Advanced Graph Algorithms Jan-Apr 2014

Lemma of Gessel Viennot

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1 Overview

In the last few lectures, we had learnt different algorithms using matrix multiplication. For example strassen's algorithm for matrix multiplication, finding max cut problem of a weighted graph, finding witness for boolean matrix multiplication, min plus product of two matrices etc etc. Today, we will learn **Gessel Viennot Lemma**. It is one that establishes a connection between lattice paths and determinants. It was first proved by Lindstrom, but its combinatorial significance was first seen by Gessel and Viennot. They establishes a connection between the **path systems** and determinant.

2 Definitions

Let $G = (V, E)$ be a directed acyclic graph, a weight function $w : E \to R$ and $\mathcal{A} = (A_1, A_2, \ldots, A_n)$, $\mathcal{B} = (B_1, B_2, \ldots, B_n)$ be two sets of vertices which are not necessarily disjoint.

Definition 1. *A* ***path system*** $\mathcal{P}$ *is given by* $\sigma \in S_n$ *and* n *paths* $P_1 : A_1 \to B_{\sigma(1)}, P_2 : A_2 \to B_{\sigma(2)}, \ldots, P_n : A_n \to B_{\sigma(n)}$. *Weight of a path system* $\mathcal{P}$ *is denoted as*

$$w(\mathcal{P}) = \prod_{i=1}^{n} w(P_i)$$

and $sign(\mathcal{P}) = sign(\sigma)$.
Weight of a path P *is given by the product of the edges in the path.*

$$w(P) = \prod_{e \in P} w(e)$$

When the path P *is from vertex* v *to vertex* v*, it does not contain any edge since the graph is directed acyclic graph, we consider* $w(P) = 1$.

Definition 2. *A path system* $\mathcal{P}$ *is said to be* ***vertex disjoint*** *if for any two paths* $P_i, P_j \in \mathcal{P}$*, there is no vertex in common.*

Definition 3. ***Path Matrix*** $M = [m_{i,j}]_{i,j=1}^{n}$ *from* $\mathcal{A}$ *to* $\mathcal{B}$ *is defined as follows.*

$$m_{i,j} = \sum_{P : A_i \to B_j} w(P)$$

3 Interpretation of a square matrix as a path matrix of a bipartite graph

Let M be a square matrix. Construct a directed bipartite graph $G = (\mathcal{A}, \mathcal{B}, E)$ with $\mathcal{A} = (A_1, \ldots, A_n)$ that corresponds to the rows of M and $\mathcal{B} = (B_1, B_2, \ldots, B_n)$ that corresponds to columns of M. $E = \{(A_i, B_j) | i, j \in [n]\}$ and $w(A_i, B_j) = m_{i,j}$.

In this graph, consider A_i and B_j, there is only one path from that goes from A_i to B_j which is just the edge (A_i, B_j) and $w(P) = w(A_i, B_j) = m_{i,j}$.

Now, lets consider the permutation description of determinant.

$$det(M) = \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}$$

Define $\psi : AllPathSystems \to S_n$ as $\psi(\mathcal{P}) = \sigma$ where $\mathcal{P}$ is defined as σ and n paths.

Lemma 4. *ψ is a bijection.*

Proof. Consider any path system $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{B}$. For that $\mathcal{P}$ there is unique $\sigma \in S_n$.

Let $\psi(\mathcal{P}) = \sigma_1$ and $\psi(\mathcal{Q}) = \sigma_2$ and $\sigma_1 = \sigma_2$, then $\forall i \in [n] : \sigma_1(i) = \sigma_2(i)$. Then, let $\mathcal{P}$ is given by σ_1 and n unique paths. $\mathcal{Q}$ also gives σ_2 and n unique paths. Now, $\sigma_1 = \sigma_2$. Now, fix any $\sigma \in S_n$. Then for every $i \in [n]$, there is only one path from A_i to $B_{\sigma(i)}$ in G. Therefore, $\mathcal{P} = \mathcal{Q}$ since $\sigma_1 = \sigma_2$. Therefore, ψ is injective.

