplus.

Exercise 4. Prove that the Knapsack auction allocation rule x^G induced by the greedy $(1/2)$ -approximation algorithm covered in the lecture is monotone.

*Information about the course can be found at <http://kam.mff.cuni.cz/~balko/>