Exercise sheet #6
Set Theory 2024

Definition 1. A binary relation $R \subseteq X \times X$ is

1. *total* if $\forall a \in X \exists b \in X(R(a,b))$ and
2. *well-founded* if $\forall S \subset X(S \neq \varnothing \Rightarrow \exists m \in S \forall s \in S(\neg R(s,m))$.

Exercise 1. The following statement is known as the Axiom of Dependent Choice (DC): For every nonempty X and every total binary relation $R \subset X \times X$ there exists a sequence of elements of X $(x_n)_{n\in\omega}$ such that for all $n \in \omega$ $R(x_n, x_{n+1})$. Use DC to prove that a binary relation $R \subseteq X \times X$ is well-founded if and only if there is no infinite sequence $(x_n)_{n\in\omega}$ such that $R(x_{n+1}, x_n)$ for all $n \in \omega$.

Exercise 2. Suppose that A is at most countable and B is uncountable. Prove that $B \setminus A$ is uncountable.

Exercise 3. If S is uncountable and $S \subseteq T$, then T is uncountable.

Exercise 4. If R well-orders A and $X \subseteq A$, then R well-orders X.

Exercise 5. If $<$ and $\prec$ are well-orders on S, T, respectively, then their lexicographic product is a well-order of $S \times T$.

Exercise 6. If A is finite and R is a binary relation on A, then R is well-founded if and only if R is acyclic on A.

Exercise 7. Present an example of an infinite set A and an acyclic binary relation R on A which is not well-founded.

Exercise 8. In this exercise, the notation $\forall \alpha \in \mathrm{ON}(\varphi(\alpha))$ abbreviates $\forall \alpha(\alpha \textit{ is an ordinal} \to \varphi(\alpha))$. Prove:

1. $\forall \alpha \in \mathrm{ON} \forall z \in \alpha(z \in \mathrm{ON})$
2. $\forall \alpha, \beta \in \mathrm{ON}(\alpha \cap \beta \in \mathrm{ON})$
3. $\forall \alpha, \beta \in \mathrm{ON}(\alpha \subseteq \beta \leftrightarrow (\alpha \in \beta \vee \alpha = \beta))$
4. $\forall \alpha, \beta, \gamma \in \mathrm{ON}(\alpha \in \beta \wedge \beta \in \gamma \to \alpha \in \gamma)$
5. $\forall \alpha \in \mathrm{ON}(\alpha \notin \alpha)$
6. $\forall \alpha, \beta \in \mathrm{ON}(\alpha \in \beta \vee \beta \in \alpha \vee \alpha = \beta)$
7. Every nonempty set of ordinals has a $\in$-least element.
8. $\{x : x \textit{ is an ordinal}\}$ is not a set.