Take any $\sigma \in S_n$. By the construction of the graph, clearly, for this σ, there is a path system $\mathcal{P}_\sigma$ with n paths $P_1 : A_1 \to B_{\sigma(1)}, P_2 : A_2 \to B_{\sigma(2)}, \ldots, P_n : A_n \to B_{\sigma(n)}$ where weights the paths are $w(P_k) = w(A_k, B_{\sigma(k)}) = m_{k,\sigma(k)}$. Therefore, ψ is surjective.

Therefore, ψ is a bijection. □

Similarly consider any $\sigma \in S_n$, then there is a unique path system $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{B}$. $w(\mathcal{P}) = \prod_{i=1}^{n} w(A_i, B_{\sigma(i)}) = \prod_{i=1}^{n} m_{i,j}$

$det(M) = \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)} = \sum_{\mathcal{P}} sign(\mathcal{P}) \prod_{i=1}^{n} m_{i,j} = \sum_{\mathcal{P}} sign(\mathcal{P}) w(\mathcal{P})$.

This summation is over all path systems. In this way, we interpret a given square matrix as path matrix of a directed bipartite graph.

4 Gessel Viennot Lemma

Gessel Viennot Lemma generalizes this interpretation from bipartite graph to general directed acyclic graph. Statement of the lemma is as follows.

Theorem 5. *Let $G = (V, E)$ be a directed acyclic graph, a weight function $w : E \to R$ and $\mathcal{A} = (A_1, A_2, \ldots, A_n)$, $\mathcal{B} = (B_1, B_2, \ldots, B_n)$ be two sets of vertices which are not necessarily disjoint. Let M be the path matrix from (A) to (B) and let VD be the set of all vertex disjoint path systems from $\mathcal{A}$ to $\mathcal{B}$. Then*

$$det(M) = \sum_{\mathcal{P} \in VD} sign(\mathcal{P}) w(\mathcal{P})$$

Proof. Lets consider the permutation description of determinant.

$$det(M) = \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}$$

Fix arbitrary $\sigma \in S_n$. Consider $sign(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}$.

$$sign(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)} = sign(\sigma)[\sum_{P_1 : A_1 \to B_{\sigma(1)}} w(P_1)] \ldots [\sum_{P_n : A_n \to B_{\sigma(n)}} w(P_n)]$$

Now, for a given σ there are collection of path systems from $\mathcal{A}$ to $\mathcal{B}$. Define

$$\mathcal{P}_\sigma = \{\mathcal{P} | \mathcal{P} \text{ is a path system from } \mathcal{A} \text{ to } \mathcal{B} \text{ given by } \sigma\}$$

Now, we have

$$sign(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)} = sign(\sigma)[\sum_{P_1 : A_1 \to B_{\sigma(1)}} w(P_1)] \ldots [\sum_{P_n : A_n \to B_{\sigma(n)}} w(P_n)]$$

$$= \sum_{\mathcal{P} \in \mathcal{P}_\sigma} sign(\mathcal{P}) w(\mathcal{P})$$

We know that $\bigcup_{\sigma \in S_n} \mathcal{P}_\sigma = AllPathSystems$.

Therefore, we get

$$\sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)} = \sum_{\sigma \in S_n} \sum_{\mathcal{R} \in \mathcal{P}_\sigma} sign(\mathcal{R}) w(\mathcal{R})$$

$$= \sum_{\mathcal{P} \in AllPathSystems} sign(\mathcal{P}) w(\mathcal{P})$$

we get summation over all path systems $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{B}$. Therefore, we have

$$det(M) = \sum_{\mathcal{P}} sign(\mathcal{P}) w(P)$$

Now, we have to prove that $\sum_{\mathcal{P}} sign(\mathcal{P}) w(P) = \sum_{\mathcal{P} \in VD} sign(\mathcal{P}) w(P)$. Proving the following lemma will prove this theorem. □

Lemma 6. $\sum_{\mathcal{P}} sign(\mathcal{P})w(P) = \sum_{\mathcal{P}\in VD} sign(\mathcal{P})w(P)$

