Exercise sheet #7
Set Theory 2024

Exercise 1. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partially ordered sets. A function $f\colon P \to Q$ is an *embedding of partial orders* if it satisfies $\forall x, y \in P(x \leq_P y \Leftrightarrow f(x) \leq_Q f(y))$. Show that an order embedding is always injective.

Exercise 2. Show that every countable linear order can be embedded into $(\mathbb{Q}, <)$.

Exercise 3.

1. Let $\varepsilon_0 = \lim_{n\to\omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$ for all n. Show that ε_0 is the least ordinal ε such that $\omega^\varepsilon = \varepsilon$.
2. ε_0 is countable.
3. Find a subset of $\mathbb{Q}$ isomorphic to ε_0.

Exercise 4. Let $(P, \leq_P)$ be a partially ordered set. For each subset A of P, define

$$A^\uparrow := \{x \in P : \forall a \in A(a \leq_P x)\} \text{ and}$$
$$A^\downarrow := \{x \in P : \forall a \in A(x \leq_P a)\}$$

A *cut* of $(P \leq_P)$ is a pair $(A, B) \in \mathcal{P}(P)^2$ such that $A^\uparrow = B$ and $B^\downarrow = A$. Let $\mathcal{C}(P)$ denote the set of all cuts of P.

1. Show that $(\mathcal{C}(P), \leq)$ is a partially ordered set, where $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \subseteq A_2$.
2. Show that if (A, B) is a cut, then $(A^\uparrow)^\downarrow = A$.
3. Show that if $(A^\uparrow)^\downarrow = A$, then $(A, A^\uparrow)$ is a cut.
4. Show that the function $\ell\colon P \to \mathcal{C}(P)$ given by $\ell(p) = (\{x \in P : x \leq p\}, \{x \in P : x \leq p\}^\uparrow)$ is an embedding.
5. Show that for all $Q \subseteq P$, the set $\ell[Q]$ has a supremum and an infimum in $(\mathcal{C}(P), \leq)$.

Exercise 5. A set A can be well-ordered if and only if there exists a choice function on $\mathcal{P}(A) \setminus \{\varnothing\}$.

Exercise 6. (AC) Every infinite set has a countable subset.

Exercise 7. (AC) For every infinite set A there exists an ordinal α such that $|A| = \aleph_\alpha$.

Exercise 8. (AC) The union of a countable collection of countable sets is countable.

Exercise 9. (AC) If f is a function and $\mathrm{dom}(f) = A$, then $|f[A]| \leq |A|$.

Exercise 10. (AC) A set is finite if and only if it is Dedekind-finite.

Exercise 11. (AC) Every partial ordering can be extended to a linear ordering.

Exercise 12. The following are equivalent:

1. (AC) A choice function exists for every system of nonempty sets.
2. If $X_i \neq \varnothing$ for all $i \in I$, then $\prod X_i \neq \varnothing$.
3. (Zorn's Lemma) Let $(P, \leq)$ be a partially ordered set. If every chain in P has an upper bound, then P has a maximal element.
4. Every set can be well-ordered.
5. (Tukey's Lemma). Every system of sets with finite character has a $\subseteq$-maximal element.(A system of sets A has finite character if $X \in A$ holds iff every finite subset of X belongs to A.)