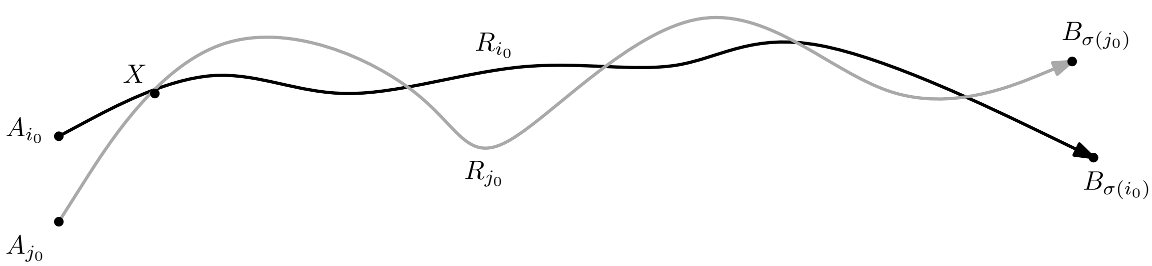
Proof. Let ND be the set of path systems that are not vertex disjoint path systems from $\mathcal{A}$ to $\mathcal{B}$. We break the left hand side as follows since the set of all path systems is partitioned into VD and ND.

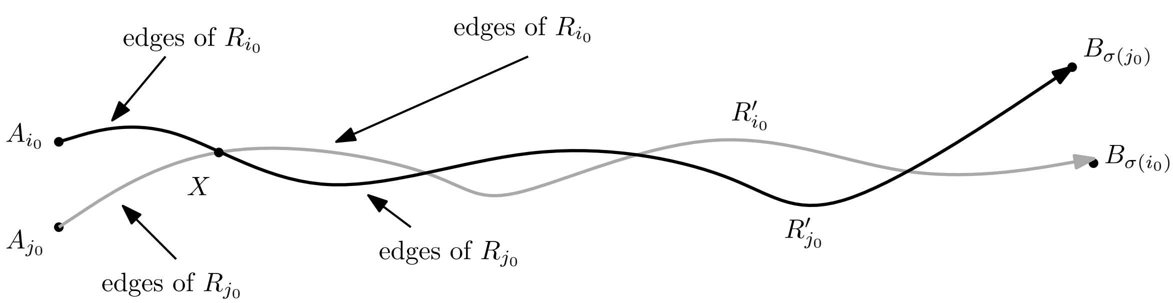
$$\sum_{\mathcal{P}} sign(\mathcal{P})w(P) = \sum_{\mathcal{P}\in VD} sign(\mathcal{P})w(P) + \sum_{\mathcal{P}\in ND} sign(\mathcal{P})w(P)$$

Pick any $\mathcal{R} \in ND$. $\mathcal{R} = (R_1, \ldots, R_n)$. Among the crossing paths, we define the following things:

i_0 = smallest index such that R_{i_0} crosses with some R_j with $j > i_0$.
X = first vertex at which R_{i_0} is intersected by some other path in $\mathcal{R}$.
j_0 = smallest index of all the paths in $\mathcal{R}$ that intersect R_{i_0} at X (or equivalently smallest index of all paths such that $X \in R_{i_0} \cap R_{j_0}$)

Now, we define involution on ND as follows.
Define $\phi : ND \to ND$ by $\phi(\mathcal{R}) = \mathcal{T} = (T_1, \ldots, T_n)$ where
$T_k = R_k$ when $k \neq i_0, j_0$
T_{i_0} : From A_{i_0}, traverse the edges of R_{i_0} till it reaches X. After that it traverses the edges of R_{j_0} from X to reach $B_{\sigma(j_0)}$.
T_{j_0} : From A_{j_0}, traverse the edges of R_{j_0} till it reaches X. After that it traverses the edges of R_{i_0} from X to reach $B_{\sigma(i_0)}$.





Therefore, for the path system $\mathcal{R}'$, we get $\sigma' \in S_n$ where $\sigma' = \sigma \cdot (i_0 j_0)$. Therefore, $sign(\sigma') = sign(\sigma)sign(i_0\ j_0) = -sign(\sigma)$. Therefore, $-sign(\mathcal{R}) = sign(\mathcal{T})$. Also both $\mathcal{T}$ and $\mathcal{R}$ consists of same set of edges. Therefore,

$$w(\mathcal{T}) = \prod_{i=1}^{n} w(T_i) = [\prod_{k\in[n]\setminus\{i_0,j_0\}} w(T_k)].w(T_{i_0}).w(T_{j_0}) = [\prod_{k\in[n]\setminus\{i_0,j_0\}} w(R_k)].w(T_{i_0}).w(T_{j_0})$$

Now, set of edges that are there in T_{i_0}, T_{j_0} are the same set of edges that are there in R_{i_0}, R_{j_0}. Therefore

$$w(T_{i_0})w(T_{j_0}) = \prod_{e\in T_{i_0}\cup T_{j_0}} w(e) = \prod_{e\in R_{i_0}\cup R_{j_0}} w(e) = w(R_{i_0})w(R_{j_0})$$

Therefore, $w(\mathcal{R}) = w(\mathcal{T})$.

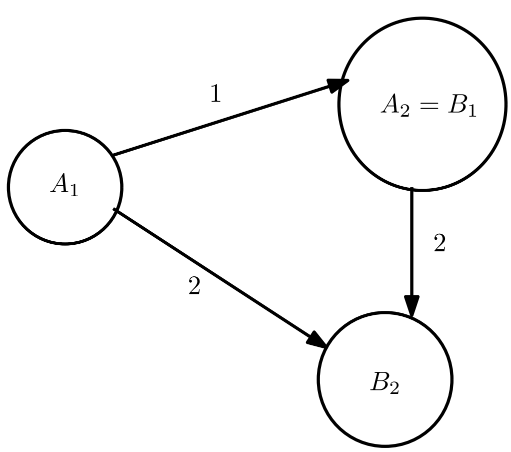
Now, if we apply ϕ again, then let $\phi(\mathcal{T}) = \mathcal{Z}$.
$Z_k = T_k = R_k$ when $k \neq i_0, j_0$
Z_{i_0} : From A_{i_0}, traverse the edges of T_{i_0} till it reaches X. After that it traverses the edges of T_{j_0} from X to reach $B_{\sigma'(j_0)} = B_{\sigma(i_0)}$. This yields path which is same as R_{i_0}.
Z_{j_0} : From A_{j_0}, traverse the edges of T_{j_0} till it reaches X. After that it traverses the edges of T_{i_0} from X to reach $B_{\sigma'(i_0)} = B_{\sigma(j_0)}$. This yields path R_{j_0}.

Therefore, $\mathcal{Z} = \mathcal{R}$. Therefore, $\phi(\phi(\mathcal{R})) = \mathcal{R}$. Since $\phi = \phi^{-1}$, therefore, ϕ is a bijection. Therefore, we have found an one to one correspondance to obtain pairs of path systems $(\mathcal{R}, \phi(\mathcal{R}))$ in ND where for every pairs of path systems with $w(R) = w(\phi(\mathcal{R}))$ and $sign(\mathcal{R}) = sign(\phi(\mathcal{R}))$. Therefore, we get that $\sum_{\mathcal{P}\in ND} sign(\mathcal{P})w(P) = 0$.
It implies that $\sum_{\mathcal{P}} sign(\mathcal{P})w(P) = \sum_{\mathcal{P}\in VD} sign(\mathcal{P})w(P)$. □

Proving the lemma proves that $det(M) = \sum_{\mathcal{P}\in VD} sign(\mathcal{P})w(P)$ which proves the theorem.

4.1 Example of a graph and path matrix

We give an example of a directed acyclic graph and its corresponding path matrix. Let G be the graph as drawn in figure. Notice that $\mathcal{A}$ and $\mathcal{B}$ are not disjoint. In fact, in this path matrix, there are path systems which are not vertex disjoint(common vertex can be present even at endpoints also).



The path systems are as follows.
$\mathcal{P}_1 = [(\{A_1 \to B_1\}, \{A_2 \to B_2\}), (1)(2)]$
$\mathcal{P}_2 = [(\{A_1 \to B_1 \to B_2\}, \{A_2 \to B_1\}), (12)]$
$\mathcal{P}_3 = [(\{A_1 \to B_2\}, \{A_2 \to B_1\}), (12)]$

Only P_3 is vertex disjoint path system.
Now the elements of the path matrix M are as follows:
$m_{11} = w(A_1 \to B_1) = 2$
$m_{12} = w(A_1 \to B_1 \to B_2) + w(A_1 \to B_2) = 1.2 + 2 = 4$
$m_{21} = w(A_2 \to B_1) = 1$
$m_{22} = w(A_2 \to B_2) = 2$
Therefore, $det(M) = 2 - 4 = -2$ and

$$\sum_{\mathcal{P} \in VD} sign(\mathcal{P}) \prod_{i=1}^{n} w(P_i) = sign(\mathcal{P}_3).w(\mathcal{P}_3) = (-1).w(\{A_1 \to B_2\}).w((\{A_2 \to B_1\}) = (-1).2.1 = -2$$
.

Therefore, we can see that the Gessel Viennot Lemma holds.

5 Applications of Gessel Viennot Lemma to Matrix Properties

It has applications to Matrix Theorems as well as some combinatorial applications. Consider these applications in matrix theorems.

5.1 Application 1:

For any square matrix M, $det(M) = det(M^T)$.

Proof. For M, we consstructed a directed bipartite graph $G = (\mathcal{A}, \mathcal{B}, E)$ with bipartitions $\mathcal{A}, \mathcal{B}$ and

$$E = \{(A_i, B_j) | A_i \in \mathcal{A}, B_j \in \mathcal{B}\}$$

Now, $m^T_{ij} = m_{ji}$. Now, we construct the bipartite graph $H = (\mathcal{C}, \mathcal{D}, F)$ where $\mathcal{C} = \mathcal{A}, \mathcal{D} = \mathcal{B}$ and

$$F = \{(D_i, C_j) | D_i \in \mathcal{D}, C_j \in \mathcal{C}\}$$

Now, since $sign(\sigma) = sign(\sigma^{-1})$, therefore we get

$$det(M^T) = \sum_{\mathcal{P}} sign(\mathcal{P}) w(\mathcal{P}) = \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} m^T_{i,\sigma(i)}$$

$$= \sum_{\sigma \in S_n} sign(\sigma) \prod_{k=1}^{n} m_{k,\sigma^{-1}(k)}$$

$$= \sum_{\sigma \in S_n} sign(\sigma^{-1}) \prod_{k=1}^{n} m_{k,\sigma^{-1}(k)}$$

$$= det(M)$$

□

5.2 Application 2:

For any two square matrices of same size M_1, M_2, $det(M_1.M_2) = det(M_1)det(M_2)$.

Proof. Construct these set of vertices $\mathcal{A} = (A_1, \ldots, A_n), \mathcal{B} = (B_1, \ldots, B_n), \mathcal{C} = (C_1, \ldots, C_n)$. We construct the directed graph with vertex $\mathcal{A}, \mathcal{B}, \mathcal{C}$ where edges are directed from $\mathcal{A}$ to $\mathcal{B}$ and also from $\mathcal{B}$ to $\mathcal{C}$ with $w(A_i, B_j) = m_1[i, j]$ and $w(B_j, C_k) = m_2[j, k]$.

$\mathcal{A}$ corresponds to the rows of M_1. $\mathcal{B}$ corresponds to the columns of M_1 as well as to the rows of M_2. $\mathcal{C}$ corresponds to the columns of M_2.

Let $M = M_1.M_2$. Then, $m[i, j] = \sum_{k=1}^{n} m_1[i, k].m_2[k, j]$. Now, we consider an arbitrary vertexd disjoint path systems $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{C}$. $\mathcal{P}$ must come through vertices of $\mathcal{B}$. Since $\mathcal{P}$ is vertex disjoint, therefore any path system which is a subpath system of $\mathcal{P}$ is also vertex disjoint. Now, we see that any vertex disjoint path system $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{C}$ is divided into two parts $\mathcal{Q}$ and $\mathcal{R}$ where $\mathcal{Q}$ is a vertex disjoint path system from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{R}$ is a vertex disjoint path system from $\mathcal{B}$ to $\mathcal{C}$. Let W be the set of all vertex disjoint path systems from $\mathcal{A}$ to $\mathcal{B}$ and Z be the set of all vertex disjoint path systems from $\mathcal{B}$ to $\mathcal{C}$. Now consider

$$det(M_1).det(M_2) = \sum_{\mathcal{Q} \in W} sign(\mathcal{Q})w(\mathcal{Q}) \sum_{\mathcal{R} \in Z} sign(\mathcal{R})w(\mathcal{R})$$

$$= \sum_{\mathcal{P} \in W \times Z} sign(\mathcal{R})sign(\mathcal{Q})w(\mathcal{R})w(\mathcal{Q})$$

Now $W \times Z$ is the set of all ordered pair of vertex disjoint path systems where the for every pair $(\mathcal{Q}, \mathcal{R})$ where $\mathcal{Q}$ is a vertex disjoint path system from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{R}$ is a vertex disjoint path system from $\mathcal{B}$ to $\mathcal{C}$.

Now, $\mathcal{Q} \simeq (\sigma_Q, Q_1, Q_2, \ldots, Q_n)$ where $\forall i \in [n] : Q_i : A_i \to B_{\sigma_R(i)}$.
$\mathcal{R} \simeq (\sigma_R, R_1, R_2, \ldots, R_n)$ where $\forall i \in [n] : R_i : B_i \to C_{\sigma_R(i)}$.

Now, consider $P_i : A_i \to B_{\sigma_Q(i)} \to C_{\sigma_R(\sigma_Q(i))}$. Composition of two permutations is also a permutation. Let $\sigma_Q \circ \sigma_R = \sigma$. So, composition of $\mathcal{Q}$ and $\mathcal{R}$ gives a path system $\mathcal{P}$.

We have to show that $W \times Z$ exhausts the set of all vertex disjoint path systems from $\mathcal{A}$ to $\mathcal{C}$.

Take any vertex disjoint path system $\mathcal{P}$ given by $\sigma \in S_n$ from $\mathcal{A}$ to $\mathcal{C}$. Since it is vertex disjoint and every path from $\mathcal{A}$ to $\mathcal{C}$ comes through $\mathcal{B}$ and $\mathcal{P}$ exhausts all vertices of $\mathcal{B}$ (otherwise $\mathcal{P}$ would not be vertex disjoint). Therefore, $\mathcal{P}$ is decomposed into $\mathcal{Q}$ (vertex disjoint path system from $\mathcal{A}$ to $\mathcal{B}$) and $\mathcal{R}$ (vertex disjoint path system from $\mathcal{B}$ to $\mathcal{C}$). It easily follows that $w(\mathcal{P}) = w(\mathcal{Q})w(\mathcal{R})$. Also, since $sign(\sigma) = sign(\sigma_Q \circ \sigma_R) = sign(\sigma_Q)sign(\sigma_R)$, therefore, $sign(\mathcal{P}) = sign(\mathcal{Q}).sign(\mathcal{R})$.

Therefore, we have the followings.

$$det(M_1).det(M_2) = \sum_{\mathcal{P} \in W \times Z} sign(\mathcal{R})sign(\mathcal{Q})w(\mathcal{R})w(\mathcal{Q}) = \sum_{\mathcal{P}} sign(\mathcal{P})w(\mathcal{P}) = det(M_1.M_2)$$

□

5.3 Application 3

This application is a generalization of application 2. It is called **Cauchy Binet Formula**.

Theorem 7. *Let M_1 be an $n \times r$ matrix and M_2 be an $r \times n$ matrix with $n \leq r$. Then we have the following*

$$det(M_1.M_2) = \sum_{X \subseteq [r], |X| = n} det(M_1[X]).det(M_2[X])$$

where $M_1[X]$ is the matrix restricted to the columns indexed by X and $M_2[X]$ is the matrix restricted to the rows indexed by X.

Proof. Construct the directed graph $G = (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}, E)$ where $\mathcal{A} = (A_1, \ldots, A_n), \mathcal{B} = (B_1, \ldots, B_r), \mathcal{C} = (C_1, \ldots, C_n)$. $E = \{(A_i, B_j) | i \in [n], j \in [r]\} \cup \{(B_j, C_k) | j \in [r], k \in [n]\}$. Also, we assign weights to the edges as $w(A_i, B_j) = m_1[i, j]$ and $w(B_j, C_k) = m_2[k, j]$.

Let $M = M_1.M_2$. $m[i, j] = \sum_{k=1}^{r} m_1[i, k]m_2[k, j]$.

Fix any arbitrary $X \subseteq [r]$. Let $\mathcal{P}_{AX}$ be the set of all vertex disjoint path systems from $\mathcal{A}$ to $\mathcal{B}[\mathcal{X}]$($B[X]$ be the subset of B restricted to the indices denoted by X) and $\mathcal{P}_{XB}$ be the set of all vertex disjoint path systems from $\mathcal{B}[\mathcal{X}]$ to $\mathcal{C}$. Then, consider $det(M_1[X]).det(M_2[X])$.

$$det(M_1[X]).det(M_2[X]) = \sum_{\mathcal{Q} \in \mathcal{P}_{AX}} sign(\mathcal{Q})w(\mathcal{Q}). \sum_{\mathcal{R} \in \mathcal{P}_{XB}} sign(\mathcal{R})w(\mathcal{R})$$

Now, similar to the *Lemma 6*, we know that $\mathcal{P}_{AX} \times \mathcal{P}_{XB}$ exhausts the set of all vertex disjoint path systems from $\mathcal{A}$ to $\mathcal{C}$ that passes through this $B[X]$. Therefore we have that

$$\sum_{\mathcal{Q} \in \mathcal{P}_{AX}} sign(\mathcal{Q})w(\mathcal{Q}). \sum_{\mathcal{R} \in \mathcal{P}_{XB}} sign(\mathcal{R})w(\mathcal{R}) = \sum_{\mathcal{P} \in \mathcal{P}_{AX} \times \mathcal{P}_{XB}} sign(\mathcal{P})w(\mathcal{P})$$

Now, in order to get $\sum_{X \subseteq [r], |X|=n} det(M_1[X]).det(M_2[X])$, we have to sum the expression

$$[\sum_{\mathcal{P} \in \mathcal{P}_{AX} \times \mathcal{P}_{XB}} sign(\mathcal{P})w(\mathcal{P})]$$

over all $X \subseteq [r]$.

This summation will also give the summation over the set of all vertex disjoint path systems from $\mathcal{A}$ to $\mathcal{C}$.
Therefore we have,

$$\sum_{X \subseteq [r], |X|=n} det(M_1[X]).det(M_2[X]) = \sum_{X \subseteq [r], |X|=n} [\sum_{\mathcal{P} \in \mathcal{P}_{AX} \times \mathcal{P}_{XB}} sign(\mathcal{P})w(\mathcal{P})]$$

$$= \sum_{\mathcal{P}} sign(\mathcal{P})w(\mathcal{P}) = det(M_1.M_2)$$

□

6 Combinatorial Applications of Gessel Viennot Lemma

Gessel Viennot Lemma also has some interesting combinatorial applications such as **Binomial Determinants in terms of Lattice Paths**, **Rhombic Tilings** are all available in [1].

References

[1] Martin Aigner: *A Course in Enumeration*, Springer.

[2] Justin Chan: *The Gessel Viennot Lemma and its Applications to Combinatorics*, Math 821, December 14, 2010.

[3] Martin Aigner, Gunter M. Zeigler: *Proofs From THE BOOK*, Springer, Fourth Edition